KOÇ UNIVERSITY

MATH 102 - CALCULUS

Midterm I May 10, 2010

Duration of Exam: 90 minutes

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name: ANSWER KEY

Surname: ______________________

Signature: ______________________

Section (Check One):

Section 1: Sultan Erdoğan M-W (14:00) —
Section 2: Benjamin Smith M-W (17:00) —
Section 3: Selda Küçükçifçi T-Th (11:00) —
Section 4: Selda Küçükçifçi T-Th (14:00) —
Section 5: Sultan Erdoğan M-W(12:30) —

PROBLEM	POINTS	SCORE
1	25	
2	12	
3	20	
4	18	
5	25	
TOTAL	**100**	

1. Let $f(x) = x^4 - 2x^2 + 1$.

(a) (7 points) Find the critical numbers of f and the intervals on which f is increasing/decreasing.

$f'(x) = 4x^3 - 4x = 0 \Rightarrow 4x(x^2-1) = 0$ so, $x=0$, $x = \pm 1$ are the critical points of f.

If $x \in (-\infty, -1)$; $f'(x) < 0$ decreasing
If $x \in (-1, 0)$; $f'(x) > 0$ increasing
If $x \in (0, 1)$; $f'(x) < 0$ decreasing
If $x \in (1, \infty)$; $f'(x) > 0$ increasing

x		-1		0		1	
f'	$-$	0	$+$	0	$-$	0	$+$
f	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

(b) (3 points) Find the local maximum/minimum point(s) of f. (Specify x and y-coordinates of each point.)

$x = -1$; $f(-1) = 0$
$x = 0$; $f(0) = 1$
$x = 1$; $f(1) = 0$

So, $(-1, 0)$ is a local minimum
$(0, 1)$ is a local maximum
$(1, 0)$ is a local minimum

(c) (5 points) Find the intervals on which f is concave up/down.

$f''(x) = 12x^2 - 4 = 0 \Rightarrow 12x^2 = 4 \Rightarrow x^2 = 1/3 \Rightarrow x = \pm\sqrt{1/3}$

If $x \in (-\infty, -\sqrt{1/3})$; $f''(x) > 0$ concave up
If $x \in (-\sqrt{1/3}, \sqrt{1/3})$; $f''(x) < 0$ concave down
If $x \in (\sqrt{1/3}, \infty)$; $f''(x) > 0$ concave up

		$-1/\sqrt{3}$		$1/\sqrt{3}$	
f''	$+$	0	$-$	0	$+$
f	C.U		C.D		C.U

(d) (2 points) Find the inflection point(s) of f. (Specify x and y-coordinates of each point.)

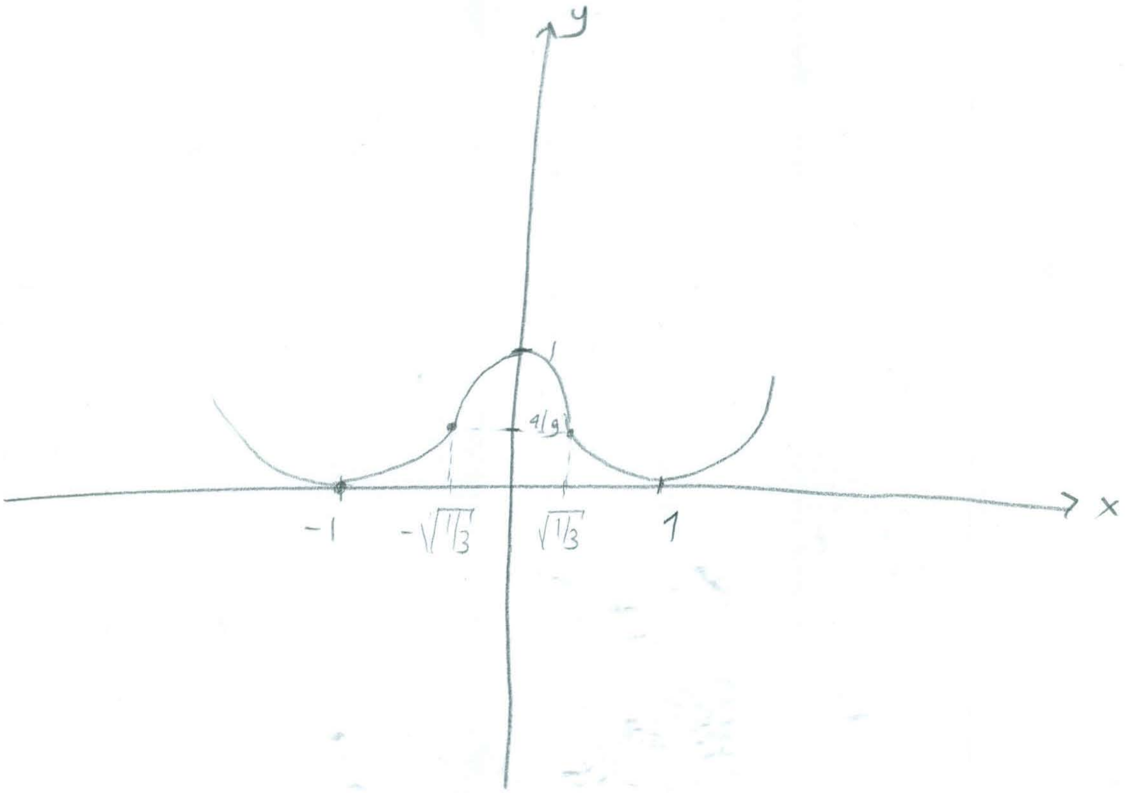
$$f\left(-\sqrt{\tfrac{1}{3}}\right) = \frac{1}{9} - 2\cdot\frac{1}{3} + 1 = \frac{1-6+9}{9} = 4/9$$

