SPRING 2007 FINAL

Question 1: (20 Points)

Evaluate the following integrals :

(a) $\int \sin^3(\frac{x}{2})dx$ (b) $\int \sin(2x)\cos(2x)dx$ (c) $\int (\tan^2(x)+\tan(x))dx$

a) $\cos^2\frac{x}{2}+\sin^2\frac{x}{2}=1 \Rightarrow \sin^2\frac{x}{2}=1-\cos^2\frac{x}{2}$

So $\int \sin^3\frac{x}{2}\,dx = \int (1-\cos^2\frac{x}{2})\sin\frac{x}{2}\,dx = \int \sin\frac{x}{2}-\sin\frac{x}{2}\cdot\cos^2\frac{x}{2}\,dx$

$$= \int \sin\frac{x}{2}\,dx - \int \sin\frac{x}{2}\cdot\cos^2\frac{x}{2}\,dx$$

$$= -2\cos\frac{x}{2} + C_1 + \frac{2}{3}\cos^3\frac{x}{2} + C_2$$

$$= -2\cos\frac{x}{2} + \frac{2}{3}\cos^3\frac{x}{2} + C$$

b) $\int \sin 2x\cos 2x\,dx = \int \frac{1}{2}\sin 4x\,dx$

as $2\sin 2x\cos 2x = \sin 4x$

So $\int \frac{1}{2}\sin 4x\,dx = -\frac{1}{8}\cos 4x + C$

c) $\int (\tan^2 x+\tan x)dx = \int \left(\frac{\sin^2 x}{\cos^2 x}+\frac{\sin x}{\cos x}\right)dx = \int \frac{\sin^2 x+\sin x\cos x}{\cos^2 x}\,dx$

$$= \int \frac{\sin^2 x}{\cos^2 x}\,dx + \int \frac{\sin x\cos x}{\cos^2 x}\,dx$$

$$= \int \frac{1-\cos^2 x}{\cos^2 x}\,dx + \int \frac{\sin x\cos x}{\cos^2 x}\,dx$$

$$= \int \frac{1}{\cos^2 x}\,dx - \int dx + \int \frac{\sin x\cos x}{\cos^2 x}\,dx$$

$$= \tan x - x - \frac{1}{2}\ln|\cos^2 x| + D$$

$\cos^2 x = u$

$-2\sin x\cos x = du$

$\int \frac{\sin x\cos x}{\cos^2 x}\,dx = \int -\frac{1}{2u}\,du$

$= -\frac{1}{2}\ln|u| + C$

$= -\frac{1}{2}\ln|\cos^2 x| + C$

Question 2: (15 Points)
Evaluate

$$\int \frac{-x^3+2x^2-3x+6}{(x^2+1)(x-1)^2}dx$$

$$\frac{-x^3+2x^2-3x+6}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$= \frac{(Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)}{(x^2+1)(x-1)^2}$$

$$= \frac{Ax^3 + (B-2A)x^2 + (A-2B)x + B + Cx^3 - Cx^2 + Cx + C + Dx^2 + 1}{(x^2+1)(x-1)^2}$$

$$= \frac{(A+C)x^3 + (B-2A-C+D)x^2 + (A-2B+C)x + B + C + 1}{(x^2+1)(x-1)^2}$$

$\Rightarrow$ $A+C=-1$ $\qquad\Rightarrow$ $A=-5$

$B-2A-C+D=2$ $\qquad B=1$

$A-2B+C=-3$ $\qquad C=5$

$B+C+1=6$ $\qquad D=-5$

$$\int \frac{-x^3+2x^2-3x+6}{(x^2+1)(x-1)^2}dx = \int \frac{-5x+1}{x^2+1} + \frac{5}{x-1} - \frac{5}{(x-1)^2}\,dx$$

$$= \int \frac{-5x+1}{x^2+1}dx + \int \frac{5}{x-1}dx - \int \frac{5}{(x-1)^2}dx$$

$$= \int -\frac{5x}{x^2+1}dx + \int \frac{1}{x^2+1}dx + \int \frac{5}{x-1}dx - \int \frac{5}{(x-1)^2}dx$$

$$= -\frac{5}{2}\ln|x^2+1| + \arctan x + 5\ln|x| + \frac{5}{(x-1)} + C$$

Question 3: (10 Points + 5 Points):

Show that $\int e^{-\sqrt{x}}dx = -2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + c$ by

(a) integrating the left side using a substitution and/or Integration by Parts.

