MATH- 102/Spring 2005, Final Exam Solutions

1) Each piece of this piecewise defined function f is continuous, so f is continuous if and only if f is continuous at $x = 0$. f is continuous at $x = 0$ if and only if $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} = \lim_{x\to 0^-} x + a = a \Rightarrow a = \frac{\pi}{2}$.

2) $f(x) = \int_0^{x^2} \sqrt{1+t^3}\cdot dt \Rightarrow \left.\frac{df(x)}{dx}\right|_{x=\sqrt{2}} = \frac{d}{dx}\left(\int_0^{x^2}\sqrt{1+t^3}\cdot dt\right)_{x=\sqrt{2}} = \left(\sqrt{1+\left(x^2\right)^3}\cdot 2x\right)_{x=\sqrt{2}}$

$$= \sqrt{1+\left(\left(\sqrt{2}\right)^2\right)^3}\cdot 2\cdot\sqrt{2} = \sqrt{1+8}\cdot 2\cdot\sqrt{2} = 6\cdot\sqrt{2}\,.$$

So, the equation of the tangent line at $x = \sqrt{2}$ is $y - f\left(\sqrt{2}\right) = 6\cdot\sqrt{2}\cdot\left(x-\sqrt{2}\right)$.

3) If $f(x) = \frac{u(x)}{v(x)}$ then $f'(x) = \frac{u'(x)\cdot v(x) - v'(x)\cdot u(x)}{\left(v(x)\right)^2}$.

$u(x) = \sin(\ln x) \Rightarrow u'(x) = \cos(\ln x)\cdot\frac{1}{x}$ and $v(x) = \cos\left(e^x\right) \Rightarrow v'(x) = -\sin\left(e^x\right)\cdot e^x$. So

$$f'(x) = \frac{\cos(\ln x)\cdot\frac{1}{x}\cdot\cos\left(e^x\right) - \left(-\sin\left(e^x\right)\right)\cdot e^x\cdot\sin(\ln x)}{\left(\cos\left(e^x\right)\right)^2} \text{ and}$$

$$f'(1) = \frac{\cos(\ln 1)\cdot 1\cdot\cos(e) + \sin(e)\cdot\sin(\ln 1)\cdot 1}{\cos^2(e)} = \frac{\cos(e)}{\cos^2(e)} = \frac{1}{\cos(e)}\,.$$

4) Since $\left(\frac{3+n}{n}\right)^{2n} = e^{\ln\left(\frac{3+n}{n}\right)^{2n}} = e^{(2n)\cdot\ln\left(\frac{3+n}{n}\right)}$ and $f(x) = e^x$ is a continuous function, we have

$$\lim_{n\to\infty}\left(\frac{3+n}{n}\right)^{2n} = \lim_{n\to\infty} e^{(2n)\cdot\ln\left(\frac{3+n}{n}\right)} = e^{\lim_{n\to\infty}(2n)\cdot\ln\left(\frac{3+n}{n}\right)}\,.$$

$$\lim_{n\to\infty}(2n)\cdot\ln\left(\frac{3+n}{n}\right)=\lim_{n\to\infty}\frac{\ln\left(\frac{3+n}{n}\right)}{\frac{1}{2n}}\underset{\substack{\uparrow\\ L'H}}{=}\lim_{n\to\infty}\frac{\frac{n}{3+n}\cdot\left(-\frac{3}{n^2}\right)}{-\frac{1}{2n^2}}=\lim_{n\to\infty}\frac{6n}{3+n}=6.\ \text{So, } \lim_{n\to\infty}\left(\frac{3+n}{n}\right)^{2n}=e^6.$$

Another solution:

Since $\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n=e^k$, $\lim_{n\to\infty}\left(\frac{3+n}{n}\right)^{2n}=\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n}=\left(\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n\right)^2=\left(e^3\right)^2=e^6$.

5) In order to evaluate this integral, we will use Integration by Parts;

$u=x^2$, $\quad dv=e^x$

$du=2x\cdot dx$, $\quad v=e^x$ and

$$\int_a^b u\cdot dv=(u\cdot v)\Big|_a^b-\int_a^b v\cdot du\Rightarrow\int_0^1 x^2\cdot e^x\cdot dx=\left(x^2\cdot e^x\right)\Big|_0^1-=e-0-\int_0^1 2x\cdot e^x\cdot dx.$$

To evaluate $\int_0^1 2x\cdot e^x\cdot dx$, we will use integration by parts again;

$u=2x$, $\quad dv=e^x\cdot dx$

$du=2\cdot dx$, $\quad v=e^x$ and

$$\int_0^1 2x\cdot e^x\cdot dx=\left(2x\cdot e^x\right)\Big|_0^1-\int_0^1 2\cdot e^x\cdot dx=2e-0-\left(2e^x\right)\Big|_0^1=2e-(2e-2)=2.$$

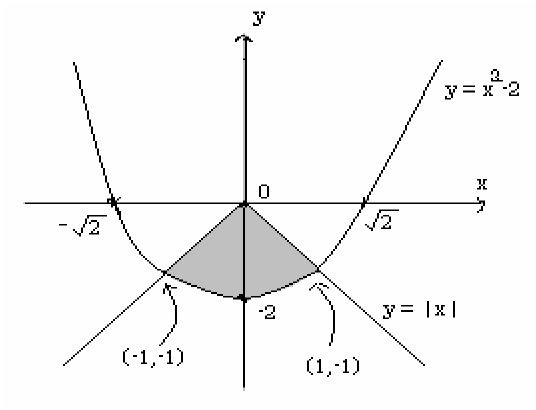
Therefore $\int_0^1 x^2\cdot e^x\cdot dx=e-2$

