# KOÇ UNIVERSITY

# MATH 102

## SECOND MIDTERM DECEMBER 24, 2010

**Duration of Exam: 75 minutes**

INSTRUCTIONS:

- No calculators may be used on the test.
- No books, no notes, no questions, and talking allowed.
- You must always explain your answers and show your work to receive full credit.
- Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

SURNAME, Name: ________________

Student ID no: KEY

Signature: ________________

(Check One):

| | | | |
|---|---|---|---|
| (Barış Coşkunüzer | – | TTh 14:00-15:15) | : __ |
| (Tolga Etgü | – | MW 14:00-15:15) | : __ |
| (Tolga Etgü | – | MW 17:00-18:15) | : __ |

| PROBLEM | 1 | 2 | 3 | 4 | TOTAL |
|---|---|---|---|---|---|
| POINTS | 25 | 30 | 20 | 35 | 110 |
| SCORE | | | | | |

**Problem 1 (25 pts)** Let $f(x) = x^4 - 4x^3$ .

a) Find all the critical points, and the intervals on which $f$ is increasing & decreasing.

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) \Rightarrow$ Critical points $= \{0,3\}$

| $x$ | $-\infty$ | 0 | 3 | $\infty$ |
|---|---|---|---|---|
| $f$ | $-$ | $-$ | $+$ | |
| $f'$ | $\searrow$ | $\searrow$ | $\nearrow$ | |

$\Rightarrow$ $f$ is increasing on $(3,\infty)$

$f$ is decreasing on $(-\infty,3)$

b) Find the inflection points, and the intervals on which $f$ is concave up & concave down.

$f''(x) = 12x^2 - 24x = 12x(x-2) \Rightarrow$ inflection points $=$ $(0,0)$ and $(2,-16)$

| $x$ | $-\infty$ | 0 | 2 | $\infty$ |
|---|---|---|---|---|
| $f''$ | $+$ | $-$ | $+$ | |
| $f$ | CU | CD | CU | |

$\Rightarrow$ $f$ is concave up on $(-\infty,0)$ and $(2,\infty)$

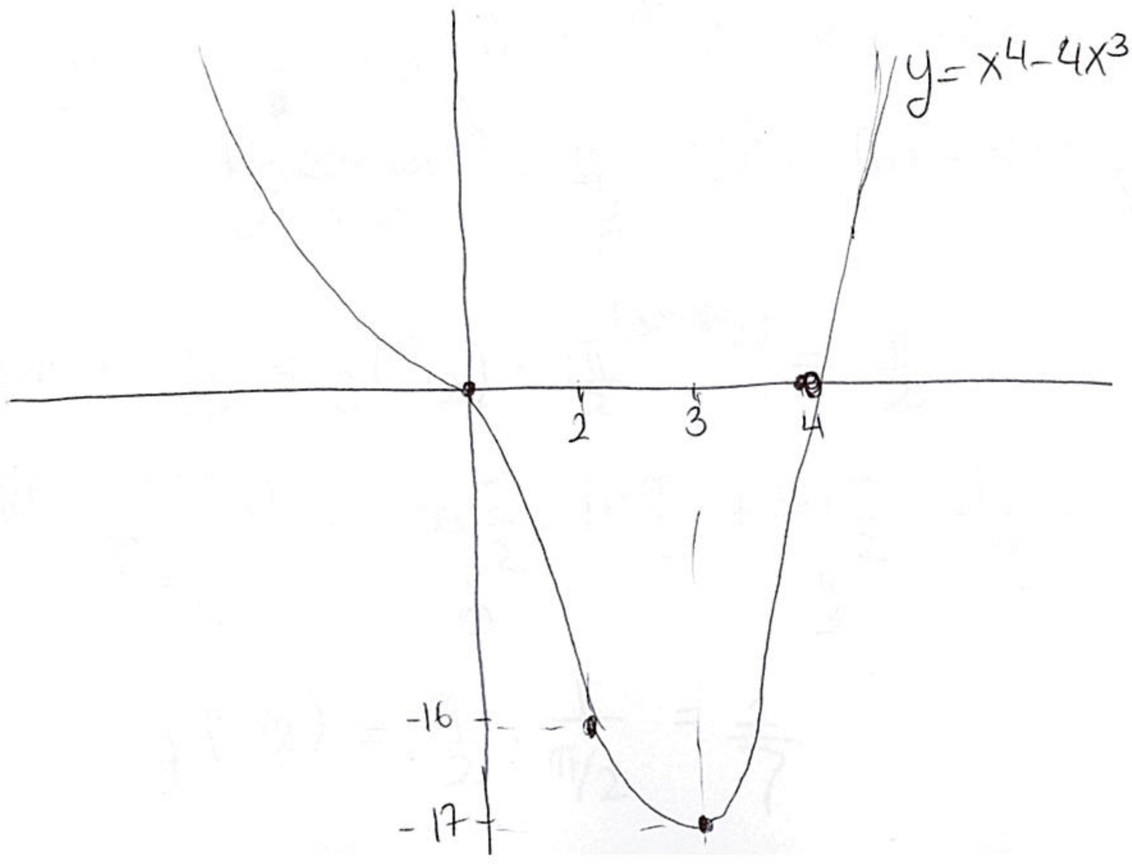
$f$ is concave down on $(0,2)$

c) Find the local max & min, if exists.

f has no local max.

f has a local min. at $x=3$.

d) Sketch the graph of $f$.



