KOÇ UNIVERSITY

MATH 102

FINAL May 24, 2014

Duration of Exam: 120 minutes

INSTRUCTIONS:

- No calculators may be used on the test.
- No books, no notes, no questions, and no talking allowed.
- You must always explain your answers and SHOW YOUR WORK to receive full credit.
- Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

KEY

SURNAME, Name: ______________________

Student ID no: ______________________

Signature: ______________________

(Check One):

(Şule Yazıcı	–	MWF 11:00-12:15)	:	___
(Şule Yazıcı	–	MWF 12:30-13:45)	:	___
(Şule Yazıcı	–	MWF 15:30-16:45)	:	___
(Fatih Demirkale	–	TuTh 12:30-13:45)	:	___
(Fatih Demirkale	–	TuTh 14:00-15:15)	:	___

PROBLEM	1	2	3	4	5	6	7	TOTAL
POINTS	10	10	10	10	48	10	12	110
SCORE								

Problem 1 (10 pts) Use implicit differentiation to find dy/dx if

$$\sin(x^2y^2) = x.$$

$\cos(x^2y^2).(x^2y^2)' = 1$

$\Rightarrow \cos(x^2y^2).(2xy^2 + x^2 2y \cdot y') = 1$

$\Rightarrow \quad y' = \frac{dy}{dx} = \frac{1 - 2xy^2 . \cos(x^2y^2)}{x^2 2y . \cos(x^2y^2)}$

Problem 2 (10 pts) Let $f(x) = \frac{x^4}{2} + px + q$. Find the values of p and q such that $f(1) = \frac{1}{2}$ is an extreme value of f. Is this value a maximum or a minimum?

$f'(x) = 2x^3 + p = 0 \quad \Rightarrow \quad p = -2x^3$

$\Rightarrow \quad p = -2$ when $x = 1$.

$f(1) = 1/2 = \frac{1}{2} - 2 + q \quad \Rightarrow \quad q = 2.$

$f''(x) = 6x^2, \quad f''(1) = 6 > 0$. So $(1, 1/2)$ is a local min.

Problem 3 (10 pts) Find $f(0)$ if f is a continuous function on $(-\infty, \infty)$ and $f(x) = x^x$ for all $x \neq 0$.

$f(0) = \lim_{x\to 0} f(x)$. Let $y = x^x$ and $\ln y = \ln x^x = x \cdot \ln x$.

$$\lim_{x\to 0} x \cdot \ln x = \lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -x = 0.$$

So, $\lim_{x\to 0} \ln y = 0$ and $\lim_{x\to 0} y = e^0 = 1$.

Thus, $f(0) = 1$.

Problem 4 (10 pts) Calculate $\lim_{x\to\infty} \frac{\ln(x^2)}{\ln(3x)}$.

$$\lim_{x\to\infty} \frac{\ln x^2}{\ln 3x} = \lim_{x\to\infty} \frac{\frac{2x}{x^2}}{\frac{3}{3x}} = \lim_{x\to\infty} 2 = 2.$$

Problem 5 Evaluate the following integrals.

(a) (8 pts) $\int_0^1 \frac{x^2}{x^2+1}\,dx$ $= \int_0^1 \left(1 - \frac{1}{x^2+1}\right) dx$

$= \left(x - \arctan x\right)\Big|_0^1$

$= 1 - \pi/4 - 0$

$= 1 - \pi/4$.

(b) (8 pts) $\int x \cos x\, dx$ $= x\sin x - \int \sin x\, dx$

$= x\sin x + \cos x + C$.

