KOÇ UNIVERSITY

MATH 102

FIRST MIDTERM March 24, 2012

Duration of Exam: 75 minutes

INSTRUCTIONS:

- No calculators may be used on the test.
- No books, no notes, no questions, and no talking allowed.
- You must always explain your answers and SHOW YOUR WORK to receive full credit.
- Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

SURNAME, Name: KEY

Student ID no: __________

Signature: __________

(Check One):

(Selda Küçükçifçi - MWF 9:30-10:20) : —
(Selda Küçükçifçi - MWF 11:30-12:20) : —
(Şule Yazıcı - MWF 10:30-11:20) : —
(Ali Göktürk - MWF 12:30-13:20) : —
(Ali Göktürk - MWF 15:30-16:20) : —

PROBLEM	1	2	3	4	5	TOTAL
POINTS	36	24	15	15	12	**102**
SCORE						

Problem 1 (36 pts) Find the following limits. Do not use l'Hospital's rule.

(a) $\lim_{x \to 9^-} \left(\frac{3+\sqrt{x}}{x-9} \right)$ $= -\infty$ since as $x \to 9^-$

$3 + \sqrt{x} \to 6 > 0$

& $x - 9 \to 0^-$

(b) $\lim_{x \to -\infty} \frac{(3x+5)(2x^2-x^3)}{(-2x^2+4x-1)(x-7)}$ $= \lim_{x \to -\infty} \frac{\frac{(3x+5)(2x^2-x^3)}{x^3}}{\frac{(-2x^2+4x-1)(x-7)}{x^3}}$

$= \lim_{x \to -\infty} \frac{\frac{(3x+5)}{x} \frac{(2x^2-x^3)}{x^2}}{\frac{(-2x^2+4x-1)}{x^2} \frac{(x-7)}{x}}$ $= \lim_{x \to -\infty} \frac{(3+\underbrace{\frac{5}{x}}_{\to 0})(2-x)}{(-2+\underbrace{\frac{4}{x}}_{\to 0}-\underbrace{\frac{1}{x^2}}_{\to 0})(1-\underbrace{\frac{7}{x}}_{\to 0})}$

$= -\infty$

(c) $\lim_{x \to 2} \frac{\sqrt{x^2-3}-1}{x^2-2x}$

$= \lim_{x \to 2} \frac{(\sqrt{x^2-3}-1)(\sqrt{x^2-3}+1)}{(x^2-2x)(\sqrt{x^2-3}+1)}$ $= \lim_{x \to 2} \frac{x^2-3-1}{x(x-2)(\sqrt{x^2-3}+1)}$

$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{x\cancel{(x-2)}(\sqrt{x^2-3}+1)}$ $= \lim_{x \to 2} \frac{x+2}{x(\sqrt{x^2-3}+1)}$ $= \frac{4}{2 \cdot 2} = 1$

(d) $\lim_{x\to\infty} \dfrac{\sin x}{|x|}$

$-1 \le \sin x \le 1$

$\dfrac{-1}{|x|} \le \dfrac{\sin x}{|x|} \le \dfrac{1}{|x|}$ since $|x|>0$

$\lim_{x\to\infty} \dfrac{-1}{|x|} = 0$ and $\lim_{x\to\infty} \dfrac{+1}{|x|} = 0$ So $\lim_{x\to\infty} \dfrac{\sin x}{|x|} = 0$

(e) $\lim_{h\to 0} \dfrac{3^{(1+h)^2}-3}{h}$ $= f'(a)$ where $f(x) = 3^{x^2}$ and $a=1$

$f(1) = 3$

$f'(x) = 3^{x^2} \ln 3 \, (2x)$

$f'(1) = 3(\ln 3) \cdot 2 = 6 \ln 3$

So $\lim_{h\to 0} \dfrac{3^{(1+h)^2}-3}{h} = 6\ln 3$.

(f) Give an example of a function f satisfying $\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = +\infty$. (A graph will not be sufficient, write f as a function of x.)

Let $f(x) = \dfrac{1}{x-1}$

$\lim_{x\to 1^-} \dfrac{1}{x-1} = -\infty$

$\lim_{x\to 1^+} \dfrac{1}{x-1} = +\infty$.

Problem 2 (24 pts) Find the derivative of the function f in (a) – (c).

(a) $f(x) = \dfrac{x \sin x}{x^2+1}$

$$f'(x) = \frac{(\sin x + x\cos x)(x^2+1) - (x\sin x)(2x)}{(x^2+1)^2}$$

(b) $f(x) = \tan^{-1}(\sqrt{x^3+1})$

$$f'(x) = \frac{1}{1+(x^3+1)} \cdot \frac{1}{2\sqrt{x^3+1}} \cdot 3x^2$$

(c) $f(x) = x^{\ln x}$ $= y$

$$(\ln x)(\ln x) = \ln y$$

$$(\ln x)^2 = \ln y$$

$$2(\ln x) \cdot \frac{1}{x} = \frac{1}{y} \cdot y'$$

$$y' = \frac{2\ln x}{x} \cdot x^{\ln x}$$

Problem 3 (15 pts) Let f be a continuous function on $[-1, 3]$ such that $f(-1) = 2$, $f(1) = 5$ and $f(2) = 4$. By using the Intermediate Value Theorem show that f is not one-to-one.

Since f is continuous on $[-1, 3]$ and $f(-1) = 2 < 4 < 5 = f(1)$ by Intermediate Value Theorem there is $c \in (-1, 1)$ such that $f(c) = 4$. But $f(2) = 4$, too.

So f is not one-to-one.

Problem 4 (15 pts) Using implicit differentiation, find the points on the ellipse $x^2 + 2y^2 = 1$, where the tangent line has slope 1.

$$2x + 4y\,y' = 0 \Rightarrow y' = \frac{-2x}{4y} = \frac{-x}{2y}$$

$$\frac{-x}{2y} = 1 \Rightarrow -x = 2y \Rightarrow x = -2y$$

$$\begin{cases} x = -2y \\ x^2 + 2y^2 = 1 \end{cases}$$

$$(-2y)^2 + 2y^2 = 1 \Rightarrow 6y^2 = 1 \Rightarrow y^2 = \frac{1}{6} \Rightarrow y = \pm\frac{1}{\sqrt{6}}$$

$$y = \frac{1}{\sqrt{6}} \qquad x = -\frac{2}{\sqrt{6}}$$

$$y = -\frac{1}{\sqrt{6}} \qquad x = \frac{2}{\sqrt{6}}$$

So the points are $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ & $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

Problem 5 (12 pts) Use linear approximation to estimate $e^{0.01}$.

Let $f(x) = e^x$ and $L(x)$ be a linear approximation of $f(x)$ at $x=0$. Then

$$L(x) = f(0) + f'(0)(x-0)$$

$f(0) = 1$ $\quad$ $f'(x) = e^x$ $\quad$ $f'(0) = 1$

then $L(x) = 1 + (x-0) = 1+x$

Since $e^{0.01} = f(0.01) \approx L(0.01) = 1 + 0.01 = 1.01$

$$e^{0.01} \approx 1.01.$$