KOÇ UNIVERSITY

FALL 2010 FINAL EXAM

MATH 102 JANUARY 21, 2011

Duration of Exam: 105 minutes

INSTRUCTIONS:

- No calculators may be used on the test.
- No books, no notes, no questions, and talking allowed.
- You must always explain your answers and show your work to receive full credit.
- Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

SURNAME, Name: ______________________

Student ID no: ______________________

Signature: ______________________

(Check One):

(Barış Coşkunüzer – TTh 14:00-15:15) : ___
(Tolga Etgü – MW 14:00-15:15) : ___
(Tolga Etgü – MW 17:00-18:15) : ___

PROBLEM	1	2	3	4	5	TOTAL
POINTS	20	20	15	15	41	111
SCORE						

Problem 1 (20 pts) Evaluate the limits in parts (a)-(c).

(a) $\lim\limits_{x\to\infty} \frac{\sqrt{x^2+2x}}{x+1}$ $= \lim\limits_{x\to\infty} \frac{\overbrace{|x|}^{=x \text{ as } x\to\infty}\sqrt{1+\frac{2}{\cancel{x}}}^{\to 0}}{\cancel{x}\left(1+\frac{1}{\cancel{x}}\right)^{\to 0}} = 1$

(b) $\lim\limits_{x\to 0} \frac{\sin(3x^2)}{x^4+x^2} \overset{\frac{0}{0}}{\underset{LH}{=}} \lim\limits_{x\to 0} \frac{\cos(3x^2)\cdot 6x}{4x^3+2x} = \lim\limits_{x\to 0} \frac{3\cos(3x^2)^{\to 1}}{2x^2+1} = 3.$

(c) $\lim\limits_{x\to 2} \frac{\sqrt{x+2}-2}{x^2-4} \overset{\frac{0}{0}}{\underset{L.H}{=}} \lim\limits_{x\to 2} \frac{\frac{1}{2}(x+2)^{-1/2}}{2x} = \lim\limits_{x\to 2} \frac{1}{4}\,\frac{1}{\cancel{x}\sqrt{x+2}} = \frac{1}{16}$

(d) Determine a and b which make the following function continuous everywhere.

$$f(x) = \begin{cases} e^x + a & , \text{ if } x < 0; \\ 3 & , \text{ if } x = 0; \\ x^2 + b & , \text{ if } x > 0. \end{cases}$$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} e^x + a = a+1$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x^2+b = b$

To make $f(x)$ cont. everywhere we should have $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(x)$

i.e $b = a+1 = 3 \Rightarrow \boxed{\begin{array}{l} b=3 \\ a=2 \end{array}}$

Problem 2 (20 pts)

(a) Let $f(x) = \frac{1}{\sqrt{3x-2}}$. Find $f'(2)$.

$f(x) = (3x-2)^{-1/2}$

$f'(x) = -1/2\ (3x-2)^{-3/2} \cdot 3$

$f'(2) = \frac{-3}{2}\ (4)^{-3/2} = \frac{-3}{2\sqrt{4^3}}$

(b) Let $g(x) = 2^{x\ln x}$. Find $g'(e)$.

$g'(x) = 2^{x\ln x} \cdot \ln 2 \cdot (\ln x + 1)$

$g'(e) = 2^e \cdot \ln 2 \cdot 2 = \ln 2 \cdot 2^{(e+1)}$

(c) Let $h(x) = \frac{\arctan x}{x^2+1}$. Find $h'(1)$.

$h'(x) = \frac{\frac{1}{1+x^2} \cdot (x^2+1) + \arctan x \cdot 2x}{(x^2+1)^2} = \frac{\arctan x \cdot 2x}{(x^2+1)^2}$

$h'(1) = \frac{\arctan 1 \cdot \cancel{2}}{2^{\cancel{2}}} = \frac{\arctan 1}{2} = \frac{\pi}{8}$

(d) Let $k(x) = \sqrt[3]{x} - \frac{2}{\sqrt[4]{x}}$. Find $k'(1)$.

$k'(x) = \frac{1}{3} x^{-2/3} - 2\left(\frac{-1}{4}\right) \cdot x^{-5/4} = \frac{1}{3} x^{-2/3} + \frac{1}{2} x^{-5/4}$

$k'(1) = \frac{1}{3} + \frac{1}{2} = 5/6$

Problem 3 (15 pts) Sketch the graph of f using the following information. Indicate the monotonicity, concavity, asymptotes, local extrema and inflection points clearly.

