KOÇ UNIVERSITY

MATH 102

FIRST MIDTERM NOVEMBER 10, 2010

Duration of Exam: 90 minutes

INSTRUCTIONS:

- No calculators may be used on the test.
- No books, no notes, no questions, and talking allowed.
- You must always explain your answers and show your work to receive full credit.
- Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

SURNAME, Name: ____________

Student ID no: ____ Answer Key

Signature: ____________

(Check One):

(Barış Coşkunüzer – TTh 14:00-15:15) : ___
(Tolga Etgü – MW 14:00-15:15) : ___
(Tolga Etgü – MW 17:00-18:15) : ___

PROBLEM	1	2	3	4	5	TOTAL
POINTS	30	20	20	20	20	110
SCORE						

Problem 1 Find the following limits. Do not use L'hospital rule. Show your work.

1a (5 pts) $\lim_{x\to 2} \frac{x^2-x-2}{x^2-4}$

$$\lim_{x\to 2} \frac{x^2-x-2}{x^2-4} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x+2)} = \frac{3}{4}$$

1b (5 pts) $\lim_{x\to\infty} \frac{\sqrt{x}+3x^3-2}{2\sqrt{x}-5x^3+x}$

$$\lim_{x\to\infty} \frac{x^3\left(3+1/x^2\sqrt{x} - 2/x^3\right)}{x^3\left(-5+2/x^2\sqrt{x} + 1/x^2\right)} = \frac{-3}{5}$$

1c (5 pts) $\lim_{x\to 3^-} \frac{|x-3|}{x^2-9}$

$$\lim_{x\to 3^-} \frac{|x-3|}{x^2-9} = \lim_{x\to 3^-} \frac{\cancel{3-x}}{\cancel{(x-3)}(x+3)} = \frac{-1}{6}$$

1d (5 pts) $\lim_{x\to 0} \frac{x^2}{1-\cos x}$

$$\lim_{x\to 0} \frac{x^2}{1-\cos x} = \lim_{x\to 0} \frac{x^2}{1-\cos x} \cdot \frac{(1+\cos x)}{(1+\cos x)} = \lim_{x\to 0} \frac{x^2(1+\cos x)}{1-\cos^2 x}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}(1+\cos x)}{\cancel{\sin^2 x}} = 2$$

1e (10 pts) $\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{4}+h\right)-\tan\frac{\pi}{4}}{h}$

$$\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{4}+h\right)-\tan\frac{\pi}{4}}{h} = f'\left(\frac{\pi}{4}\right) \text{ where } f(x)=\tan x$$

$$f'(x) = \frac{1}{\cos^2 x}$$

$$\Rightarrow f'\left(\frac{\pi}{4}\right) = \frac{1}{\cos^2\frac{\pi}{4}} = \frac{1}{\frac{1}{2}} = 2$$

Problem 2 (20 pts) Let f be as follows.

$$f(x) = \begin{cases} x^2 & x < 0 \\ x+1 & 0 \le x < 2 \\ ax - 3 & 2 \le x \end{cases}$$

2a (10 pts) Is f [redacted] continuous at $x = 0$? Show your work.

NO.

$$\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} x^2 = 0 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} x+1 = 1 \end{aligned}\right\} \; 0 \neq 1 \Rightarrow \text{Not continuous.}$$

2b (10 pts) If f is continuous at $x = 2$, what is a? Show your work.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x+1 = 3$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax-3 = 2a-3 \Rightarrow 2a-3 = 3 \Rightarrow a = 3$$

Problem 3 (20 pts) Find the following derivatives.

3a (5 pts) Given $f(x) = x^3 + 4x + 5$, find $f'(2)$.

$f'(x) = 3x^2 + 4 \Rightarrow f'(2) = 3.(2)^2 + 4 = 16$

3b (5 pts) Given $f(x) = x^2 \cdot \sin x$, find $f'(x)$.

$f'(x) = 2x . \sin x + x^2 . \cos x$

3c (5 pts) Given $f(x) = \frac{e^{2x}}{e^x + 3x}$, find $f'(0)$.

$f'(x) = \frac{2e^{2x}(e^x + 3x) - e^{2x}(e^x + 3)}{(e^x + 3x)^2} \Rightarrow f'(0) = \frac{2(1+0) - 1(1+3)}{(1+0)^2}$

$= \frac{2-4}{1} = 2$

3d (5 pts) Given $f(x) = \sin(x^2 + \sqrt{x+5})$, find $f'(x)$.

$f'(x) = \cos(x^2 + \sqrt{x+5}) . \left(2x + \frac{1}{2\sqrt{x+5}}\right)$

Problem 4 (20 pts) Let $F(x) = f \circ g(x)$. Find $f'(3)$ by using the following information:

- The equation of the tangent line of the graph of the function $g(x)$ at the point $(5,3)$ is $y = 4x - 17$,
- $\lim\limits_{h \to 0} \frac{F(5+h) - F(5)}{h} = 2.$

$\{ g(5) = 3, \; g'(5) = 4$

$\{ F'(5) = 2$

$F'(5) = f'(g(5)) \cdot g'(5)$

$2 = f'(3) \cdot 4 \quad \Rightarrow \quad f'(3) = \frac{2}{4} = \frac{1}{2}$

Problem 5 (20 pts) Show that there is a solution of the equation $4x^3 - 2^x = 1$.

Let $f(x) = 4x^3 - 2^x$

Since f is the difference of a polynomial and an exponential function, it is continuous everywhere.

$f(0) = 0 - 1 = -1$

$f(1) = 4 - 2 = 2$

Since f is continuous on $[0,1]$ and $f(0) < 1 < f(1)$, by the Intermediate Value Theorem, there exists c in $(0,1)$ such that $f(c) = 1$. Such a number c is clearly a solution of the equation $4x^3 - 2^x = 1$