KOÇ UNIVERSITY

MATH 102

FIRST MIDTERM APRIL 1, 2011

Duration of Exam: 75 minutes

INSTRUCTIONS:

- No calculators may be used on the test.
- No books, no notes, no questions, and no talking allowed.
- You must always explain your answers and SHOW YOUR WORK to receive full credit.
- Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

SURNAME, Name: ______________________

Student ID no: ______________________

Signature: ______________________

(Check One):

(Selda Küçükçifçi	–	MW 9:30-10:45)	:	—
(Selda Küçükçifçi	–	MW 12:30-13:45)	:	—
(Tolga Etgü	–	TTh 14:00-15:15)	:	—
(Tolga Etgü	–	TTh 17:00-18:15)	:	—

PROBLEM	1	2	3	4	TOTAL
POINTS	36	30	24	20	110
SCORE					

Problem 1 (36 pts) Find the following limits. Do not use l'Hospital's rule.

a) **(6 pts)** $\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{5}{x^2+x-6}\right)$

$$= \lim_{x\to 2}\left(\frac{x+3-5}{(x-2)(x-3)}\right) = \lim_{x\to 2}\frac{1}{x+3} = \frac{1}{5}$$

b) **(6 pts)** $\lim_{x\to\infty}\frac{x\sqrt{x}-3x}{5x-7x\sqrt{x}}$

$$= \lim_{x\to\infty}\frac{x\sqrt{x}\,(1-3/\sqrt{x})}{x\sqrt{x}\,(5/\sqrt{x}-7)} = -1/7$$

c) **(6 pts)** $\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{3\cos^2 x}$

$$= \lim_{x\to\pi/2}\frac{(1-\sin x)}{3\cos^2 x}\frac{(1+\sin x)}{(1+\sin x)} = \lim_{x\to\pi/2}\frac{1-\sin^2 x}{3\cos^2 x\,(1+\sin x)} = \lim_{x\to\pi/2}\frac{\cos^2 x}{3\cos^2 x\,(1+\sin x)}$$

$$= \lim_{x\to\pi/2}\frac{1}{3(1+\sin x)}$$

$$= 1/6$$

d) **(6 pts)** $\lim_{x\to 1+}\ln\left(\frac{x^2-1}{x+1}\right)$

$$= \lim_{x\to 1^+}\ln(x-1) = -\infty$$

e) **(12 pts)** $\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2}(h+1)^{100}\right)-1}{h}$ $= f'(1)$ for $f(x) = \sin\left(\frac{\pi}{2}x^{100}\right)$

Then $f'(x) = \cos\left(\frac{\pi}{2}x^{100}\right)\frac{\pi}{2}\cdot 100x^{99}$

So $f'(1) = \cos(\pi/2)\cdot\pi/2\cdot 100 = 0$, and the limit in question is 0.

Problem 2a (15 pts) Let $f(x) = \frac{2-\sqrt{x}}{4-x}$ for $x \neq 4$ and $x \geq 0$. What should $f(4)$ be in order to make $f(x)$ continuous at $x = 4$.

f is continuous at $x=4$ if $\lim\limits_{x \to 4} f(x) = f(4)$

So $\lim\limits_{x\to 4} f(x) = \lim\limits_{x \to 4} \frac{2-\sqrt{x}}{(2-\sqrt{x})(2+\sqrt{x})} = \lim\limits_{x\to 4} \frac{1}{2+\sqrt{x}} = \frac{1}{4}$

Then $f(4)$ should be $1/4$.

2b (15 pts) Show that the equation $e^{-x} - x = 0$ has a solution in the interval $[0, 1]$.

Let $f(x) = e^{-x} - x$ be continuous on $[0,1]$.

$f(0) = 1 > 0$

$f(1) = 1/e - 1 < 0$

Since $f(0) > 0 > f(1)$, by Intermediate Value Thm there is a number c in $(0,1)$ such that $f(c) = 0$; that is $e^{-c} - c = 0$.

Problem 3 (24 pts) Find the derivative of the function f in (a) – (c).

a) **(6 pts)** $f(x) = \frac{e^{1/x}}{x^2}$ $= \frac{e^{x^{-1}}}{x^2}$

$$f'(x) = \frac{e^{x^{-1}}(-1)\,x^{-2}\,x^2 - e^{x^{-1}}.2x}{x^4} = \frac{-e^{1/x}(1+2x)}{x^4}$$

b) **(6 pts)** $f(x) = \sin(\cos(4x))$

$$f'(x) = \cos(\cos(4x))\,(-\sin(4x)).4$$
$$= -4\cos(\cos(4x))\sin(4x)$$

c) **(6 pts)** $f(x) = \sqrt[3]{x^3 + \frac{1}{x}}$ $= (x^3 + x^{-1})^{1/3}$

$$f'(x) = \frac{1}{3}(x^3 + x^{-1}) \cdot \left(3x^2 - \frac{1}{x^2}\right)$$

d) **(6 pts)** Determine $f^{(10)}(x)$ where $f(x) = (5 + x)^{-1}$

$$f'(x) = -1\,(5+x)^{-2}$$
$$f''(x) = 2\,(5+x)^{-3}$$
$$f'''(x) = (-1)\cdot 2\cdot 3\,(5+x)^{-4}$$
$$\vdots$$
$$f^{(10)}(x) = 10!\,(5+x)^{-11}$$

Problem 4 (20 pts) Suppose that $y = 2x + 1$ is tangent to $y = f(x)$ at $(1, 3)$ and $y = x + 4$ is tangent to $y = g(x)$ at $(2, 6)$. Find the slope of the tangent line to the curve given by the equation

$$f(x)g(y) = 1$$

at the point $(1, 2)$.

Given: $f(1) = 3$, $f'(1) = 2$

$g(2) = 6$, $g'(2) = 1$

Asked: slope $m = \frac{dy}{dx}$ at $x = 1$, $y = 2$

Solution: Apply implicit differentiation on $f(x)g(y) = 1$.

$$f'(x)g(y) + f(x)g'(y)\frac{dy}{dx} = 0$$

So $f'(1)g(2) + f(1)g'(2)\,m = 0$

$$m = -\frac{f'(1)g(2)}{f(1)g'(2)} = -\frac{2 \cdot 6}{3 \cdot 1} = -4$$