MATH- 102/Spring2005, MT- 1 Solutions

1- a) $\lim\limits_{x\to -2}\dfrac{\sqrt{x^2+5}-3}{x+2}=\lim\limits_{x\to -2}\dfrac{\sqrt{x^2+5}-3}{x+2}\cdot\dfrac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}=\lim\limits_{x\to -2}\dfrac{x^2+5-9}{(x+2)\cdot\left(\sqrt{x^2+5}+3\right)}$

$$=\lim_{x\to -2}\frac{x^2-4}{(x+2)\cdot\left(\sqrt{x^2+5}+3\right)}=\lim_{x\to -2}\frac{x-2}{\sqrt{x^2+5}+3}=\frac{-4}{6}=-\frac{2}{3}.$$

Another Solution:

Since $\lim\limits_{x\to -2}\left(\sqrt{x^2+5}-3\right)=0$, $\lim\limits_{x\to -2}(x+2)=0$, $\sqrt{x^2+5}-3$ and $x+2$ are differentiable

functions at $x=-2$, we have

$$\lim_{x\to -2}\frac{\sqrt{x^2+5}-3}{x+2}\underset{\substack{\uparrow\\ L'H}}{=}\lim_{x\to -2}\frac{2x}{2\cdot\sqrt{x^2+5}\cdot 1}=\frac{-2}{3}.$$

b) $x+1=0\Rightarrow x=-1$.

$\lim\limits_{x\to -1^+} y=\lim\limits_{x\to -1^+}\dfrac{2x-6}{x+1}=-\infty$ and $\lim\limits_{x\to -1^-} y=\lim\limits_{x\to -1^+}\dfrac{2x-6}{x+1}=+\infty$.

Hence $x=-1$ is the vertical asymptote.

Since $\lim\limits_{x\to\infty} y=\lim\limits_{x\to\infty}\dfrac{2x-6}{x+1}=\lim\limits_{x\to\infty}\dfrac{2-\frac{6}{x}}{1+\frac{1}{x}}=2$, $y=2$ is the horizontal asymptote.

c- i) $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{\sin^2 x}{\tan x^2}=\lim\limits_{x\to 0}\dfrac{\sin x\cdot\sin x}{\dfrac{\sin x^2}{\cos x^2}}=\lim\limits_{x\to 0}\dfrac{(\sin x)^2}{x^2}\cdot\dfrac{x^2}{\sin x^2}\cdot\cos x^2$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{x^2}{\sin x^2}\cdot\cos x^2=1\cdot 1\cdot 1\cdot 1=1.$$

Another way for finding this limit;

Since $\lim_{x\to 0}\sin^2 x = 0$, $\lim_{x\to 0}\tan x^2 = 0$, $\sin^2 x$ and $\tan x^2$ are differentiable functions at $x = 0$, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin^2 x}{\tan x^2} \underset{L'H}{\overset{\uparrow}{=}} \lim_{x\to 0}\frac{2\cdot\sin x\cdot\cos x}{\sec^2 x^2\cdot 2x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\cos x\cdot\cos^2 x^2 = 1\cdot 1\cdot 1 = 1.$$

ii) f is continuous at $x = 0$ if and only if $\lim_{x\to 0} f(x) = f(0)$.

Since $\lim_{x\to 0} f(x) = 1$ and $f(0) = c^2$, we have $c^2 = 1$.

$c^2 = 1 \Rightarrow c = \pm 1$.

2-a) $r'(\theta) = 2\cdot\dfrac{1-\sin\theta}{1+\cos\theta}\cdot\dfrac{(-\cos\theta)\cdot(1+\cos\theta)-(-\sin\theta)\cdot(1-\sin\theta)}{(1+\cos\theta)^2}$

$$\Rightarrow r'(0) = 2\cdot\frac{1-\sin 0}{1+\cos 0}\cdot\frac{(-\cos 0)\cdot(1+\cos 0)-(-\sin 0)\cdot(1-\sin 0)}{(1+\cos 0)^2}$$

$$\Rightarrow r'(0) = 2\cdot\frac{1}{2}\cdot\frac{-2}{4} = -\frac{1}{2}.$$

b) $2y\cdot y' + 2y + 2x\cdot y' + 2x = 4\cdot(x\cdot y^3 + x^2)^3\cdot(y^3 + 3\cdot x\cdot y^2\cdot y' + 2x)$

When $x = 1$ and $y = 0$:

$2\cdot y' + 2 = 4\cdot(1)^3\cdot(2) = 8 \Rightarrow y' = \dfrac{6}{2} = 3.$ So the slpoe of the line tangent to this curve at $(1,0)$ is 3.

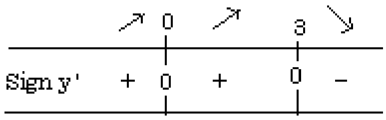
c) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\sec^2 t}{2t}$. When $t = \dfrac{\pi}{4}$: $\dfrac{dy}{dx} = \dfrac{\sec^2\left(\frac{\pi}{4}\right)}{2\cdot\frac{\pi}{4}} = \dfrac{(\sqrt{2})^2}{\frac{\pi}{2}} = \dfrac{4}{\pi}$.