$u = x$ $\quad dv = \cos x\, dx$

$du = dx$ $\quad v = \sin x$

(c) (8 pts) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \quad = \int 2e^{u}\,du$

$u=\sqrt{x}\ ,\ du=\frac{1}{2x}dx$

$= 2e^{u}+C$

$= 2e^{\sqrt{x}}+C.$

(d) (8 pts) $\int \frac{2x^2-x+1}{(x^2+1)(x-1)}\,dx \; = \int \frac{x}{x^2+1}dx + \int \frac{1}{x-1}dx$

$\frac{Ax+B}{x^2+1}+\frac{C}{x-1}=\frac{2x^2-x+1}{(x^2+1)(x-1)}$

$\Rightarrow Ax^2-Ax+Bx-B+Cx^2+C=2x^2-x+1$

$\Rightarrow \quad A+C=2 \qquad\qquad A=1$

$-A+B=-1 \quad \Rightarrow \quad B=0$

$-B+C=1 \qquad\qquad C=1$

$=\frac{1}{2}\ln(x^2+1)+\ln|x-1|+C$

(e) (8 pts) $\int_0^2 \frac{1}{(x-1)^2}\, dx$

Note that $\frac{1}{(x-1)^2}$ is not defined at $x=1$.

So, $\int_0^2 \frac{1}{(x-1)^2} dx = \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx$.

$$\int_0^1 \frac{1}{(x-1)^2} dx = \lim_{t\to 1^-} \left(\int_0^t \frac{1}{(x-1)^2} dx \right) = \lim_{t\to 1^-} \left(\frac{-1}{(x-1)} \Big|_0^t \right)$$

$$= \lim_{t\to 1^-} \left(\frac{-1}{t-1} - 1 \right) = \infty.$$

So, $\int_0^1 \frac{1}{(x-1)^2} dx$ is divergent and $\int_1^2 \frac{1}{(x-1)^2} dx$ is divergent.

(f)(8 pts) $\int_{-2}^2 \frac{\sin x}{1+x^2+x^4}\, dx$

Let $f(x) = \frac{\sin x}{1+x^2+x^4}$.

$f(x)$ is an odd function since $f(-x) = \frac{\sin(-x)}{1+(-x)^2+(-x)^4}$

$$= \frac{-\sin x}{1+x^2+x^4} = -f(x).$$

Thus, $\int_{-2}^2 \frac{\sin x}{1+x^2+x^4} dx = 0$.

Problem 6 (10 pts) Calculate $F'(x)$ if $F(x) = \int_{x^2}^{0} t \sec t \, dt$.

$$F'(x) = \frac{d}{dx} \int_{x^2}^{0} t \sec t \, dt$$

$$= -\frac{d}{dx} \int_{0}^{x^2} t \sec t \, dt$$

$$= -x^2 \sec x^2 \cdot 2x$$

$$= -2x^3 \sec x^2 \quad \text{by the Fundamental Theorem of Calculus.}$$

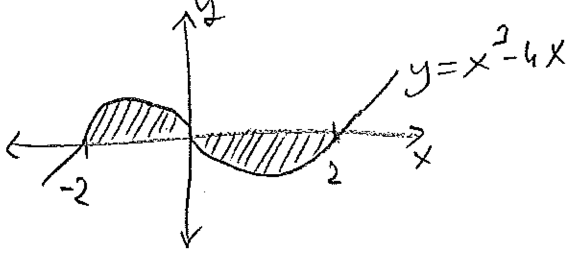
Problem 7 (12 pts) Determine the area of the region enclosed by the curve $y = x^3 - 4x$ and the x-axis.

$$y = x^3 - 4x = 0 \Rightarrow x(x-2)(x+2) = 0$$

$$\Rightarrow x = 0, \; x = 2, \; x = -2.$$

For $x \in [-2, 0]$, $x^3 - 4x \geq 0$,

$x \in [0, 2]$, $x^3 - 4x \leq 0$.



$$\text{Area} = \int_{-2}^{0} (x^3 - 4x) \, dx + \int_{0}^{2} -(x^3 - 4x) \, dx$$

$$= \left(\frac{x^4}{4} - 2x^2\right)\Big|_{-2}^{0} + \left(-\frac{x^4}{4} + 2x^2\right)\Big|_{0}^{2}$$

$$= \left(0 - \frac{16}{4} + 8\right) + \left(-\frac{16}{4} + 8 - 0\right)$$

$$= 4 + 4 = 8.$$