6)

$$\left.\begin{array}{r} y=x^2-2\\ y=-|x| \end{array}\right\}\Rightarrow\left\{\begin{array}{ll} x^2-2=-x, & if\quad x\ge 0\\ x^2-2=x, & if\quad x<0 \end{array}\right\}$$

$$x^2-2=-x\Rightarrow x^2+x-2=0\Rightarrow(x+2)\cdot(x-1)=0\Rightarrow x=1\quad or\quad x=-2.$$

$$x^2-2=x\Rightarrow x^2-x-2=0\Rightarrow(x-2)\cdot(x+1)=0\Rightarrow x=-1\quad or\quad x=2.$$



So, $Area = \int_{-1}^{0} \left[\left(-|x|\right) - \left(x^2 - 2\right)\right] \cdot dx + \int_{-1}^{0} \left[\left(-|x|\right) - \left(x^2 - 2\right)\right] \cdot dx$

$$= \int_{-1}^{0} \left(x - x^2 + 2\right) \cdot dx + \int_{0}^{1} \left(-x - x^2 + 2\right) \cdot dx = \left(\frac{x^2}{2} - \frac{x^3}{3} + 2x\right)\Bigg]_{-1}^{0} + \left(-\frac{x^2}{2} - \frac{x^3}{3} + 2x\right)\Bigg]_{0}^{1}$$

$$= \left[(0) - \left(\frac{1}{2} + \frac{1}{3} - 2\right)\right] + \left[\left(-\frac{1}{2} - \frac{1}{3} + 2\right) - (0)\right] = \frac{7}{6} + \frac{7}{6} = \frac{7}{3}.$$

7) Since $x^2 - 1 = (x-1) \cdot (x+1)$, using the method of partial fractions, we have

$$\frac{2}{x^2 - 1} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A \cdot (x+1) + B \cdot (x-1)}{(x-1) \cdot (x+1)}.$$

$$A \cdot (x+1) + B \cdot (x-1) = 2 \Rightarrow \left\{\begin{matrix} A + B = 0 \\ A - B = 2 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} A = 1 \\ B = -1 \end{matrix}\right.$$

So, $\int_{2}^{\infty} \frac{2}{x^2 - 1} \cdot dx = \lim_{b \to \infty} \int_{2}^{b} \frac{2}{x^2 - 1} \cdot dx = \lim_{b \to \infty} \left(\int_{2}^{b} \frac{1}{x-1} \cdot dx + \int_{2}^{b} \frac{-1}{x+1} \cdot dx \right) = \lim_{b \to \infty} \left(\ln|x-1| \Big]_{2}^{b} - \ln|x+1| \Big]_{2}^{b} \right)$

$$= \lim_{b \to \infty} \left(\ln\left|\frac{x-1}{x+1}\right| \Bigg]_{2}^{b} \right) = \lim_{b \to \infty} \left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{2-1}{2+1}\right| \right) = \ln 1 - \ln\left(\frac{1}{3}\right) = \ln 3 < \infty.$$

Therefore, this series is **convergent** and it converges to **ln3**.

8-a) Since the Harmonic Series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent and $\lim_{n\to\infty}\frac{\frac{n^2+2n+3}{2n^3+5n+4}}{\frac{1}{n}}=\frac{1}{2}<\infty$, by The Limit Comparison Test, this series is **divergent**.

b) Since $\lim_{n\to\infty}\cos\left(\frac{1}{n}\right)=\cos 0=1\neq 0$, by the n-th Term Test, this series is **divergent.**

c)

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{\frac{2^{n+1}\cdot(n+1)!\cdot(n+1)!}{(2(n+1))!}}{\frac{2^n\cdot n!\cdot n!}{(2n)!}}=\lim_{n\to\infty}\frac{2^{n+1}\cdot(n+1)!\cdot(n+1)!}{(2(n+1))!}\cdot\frac{(2n)!}{2^n\cdot n!\cdot n}$$

$$\lim_{n\to\infty}\frac{2^n\cdot 2\cdot(n+1)\cdot n!\cdot(n+1)\cdot n!}{(2n+2)\cdot(2n+1)\cdot(2n)!}\cdot\frac{(2n)!}{2^n\cdot n!\cdot n}=\lim_{n\to\infty}\frac{2\cdot(n+1)\cdot(n+1)}{(2n+2)\cdot(2n+1)}=\lim_{n\to\infty}\frac{2\cdot(n+1)\cdot(n+1)}{2\cdot(n+1)\cdot(2n+1)}=\frac{1}{2}<1.$$

So, by The Ratio Test , this series is **convergent**.

9) Let $a_n=\frac{x^n}{\sqrt{n}\cdot 3^n}$. By the Ratio Test, we have

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{\sqrt{n+1}\cdot 3^{n+1}}}{\frac{x^n}{\sqrt{n}\cdot 3^n}}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}\cdot\sqrt{n}\cdot 3^n}{x^n\cdot\sqrt{n+1}\cdot 3^{n+1}}\right|=\frac{|x|}{3}\cdot\lim_{n\to\infty}\left|\sqrt{\frac{n}{n+1}}\right|=\frac{|x|}{3}\text{ and}$$

$\rho<1\Rightarrow\frac{|x|}{3}<1\Leftrightarrow|x|<3\Leftrightarrow -3<x<3$. So **<u>the radius of convergence is</u> 3**.

When $x=3; a_n=\frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges, since this is a p-series with $p=\frac{1}{2}<1$.

When $x=-3; a_n=\frac{(-3)^n}{\sqrt{n}\cdot 3^n}=\frac{(-1)^n}{\sqrt{n}}$ and $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test.

$\therefore$ **<u>The interval of convergence is</u>** $[-3,3)$**.**