- The domain of f is $\mathbb{R} \setminus \{1\}$.
- $\lim_{x \to -\infty} f(x) = 2$, $\lim_{x \to \infty} f(x) = \infty$.
- $\lim_{x \to 1} f(x) = \infty$.
- $f'(x) > 0$ if x is in $(-2, 1)$ or $(4, \infty)$. $f'(x) < 0$ if x is in $(-\infty, -2)$ or $(1, 4)$.
- $f''(x) > 0$ if x is in $(-3, 1)$ or $(1, \infty)$. $f''(x) < 0$ if x is in $(-\infty, -3)$.
- $f(-3) = 0$, $f(-2) = -1$, $f(4) = -4$.

Vertical asym $x = 1$

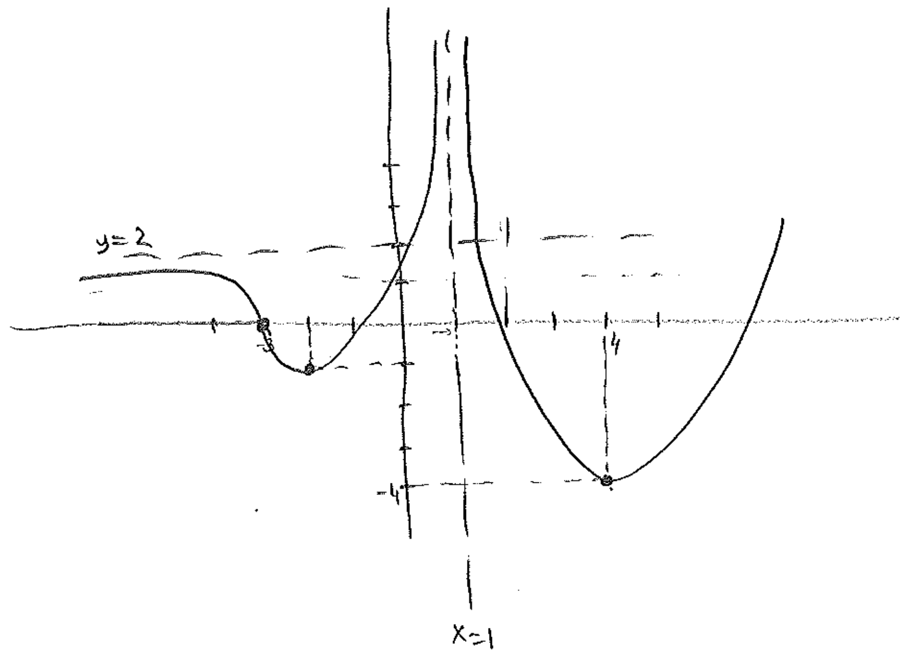
Horiz. asym $y = 2$

f is increasing on $(-2, 1)$ & $(4, \infty)$

f is decreasing on $(-\infty, -2)$ & $(1, 4)$

f is concave up on $(-3, 1)$ & $(1, \infty)$

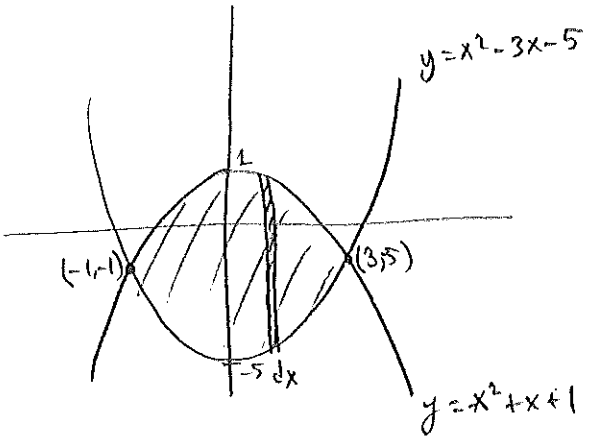
f is concave down on $(-\infty, -3)$



		-3	-2	1	4
f'	$-$	$-$	$+$	$-$	$+$
f''	$-$	$+$	$+$	$+$	$+$

Problem 4 (15 pts) Find the area of the region between the curves

$$y = -x^2 + x + 1 \ \text{ and } \ y = x^2 - 3x - 5\,.$$



$x^2 - 3x - 5 = -x^2 + x + 1$

$2x^2 - 4x - 6 = 0$

$x^2 - 2x - 3 = 0$

$\begin{matrix} -3 \\ +1 \end{matrix}$

$(x-3)(x+1) = 0$

$\left.\begin{matrix} x = 3 \\ x = -1 \end{matrix}\right\}$ are the intersection points of two curves