(3) (9)

$$f\left(+\sqrt{\tfrac{1}{3}}\right) = 4/9$$

so, $\left(-\sqrt{\tfrac{1}{3}}, \frac{4}{9}\right)$ and $\left(\sqrt{\tfrac{1}{3}}, \frac{4}{9}\right)$ are the inflection points.

(e) (8 points) Sketch a graph of f. (Indicate all maximum/minimum/inflection points.)



2. (12 points) Find the absolute maximum and the absolute minimum values of $f(x) = 2x^3 - 3x^2 - 12x + 12$ on the interval $[-2, 1]$.

$f(x) = 2x^3 - 3x^2 - 12x + 12$ is a polynomial, hence it's continuous on the interval $[-2, 1]$.

Critical points of this function can be the endpoints of the interval and the points satisfying.

$f'(x) = 6x^2 - 6x - 12 = 0 \Rightarrow 6(x^2 - x - 2) = 0$

$6.(x-2)(x+1) = 0 \Rightarrow x = 2$ or $x = -1$.

$x = 2$ isn't in the given interval. So we have, $x=-2$, $x=-1$ and $x=1$.

$f(-2) = 2.(-2)^3 - 3(-2)^2 - 12(-2) + 12 = -16 - 12 + 24 + 12 = 8.$

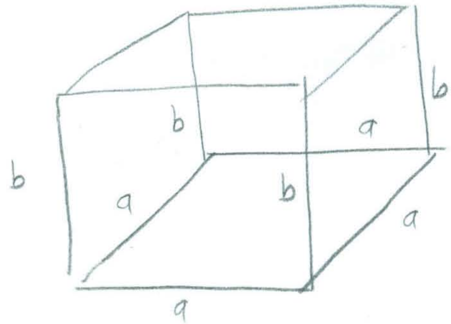
$f(-1) = 2(-1)^3 - 3(-1)^3 - 12(-1) + 12 = -2 - 3 + 12 + 12 = 19$

$f(1) = 2 - 3 - 12 + 12 = -1$

Absolute min.

Absolute max.