(b) using any other method

a) Let $u = -\sqrt{x}$, $du = -\frac{1}{2\sqrt{x}}dx \Rightarrow 2u\,du = dx$

$$\int e^{-\sqrt{x}}dx = \int 2ue^{u}du$$

Let $e^{u}du = db \Rightarrow b = e^{u}$

$2u = a \Rightarrow da = 2du$

So $\int 2ue^{u}du = a.b - \int b.da = 2ue^{u} - \int 2e^{u}du$

$= 2ue^{u} - 2e^{u} + C$

As $u = -\sqrt{x}$, $\int e^{-\sqrt{x}}dx = -2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C$

b) $\left(-2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C\right)' = -\frac{1}{\sqrt{x}}.e^{-\sqrt{x}} - 2\sqrt{x}.\left(\frac{-1}{2\sqrt{x}}\right).e^{-\sqrt{x}} - 2.\left(-\frac{1}{2\sqrt{x}}\right).e^{-\sqrt{x}}$

$= -\frac{1}{\sqrt{x}}e^{-\sqrt{x}} + e^{-\sqrt{x}} + \frac{1}{\sqrt{x}}e^{-\sqrt{x}}$

$= e^{-\sqrt{x}}$

So $\int e^{-\sqrt{x}}dx = -2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C$

Question 4: (10 Points)

Find the derivative of $f(x) = (\sin x)^{\tan x}$
(Hint: Express the function $f(x)$ as $e^{h(x)}$, or take the logarithm of both sides and use implicit differentiation)

$\ln f(x) = \tan x \cdot \ln(\sin x)$ Take the derivative of both sides

$$\Rightarrow \frac{f'(x)}{f(x)} = \sec^2 x \ln(\sin x) + \frac{\tan x \cdot \cos x}{\sin x}$$

$$\Rightarrow f'(x) = f(x)\left(\sec^2 x \ln(\sin x) + \frac{\frac{\sin x}{\cos x} \cdot \cos x}{\sin x}\right)$$

$$= (\sin x)^{\tan x}\left(\sec^2 x \ln(\sin x) + 1\right)$$

Question 5: (10 Points)
Evaluate the following integral:

$$\int \frac{2dx}{\sqrt{6x-2x^2+7}} = \int \frac{2\,dx}{\sqrt{-2x^2+6x-\frac{9}{2}+\frac{23}{2}}}$$

$$= \int \frac{2\,dx}{\sqrt{-2(x-\frac{3}{2})^2+\frac{23}{2}}} = \int \frac{2\,dx}{\sqrt{\frac{23}{2}}\left(\sqrt{1-\frac{4}{23}(x-\frac{3}{2})^2}\right)}$$

$$= \frac{2\sqrt{2}}{\sqrt{23}} \int \frac{1}{\sqrt{1-\left(\frac{2}{\sqrt{23}}x-\frac{3}{\sqrt{23}}\right)^2}}\,dx$$

Say $\frac{2}{\sqrt{23}}x - \frac{3}{\sqrt{23}} = a$. $\frac{2}{\sqrt{23}}\,dx = da$

Then $\frac{2\sqrt{2}}{\sqrt{23}} \int \frac{1}{\sqrt{1-(\frac{2}{\sqrt{23}}x-\frac{3}{\sqrt{23}})^2}}\,dx = \frac{2\sqrt{2}}{\sqrt{23}} \int \frac{1}{\sqrt{1-a^2}} \cdot \frac{\sqrt{23}}{2} \cdot da$

$$= \sqrt{2} \int \frac{1}{\sqrt{1-a^2}}\,da$$

$$= \sqrt{2} \arcsin a + C$$

$$= \sqrt{2} \arcsin\left(\frac{2}{\sqrt{23}}x - \frac{3}{\sqrt{23}}\right) + C$$