$x = 3 \quad y = -5$

$x = -1 \quad y = -1$

Area btw two curves $= \displaystyle\int_{-1}^{3} (-x^2+x+1) - (x^2-3x-5)\, dx$

$= \displaystyle\int_{-1}^{3} -2x^2 + 4x + 6 \, dx = \left(-2\frac{x^3}{3} + \frac{4x^2}{2} + 6x\right)\Bigg|_{-1}^{3}$

$= -\cancel{18} + \cancel{18} + 18 - \left(\frac{2}{3} + 2 - 6\right)$

$= 14 - \frac{2}{3}$

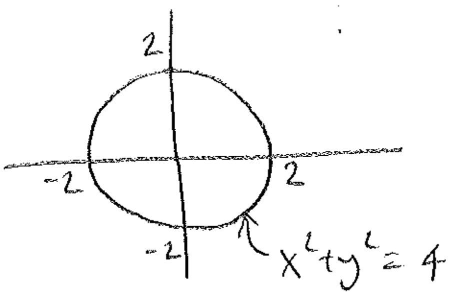
Problem 5 (41 pts) Evaluate the integrals in parts (a)-(d).

(a)(7 pts) $\int_0^{\pi/2} \sin^4 x \cos x \, dx$

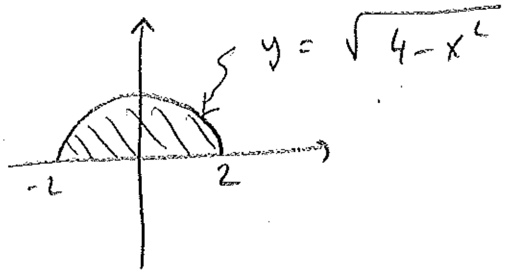
Let $u = \sin x \Rightarrow du = \cos x \, dx$

$$= \int_0^1 u^4 \, du = \frac{u^5}{5}\Big|_0^1 = \frac{1}{5}$$

(b)(7 pts) $\int_{-2}^{2} \sqrt{4 - x^2} \, dx$



So



$\int_{-2}^{2} \sqrt{4-x^2} \, dx$ is the above shaded area which is $\frac{\pi 2^2}{2} = 2\pi$.

(c)(7 pts) $\int \frac{1}{x^2+x} \, dx$

$$\frac{1}{x(x+1)} = \frac{A}{\underset{(x+1)}{x}} + \frac{B}{\underset{(x)}{x+1}} = \frac{Ax + Bx + A}{x(x+1)}$$

$A+B=0$
$A=1$
$B=-1$

$$\int \frac{1}{x(x+1)} dx = \int \frac{1}{x} - \frac{1}{x+1} \, dx$$

$$= \ln x - \ln(x+1)$$

(d)(10 pts) $\int_0^1 \arctan x \, dx$ $\overset{=}{|}$ $\left[\arctan x . x\right]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$

$u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2} dx \qquad v = x$

$u = 1+x^2$
$du = 2x dx$

$\Downarrow$

$\frac{1}{2}\int \frac{du}{u} \Rightarrow \frac{1}{2}\ln|u|$

$$= \left[\arctan x . x - \frac{1}{2}\ln|1+x^2|\right]_0^1$$

$$= \arctan 1 - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

(e)(10 pts) Determine whether the following improper integral converges or not.

$$\int_0^\infty xe^{-x} \, dx$$

$$\lim_{a\to\infty} \int_0^a xe^{-x} dx = \lim_{a\to\infty} \left(-x.e^{-x} + \int e^{-x}.dx\right)$$

$x = u$
$dx = du$
$e^{-x} dx = dv$
$v = -e^{-x}$

$$= \lim_{a\to\infty} \left(-xe^{-x} - e^{-x}\right)\Big|_0^a$$

$$= \lim_{a\to\infty} \left(-ae^{-a} - e^{-a} + 1\right) = 1$$

$\downarrow$

Converges

$$\lim_{a\to\infty} \frac{a}{e^a} \overset{\frac{\infty}{\infty}}{\underset{|}{=}} \lim_{a\to\infty} \frac{1}{e^a} = 0$$

L'Hospital