3. (20 points) If we want to make a box with a square base (kare taban) and closed top (tavanı dahil) having a volume of 8000 cm^3, find the dimensions (boyutlar) of the box that minimize the amount of material used.



$$\text{Volume} = a^2 . b = 8000 \Rightarrow b = \frac{8000}{a^2}$$

The amount of material that's used is,

$$A = a^2 + a^2 + 4ab = 2a^2 + 4ab$$

Using $b = 8000/a^2$,

$$A(a) = 2a^2 + 4\cancel{a}.\frac{8000}{a^{\cancel{2}}} = 2a^2 + \frac{32000}{a} = 2a^2 + 32000.a^{-1}$$

To find the minimum material amount,

$A'(a) = 4a - 32000a^{-2} = 0$ the roots of this equation must be computed.

$$4a = \frac{32000}{a^2} \Rightarrow a^3 = \frac{32000}{4} \Rightarrow a^3 = 8000 \Rightarrow \boxed{a = 20}$$

Then,

$$b = \frac{8000}{a^2} = \frac{80\cancel{00}}{4\cancel{00}} = 20 \quad \boxed{b = 20}$$

$A''(a) = 4 - 32000(-2)a^{-3} \Rightarrow A''(20) > 0$ so $a = 20$ minimizes $A(a)$

4. Evaluate the following limits. State the type of indeterminate form, if any.

(a) (6 points) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{0}{0} \text{ type indeterminate form.}$$

Using L'Hopital's Rule,

$$\lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2} \underbrace{\lim_{x \to 0} \frac{\sin x}{x}} = 1/2$$

We know that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

(b) (12 points) $\lim_{x \to \infty} (x^2 + 1)^{\frac{1}{x}}$

$$\lim_{x \to \infty} (x^2+1)^{1/x} = \infty^0 \text{ type indeterminate form.}$$

$$y = (x^2+1)^{1/x} \Rightarrow \ln y = \frac{1}{x} . \ln(x^2+1)$$

$$\lim_{x \to \infty} (\ln y) = \lim_{x \to \infty} \frac{\ln(x^2+1)}{x} = \frac{\infty}{\infty} \text{ type indeterminate form.}$$

Using L'Hopital's Rule,

$$\lim_{x \to \infty} \frac{1}{x^2+1} \cdot \frac{2x}{1} = \lim_{x \to \infty} \frac{2x}{x^2+1} = 0$$

So $\lim_{x \to \infty} \ln y = 0 \Rightarrow \lim_{x \to \infty} y = 1.$

That is, $\lim_{x \to \infty} (x^2+1)^{1/x} = 1$

5. (a) (5 points) Evaluate $\int_0^1 x^{1/3}(x+2)dx$.

$$\int_0^1 x^{1/3}(x+2)\,dx = \int_0^1 (x^{4/3} + 2x^{1/3})\,dx = \left[\frac{x^{7/3}}{7/3} + 2\,\frac{x^{4/3}}{4/3}\right]_0^1$$

$$= \left(\frac{3}{7}x^{7/3} + \frac{6}{4}x^{4/3}\right)\Big|_0^1 = \underset{(4)}{\frac{3}{7}} + \underset{(7)}{\frac{6}{4}} = \frac{12+42}{28} = \frac{54}{28} = \frac{27}{14}$$

(b) (8 points) Find the general indefinite integral $\int \left(\frac{x^2e^x - 1}{x^2} - \frac{2}{x}\right) dx$. (Simplify your answer.)

$$\int\left(\frac{x^2e^x - 1}{x^2} - \frac{2}{x}\right)dx = \int\left(e^x - \frac{1}{x^2} - \frac{2}{x}\right)dx = \int\left(e^x - x^{-2} - 2x^{-1}\right)dx$$

$$= e^x - \frac{x^{-1}}{-1} - 2\ln|x| + C$$

(c) (6 points) Find f where $f'(x) = 3e^x - 3$ and $f(0) = 2$.

If $f'(x) = 3e^x - 3$ then, $f(x) = 3e^x + 3x + C$

Given that $f(0) = 2$;

$f(0) = 3.1 + 3.0 + C = 3 + C = 2 \Rightarrow C = -1$ is calculated.

Then $\boxed{f(x) = 3e^x + 3x - 1}$

(d) (6 points) Find the area under the curve $y = \frac{1}{2}\sin x$ from $x = 0$ to $x = \pi$.

$$\int_0^\pi \frac{1}{2}\sin x\,dx = \frac{1}{2}\int_0^\pi \sin x\,dx = \frac{1}{2}(-\cos x)\Big|_0^\pi = \frac{-\cos x}{2}\Big|_0^\pi$$

$$= -\frac{1}{2}(-1-1) = -\frac{1}{2}\cdot(-2) = 1 \;//$$