3-a) $y = 4x^3 - x^4 \Rightarrow y' = 4 \cdot 3 \cdot x^2 - 4 \cdot x^3 \Rightarrow y' = 12 \cdot x^2 - 4 \cdot x^3$

$$y'' = (y')' = 12 \cdot 2 \cdot x - 4 \cdot 3 \cdot x^2 \Rightarrow y'' = 24 \cdot x - 12 \cdot x^2 .$$

b) The first derivative y' exists for every x in the domain. So, the points where $y' = 0$ are the only critical points:

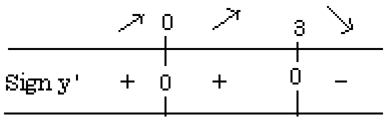
$$y' = 0 \Rightarrow 12x^2 - 4x^3 = 0 \Rightarrow 4x^2 \cdot (3 - x) = 0 \Rightarrow x = 0 \text{ or } x = 3 .$$



At $x = 0$, there is an inflection point and at $x = 3$, there is a local maximum.

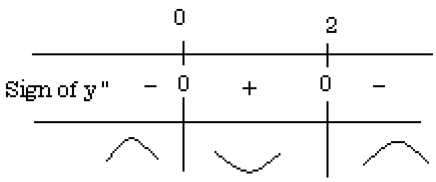
Note: We could have check the sign of y'' at $x = 0$ and at $x = 3$.
Since $y''(0) = 0$ (inflection point) and $y''(3) = -36 < 0$ (local maximum).

c) $y' = 12x^2 - 4x^3 = 0 \Rightarrow 4 \cdot x^2 \cdot (3 - x) = 0 \Rightarrow x = 0$ or $x = 3$.



The curve is increasing on $(-\infty, 0)$ and on $(0,3)$.
The curve is decreasing on $(3, \infty)$.

d) $y'' = 24 \cdot x - 12 \cdot x^2 = 0 \Rightarrow 12 \cdot x \cdot (2 - x) = 0 \Rightarrow x = 0$ or $x = 2$.

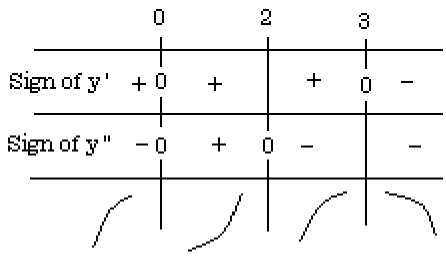


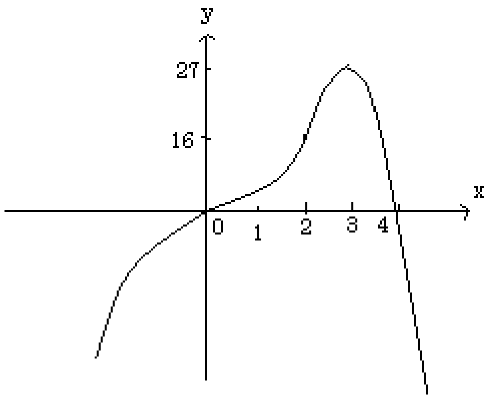
The graph of f is concave up when $0 < x < 2$ and concave down when $x < 0$ or $x > 2$.
Points of inflections (points where the curvature change directions) are at $x = 0$ and $x = 2$.

e) $y = 4x^3 - x^4 = 0 \Rightarrow x^3 \cdot (4 - x) = 0 \Rightarrow x = 0$ or $x = 4$ (x-intercepts).

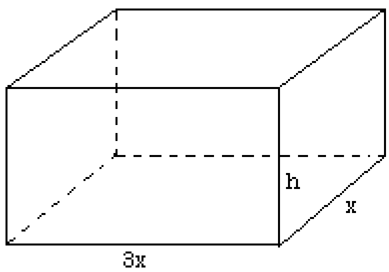
$y(0) = 0, y(2) = 16$ and $y(3) = 27$.

$\lim_{x \to \infty} y = -\infty$ and $\lim_{x \to -\infty} y = -\infty$.





4)



$$60 = 3x^2 \cdot h \Rightarrow x^2 \cdot h = 20 \Rightarrow h = \frac{20}{x^2}$$

$$c(x) = 10 \cdot \left(3x^2 + 3x^2\right) + 6 \cdot \left(2x \cdot h + 6x \cdot h\right)$$

$$\Rightarrow c(x) = 60x^2 + 48x \cdot \frac{20}{x^2}$$

$$\Rightarrow c(x) = 60x^2 + \frac{960}{x}.$$

$x \in (0, \infty)$.

$$c'(x) = 120x - \frac{960}{x^2} = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2 \text{ m.}$$

$$c'(x) = 120x - \frac{960}{x^2} = \frac{120 \cdot x^3}{x^2} - \frac{960}{x^2} = \frac{120}{x^2} \cdot \left(x^3 - 8\right) = \frac{120 \cdot (x-2) \cdot \left(x^2 + 2x + 4\right)}{x^2}$$

So $c'(x) > 0$ when $x > 2$ and $c'(x) < 0$ when $0 < x < 2$. Therefore at $x = 2$, $c(x)$ has its minimum value. Dimensions are $2 \times 6 \times 5$.

Alternative way of determining that $c(x)$ is minimum when $x = 2$;

$c''(x) = 120 + 2 \cdot \frac{960}{x^3}$. So

$c''(2) = 120 + 2 \cdot \frac{960}{2^3} > 0$ (i.e. we have minimum at $x = 2$) .