MATH 102 Midterm 1 Spring 2008

Problem 1 Find the following limits. Show all your work.

1a (6 pts) $\lim_{x\to\infty} \frac{5x^3+2x^2-1}{6x^3+4x}$

1b (7 pts) $\lim_{x\to 0} \frac{x^2-3\sin x}{2x+\cos x^2}$

1c (7 pts) $\lim_{x\to\infty} x- \frac{1}{\sin\frac{1}{x}}$

Problem 2 (15 pts) Two positive numbers are such that the sum of the first number and the square of the second number is 10. Find such numbers whose sum is the largest.

Problem 3 (25 pts) Let $f(x) = x.(x-3)^2$

a) Find all critical points, and intervals on which f is increasing & decreasing.
b) Find inflection points, and intervals on which f is concave up & concave down.
c) Find the asymptotes, if exist.
e) Sketch the graph of f.

Problem 4 (20 pts) If f is differentiable everywhere, find a and b.

$$f(x) = \begin{cases} x^3 - ax^2 + b & x > 2 \\ bx - 3a & x \leq 2 \end{cases}$$

Problem 5a (10 pts) Find the derivative of $f(x) = \cos\left(\frac{\sqrt{x}}{x^2+3}\right)$
5b (10 pts) Find the points on the curve $x^2 + xy + y^2 = 3$ where the tangent line is horizontal.
5c (5 pts) Find the points on the curve $x^2 + xy + y^2 = 3$ where the tangent line is vertical.

MATH 102 MIDTERM 1 SOLUTIONS

1. a. $\lim_{x\to\infty} \frac{5x^3+2x^2+1}{6x^3+4x} = \lim_{x\to\infty} \frac{x^3\left(5+\frac{2}{x}+\frac{1}{x^3}\right)}{x^3\left(6+\frac{4}{x^2}\right)} = \lim_{x\to\infty} \frac{5x^3}{6x^3} = \frac{5}{6}$

b. $\lim_{x\to 0} \frac{x^2-3\sin x}{2x+\cos x^2} = \lim_{x\to 0} \frac{0-0}{0+1} = \lim_{x\to 0} \frac{0}{1} = 0$

c. $\lim_{x\to\infty} x - \frac{1}{\sin\frac{1}{x}} = \lim_{h\to 0^+} \frac{1}{h} - \frac{1}{\sinh} = \lim_{h\to 0^+} \frac{\sinh - h}{h\cdot\sinh} = \lim_{h\to 0^+} \frac{\cosh-1}{\sinh+h\cosh}$

(L'hopital)

$= \lim_{h\to 0^+} \frac{-\sinh}{\cosh+\cosh-h\sinh} = \lim_{h\to 0^+} \frac{-0}{1+1-0} = \lim_{h\to 0^+} \frac{0}{2} = 0$

(L'hopital)

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2. $x+y^2=10 \Rightarrow x=10-y^2$

$S = x+y \Rightarrow S(y) = 10-y^2+y \Rightarrow S'(y) = -2y+1 \Rightarrow S'(y)=0 \Rightarrow \boxed{y=\frac{1}{2}}$

$x = 10-\frac{1}{4} = \boxed{\frac{39}{4}}$

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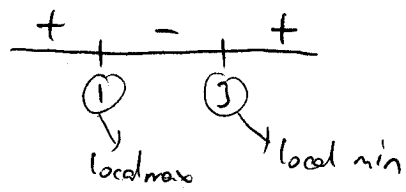
3. $f(x) = x\cdot(x-3)^2 = x\cdot(x^2-6x+9) = x^3-6x^2+9x$

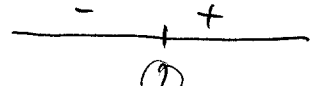
a. $f'(x) = 3x^2-12x+9 = 3(x^2-4x+3) = 3(x-1)(x-3)$



$f'>0$ $(-\infty,1)\cup(3,\infty)$ ↗

$f'<0$ $(1,3)$ ↘

b. $f''(x) = 6x-12 = 6(x-2)$

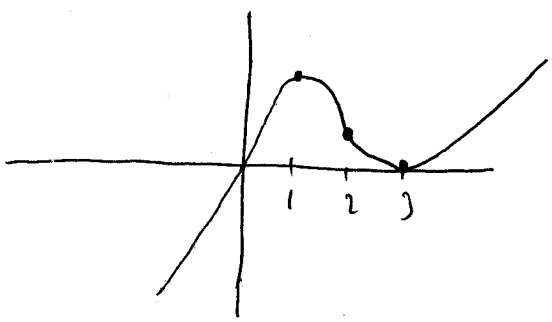


$f''<0$ $(-\infty,2)$ concave down

$f''>0$ $(2,\infty)$ concave up.

c. $\lim_{x\to\pm\infty} f(x) = \pm\infty$ No hor. asymptote. No vertical asymptote

d.



4. $f(x) = \begin{cases} x^2 - ax^2 + b & x \geq 2 \\ bx - 3a & x \leq 2 \end{cases}$

continuity at 2: $8 - 4a + b = 2b - 3a \Rightarrow a + b = 8$

diff. at 2: $x=2 \quad 3x^2 - 2ax = b$

$12 - 4a = b \Rightarrow 4a + b = 12$

$\Rightarrow a = \frac{4}{3}, \quad b = \frac{20}{3}$

5. a. $\left(\cos\left(\frac{\sqrt{x}}{x^2+3}\right)\right)' = -\sin\left(\frac{\sqrt{x}}{x^2+3}\right) \cdot \left(\frac{\frac{1}{2\sqrt{x}} \cdot (x^2+3) - \sqrt{x} \cdot 2x}{(x^2+3)^2}\right)$

b. $x^2 + xy + y^2 = 3$

$2x + y + xy' + 2yy' = 0$

$\Rightarrow y' = -\frac{(2x+y)}{x+2y}$

horizontal tangent line $y' = 0 \Rightarrow 2x + y = 0$

$y = -2x$

$x^2 + x \cdot -2x + (-2x)^2 = 3$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1 \quad y = \mp 2$

$(1, -2) \quad (-1, 2)$

c. Vertical tangent line $y' = \infty$ (or $x' = 0$)

$y' = -\frac{(2x+y)}{x+2y} \Rightarrow x + 2y = 0$

$x = -2y$

$(-2y)^2 + -2y \cdot y + y^2 = 3$

$3y^2 = 3$

$y^2 = 1$

$y = \pm 1 \Rightarrow x = \mp 2$

$(2, -1)$

$(-2, 1)$