KOÇ UNIVERSITY

MATH 102

FIRST MIDTERM March 10, 2014

Duration of Exam: 75 minutes

INSTRUCTIONS:

- No calculators may be used on the test.
- No books, no notes, no questions, and no talking allowed.
- You must always explain your answers and SHOW YOUR WORK to receive full credit.
- Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

SURNAME, Name: ______________ KEY

Student ID no: ______________

Signature: ______________

(Check One):

(Şule Yazıcı – MWF 11:00-12:15) : —
(Şule Yazıcı – MWF 12:30-13:45) : —
(Şule Yazıcı – MWF 15:30-16:45) : —
(Fatih Demirkale – TuTh 12:30-13:45) : —
(Fatih Demirkale – TuTh 14:00-15:15) : —

PROBLEM	1	2	3	4	5	BONUS	TOTAL
POINTS	42	10	15	17	16	10	110
SCORE							

Problem 1 Find the following limits.Specify infinite limits. Do not use l'Hospital's rule.

(a)(7 pts) $\lim\limits_{x\to -4}\left(\dfrac{2x+8}{x^2+x-12}\right) = \lim\limits_{x\to -4} \dfrac{2(x+4)}{(x+4)(x-3)}$

$= \lim\limits_{x\to -4} \dfrac{2}{x-3} = -\dfrac{2}{7}.$

(b)(10 pts) $\lim\limits_{x\to 0}\left(\dfrac{x}{2x+\sin x}\right) = \lim\limits_{x\to 0} \dfrac{x}{x\left(2+\frac{\sin x}{x}\right)}$

$= \lim\limits_{x\to 0} \dfrac{1}{2+\frac{\sin x}{x}}$

$= \dfrac{1}{2+\lim\limits_{x\to 0}\frac{\sin x}{x}} = \dfrac{1}{2+1} = \dfrac{1}{3}.$

(c)(10 pts) $\lim\limits_{x\to 2}\dfrac{1-x}{|x-2|} =$

$\lim\limits_{x\to 2^-} \dfrac{1-x}{|x-2|} = \lim\limits_{x\to 2^-} \dfrac{1-x}{2-x} = -\infty.$

$\lim\limits_{x\to 2^+} \dfrac{1-x}{|x-2|} = \lim\limits_{x\to 2^+} \dfrac{1-x}{x-2} = -\infty.$

So, $\lim\limits_{x\to 2} \dfrac{1-x}{|x-2|} = -\infty.$

(d) (7 pts) $\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x(x-1) = +\infty.$

(e) (8 pts) $\lim_{x\to -1} \frac{\cos(\ln(x+2))}{1 - x^2 - e^{3x}} = \frac{\cos(\ln(-1+2))}{1-(-1)^2-e^{-3}} = \frac{\cos 0}{-e^{-3}} = \frac{1}{-e^{-3}} = -e^3$

since $\ln x$ is continuous when $x>0$, and $\cos x$ and $1-x^2-e^{3x}$ are continuous on $\mathbb{R}$.

Problem 2 (10 pts) Find the values of the constants k and m if possible, that will make the function $f(x)$ continuous everywhere

$$f(x) = \begin{cases} 2x^3 + x + 7 & \text{if} \quad x \le -1 \\ m(x+1) + k & \text{if} \quad -1 < x \le 2 \\ x^2 + 5 & \text{if} \quad x > 2 \end{cases}$$

$f(x)$ is continuous on $(-\infty,-1)$, $(-1,2)$ and $(2,\infty)$ since $2x^3+x+7$, $m(x+1)+k$ and x^2+5 are polynomials.

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 2x^3+x+7 = -2-1+7 = 4.$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} m(x+1)+k = k.$ So, for $k=4$, $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1).$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} m(x+1)+k = 3m+k.$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2+5 = 4+5 = 9.$ So, for $m = 5/3$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2).$

Hence, $f(x)$ is continuous everywhere when $k=4$ and $m = 5/3$.

Problem 3 (15 pts) Find the horizontal and vertical asymptotes of

$$f(x) = \frac{x^2 - x - 2}{x^2 - 3x + 2}.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2\left(1-\frac{1}{x}-\frac{2}{x^2}\right)}{x^2\left(1-\frac{3}{x}+\frac{2}{x^2}\right)} = 1$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty} \frac{x^2\left(1-\frac{1}{x}-\frac{2}{x^2}\right)}{x^2\left(1-\frac{3}{x}+\frac{2}{x^2}\right)} = 1.$$

Hence, $y=1$ is a horizontal asymptote of $f(x)$.

Now, write $f(x)$ as $f(x) = \frac{(x-2)(x+1)}{(x-2)(x-1)}$.

So, $\mathrm{Domain}(f) = \mathbb{R} \setminus \{1, 2\}$.

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{x+1}{x-1} = -\infty. \quad \lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{x+1}{x-1} = +\infty.$$

So, $x=1$ is a vertical asymptote of $f(x)$.

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} \frac{x+1}{x-1} = 3. \quad \lim_{x\to2^+} f(x) = \lim_{x\to2^+} \frac{x+1}{x-1} = 3.$$

So, $x=2$ is NOT a vertical asymptote of $f(x)$.

Problem 4 (17 pts) (a)Find the derivative of the function $f(x) = \sqrt{x}$ using the limit definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(b) Find an equation of the tangent line to the curve $y = \sqrt{x}$ at $x = 9$.

The tangent line has the form;

$$y - y_0 = f'(x_0)(x - x_0).$$

Here, $x_0 = 9$, and $y_0 = f(9) = 3$ and $f'(x_0) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$.

Thus, we have $y - 3 = \frac{1}{6}(x-9)$; that is,

$$y = \frac{x}{6} + \frac{3}{2}.$$

Problem 5 (16 pts) Show that the equation $x = \cos x$ has at least one solution in the interval $[0, \frac{\pi}{2}]$.

Define $f(x) = \cos x - x$. Note that $f(x)$ is continuous on $\mathbb{R}$.

Now, $f(0) = \cos 0 - 0 = 1 - 0 = 1 > 0$ and

$f(\pi/2) = \cos \pi/2 - \pi/2 = 0 - \pi/2 = -\pi/2 < 0$.

Hence, there exists a number $c \in (0, \pi/2)$ such that $f(c) = 0$ by Intermediate Value Theorem.

Therefore, $\cos c - c = 0$, i.e. $c = \cos c$.

BONUS (10 pts) Suppose $f(x)$ is a function that satisfies the equation

$$f(x+y) = f(x) + f(y) + x^2 y + x y^2$$

for all real numbers x and y. Suppose also that $\lim_{x \to 0} \frac{f(x)}{x} = 1$. Find $f'(1)$.

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{f(1) + f(h) + 1^2 \cdot h + 1 \cdot h^2 - f(1)}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(h)}{h} + (1+h) \right]$$

$$= \lim_{h \to 0} \frac{f(h)}{h} + \lim_{h \to 0} (1+h)$$

$= 1 + 1 = 2$ since $\lim_{h \to 0} \frac{f(h)}{h} = 1$.