**Problem 2a (10 pts)** Given $y + x^2y + \sin(x \cdot y) = 5$ , find $y'(0)$.

Implicit differentiation:

$y' + 2xy + x^2y' + \cos(x.y)(y + xy') = 0$

When $x = 0 \Rightarrow y(0) + 0 + \sin 0 = 5 \Rightarrow \underline{y(0) = 5}$

So $y'(0) + 0 + 0 + 1 . (y(0) + 0) = 0 \Rightarrow \underline{y'(0) = -5}$

**2b (10 pts)** Find the equation of the tangent line to the curve $x^2 = \tan 2y$ at the point $(1, \frac{\pi}{8})$. By implicit differentiation: $2x = \sec^2(2y) . 2y'$

When $x = 1$, $y = \frac{\pi}{8}$ $\Rightarrow$ $2 = \underbrace{\sec^2\left(\frac{\pi}{4}\right)}_{2} . 2m$ slope of the tangent lin

$\Downarrow$

$\underline{m = 1/2}$

Eqn $\Rightarrow$ $y - \frac{\pi}{8} = \frac{1}{2}(x - 1)$ OR $y = \frac{x}{2} + \frac{\pi}{8} - \frac{1}{2}$

**2c (10 pts)** Given $y = x^{\sin x}$ , find $y'(\frac{\pi}{2})$

$y = x^{\sin x} \Rightarrow \ln y = \ln(x^{\sin x}) = \sin x . \ln x$

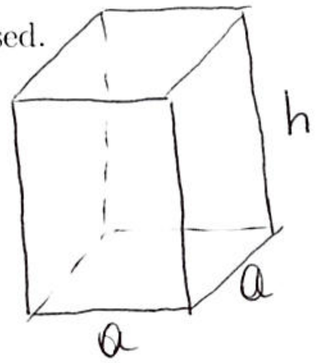
differentiate both sides $\Rightarrow \frac{y'}{y} = \cos x . \ln x + \sin x . \frac{1}{x}$

When $x = \frac{\pi}{2} \Rightarrow y(\pi/2) = \frac{\pi}{2}^{(\sin \pi/2)} = \frac{\pi}{2}$

and $\frac{y'(\pi/2)}{\pi/2} = \underbrace{\cos\frac{\pi}{2}}_{= 0} . \ln\frac{\pi}{2} + \underbrace{\sin\frac{\pi}{2}}_{= 1} . \frac{1}{\pi/2}$

$\Rightarrow y'(\pi/2) = \frac{\pi}{2} . \frac{1}{\pi/2} = \underline{1}$

**Problem 3 (20 pts)** A box with a square base and open top must have volume 32 cm$^3$. Find the dimensions of the box that minimize the amount of the material used.




$V = a^2 h$

$A = a^2 + 4ah$

$\Rightarrow a^2 h = 32 \Rightarrow h = \dfrac{32}{a^2}$

$A = a^2 + 4ah = a^2 + 4a \cdot \dfrac{32}{a^2} = a^2 + \dfrac{128}{a}$

minimize

$A' = 2a - \dfrac{128}{a^2} \Rightarrow 2a - \dfrac{128}{a^2} = 0$

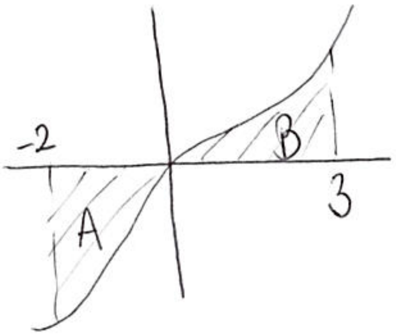
$2a = \dfrac{128}{a^2}$

$a^3 = 64$

$\underline{a = 4}$

$h = \dfrac{32}{a^2} = \dfrac{32}{16} = \underline{2}$

**Problem 4a (10 pts)** Find the area of the region between the curve $y = x^3 + 4x$ and the $x$-axis from $x = -2$ to $x = 3$.




$$-A = \int_{-2}^{0} x^3 + 4x \, dx = \frac{x^4}{4} + 2x^2 \Big]_{-2}^{0} = (0) - (4+8) = -12$$

$$\Rightarrow A = 12$$

$$B = \int_{0}^{3} x^3 + 4x \, dx = \frac{x^4}{4} + 2x^2 \Big]_{0}^{3} = \left(\frac{81}{4} + 18\right) - 0 = \frac{153}{4}$$

$$\Rightarrow B = 153/4$$

$$A + B = 12 + \frac{153}{4} = 201/4$$

**4b (10 pts)** $\int_{-\ln 2}^{0} \frac{e^x}{\sqrt{1-e^{2x}}} \, dx = ?$

$u = e^x$

$du = e^x dx$

$x = 0 \Rightarrow u = e^0 = 1$

$x = -\ln 2 \Rightarrow u = e^{-\ln 2} = \frac{1}{2}$

$$\Rightarrow \int_{1/2}^{1} \frac{du}{\sqrt{1-u^2}} = \arcsin u \Big]_{1/2}^{1} = \arcsin 1 - \arcsin 1/2$$

$$= \frac{\pi}{2} - \frac{\pi}{6} = \pi/3$$

**4c (15 pts)** $\lim_{x \to 1} \frac{\int_1^x \arctan t \, dt}{x-1} = ?$ $\frac{0}{0}$ ? $\Rightarrow$ use L'hospital!

$$\lim_{x \to 1} \frac{\left(\int_1^x \arctan t \, dt\right)'}{(x-1)'} \overset{FTC}{=} \lim_{x \to 1} \frac{\arctan x}{1} = \arctan 1$$

$$= \pi/4$$