Question 6: (10 Points)

The graph of $f(x) = x^3 + bx^2 + cx + d$ is increasing in the interval $x < -1$, decreasing in the interval $-1 < x < 3$, and increasing in the interval $x > 3$. The graph is concave down for $x < 1$, and concave up for $x > 1$. The inflection point is on the x-axis. Find the constants b, c, and d.

$f'(x) = 3x^2 + 2bx + c$

$f'(x) = 0$ when $x = -1$ and $x = 3$

So $f'(x) = k(x+1)(x-3) = 3x^2 + 2bx + c$

$= kx^2 - 2kx - 3k = 3x^2 + 2bx + c$

$\Rightarrow k = 3$ $\qquad \Rightarrow b = -3$

$-2k = 2b$ $\qquad c = -9$

$-3k = c$

Since the concavity changes when $x = 1$, f has an inflection point at $x = 1$. Since it is on the x-axis the inflection point is $(1, 0)$.

So $f(1) = 1 + b + c + d = 0$

$= 1 - 3 - 9 + d = 0$

$d = 11$

Question 7: (10 Points)

Find an expression for the volume of the solid generated by revolving the region bounded by $y = \tan(x)$, $y = -1$, $x = 0$, and $x = \pi/4$ about the line $y = -1$. Do NOT evaluate the expression.

Question 8: (5 Points)

Find $\lim\limits_{x\to\infty} \frac{x^3}{3^x}$ $\underset{L'Hospital}{\overset{\frac{\infty}{\infty}}{=}}$ $\lim\limits_{x\to\infty} \frac{3x^2}{\ln 3 . 3^x}$ $\underset{L'Hospital}{\overset{\frac{\infty}{\infty}}{=}}$ $\lim\limits_{x\to\infty} \frac{6x}{(\ln 3)^2 . 3^x}$

$\underset{L'Hospital}{\overset{\frac{\infty}{\infty}}{=}}$ $\lim\limits_{x\to\infty} \frac{6}{(\ln 3)^3 . 3^x} = 0$

Question 9: (5 Points)

Find $f'(3)$ for any function f whose domain is R satisfying the inequality $|f(x)+x^2-2x+4| \le \sin^2(x-3)$ for all real numbers x.

$$-\sin^2(x-3) \le f(x)+x^2-2x+4 \le \sin^2(x-3)$$

$$-\sin^2(x-3)-x^2+2x-4 \le f(x) \le \sin^2(x-3)-x^2+2x-4$$

$$-7 \le f(3) \le -9+6-4=-7$$

$$f'(3)=\lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$$

$$-\sin^2(x-3)-x^2+2x+3 \le f(x)-f(3) \le \sin^2(x-3)-\overbrace{x^2+2x+3}^{(x-3)(x+1)}$$

assume $x>3$

$$\frac{-\sin^2(x-3)-(x-3)(x+1)}{x-3} \le \frac{f(x)-f(3)}{x-3} \le \frac{\sin^2(x-3)-(x-3)(x+1)}{x-3}$$

by Sadwich theorem $\lim\limits_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = -4$ since

$\lim\limits_{x\to 3^+} \frac{\sin^2(x-3)-(x-3)(x+1)}{(x-3)} = -4$ & $\lim\limits_{x\to 3^+} \frac{-\sin^2(x-3)-(x-3)(x+1)}{(x-3)} = -4$

Similarly left limit is -4.
So $f'(3)=-4$.

Question 10: (10 Points)

Find the length of the curve $y = f(x)$ between $-3 \leq x \leq -2$, if $f'(x) = \sqrt{x^2 - 1}$.