FALL 2006 FINAL

Question 1 (15 Points):

Find the following limits:

(a) $\lim_{h\to 0} \dfrac{\frac{1}{2+h}-\frac{1}{2}}{h}$

(b) $\lim_{x\to 0} \dfrac{\tan(4x)}{\sin(5x)}$

(c) Let $f(x)=\dfrac{\tan(4x)}{\sin(5x)}$ for $-\pi/2<x<\pi/2,\ x\neq 0$. How would you define $f(0)$ so that $f(x)$ is continuous?

a) $\lim_{h\to 0} \dfrac{\frac{1}{2+h}-\frac{1}{2}}{h} = f'(2)$ where $f(x)=\dfrac{1}{x}$

So $f'(x) = -\dfrac{1}{x^2} \Rightarrow f'(2) = -\dfrac{1}{4}$

b) $\lim_{x\to 0} \dfrac{\tan 4x}{\sin 5x} = \lim_{x\to 0} \dfrac{\frac{\tan 4x}{4x}}{\frac{\sin 5x}{5x}} \cdot \dfrac{4}{5}$

$= \dfrac{4}{5} \cdot \dfrac{\lim_{x\to 0} \frac{\tan 4x}{4x}}{\lim_{x\to 0} \frac{\sin 5x}{5x}} = \dfrac{4}{5}\cdot\dfrac{1}{1} = \dfrac{4}{5}$

c) As $\tan(4x)$ & $\sin 5x$ are cts on $(-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2})$

& f is defined on that interval;

f is continuous on $(-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2})$.

So we just have to check for $x=0$.

In order f to be continuous at $x=0$;

$\lim_{x\to 0^+} \dfrac{\tan 4x}{\sin 5x} = \lim_{x\to 0^-} \dfrac{\tan 4x}{\sin 5x} = f(0)$

As in (b) $\lim_{x\to 0^+} \dfrac{\tan 4x}{\sin 5x} = \lim_{x\to 0^-} \dfrac{\tan 4x}{\sin 5x} = \dfrac{4}{5}$

So if $f(0)=\dfrac{4}{5}$, then f is continuous on $(-\frac{\pi}{2},\frac{\pi}{2})$

Question 2 (15 Points):

(a) $y = f(x)$ is a one-to-one function, and the point $(-1, 2)$ is on its graph. Let $f^{-1}(x)$ be the inverse function of $f(x)$, and $f'(x) = \frac{d}{dx} f(x)$ be the derivative of $f(x)$. The equation of the tangent to $y = f(x)$ at $(-1, 2)$ is $y = 2x + b$. Find the following. Justify your answers.

(i) b
(ii) $f^{-1}(2)$
(iii) $f'(-1)$
(iv) $f^{-1}(f(-1))$
(v) $\frac{d}{dx} f^{-1}(x)\Big|_{x=2}$

(b) If $\sin(x) = -\frac{1}{2}$, then what are all possible values for $\tan(x)$?

a)

(i) As $(-1,2)$ is a point on $y=2x+b$; $2 = 2.(-1)+b$

$b=4$

(ii) Since $f(-1)=2$ and f is one to one, $f^{-1}(2) = -1$

(iii) $f'(-1)$ = slope of the tangent line at $(-1,2)$.
$= 2$

(iv) $f^{-1}(f(-1)) = f^{-1}(2) = -1$ by (ii)

(v) $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$ So $\frac{d}{dx} f^{-1}(x)\Big|_{x=2} = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(-1)} = \frac{1}{2}$

b) $\tan x = \frac{\sin x}{\cos x}$

$\sin^2 x + \cos^2 x = 1 \xrightarrow{\sin x = -\frac{1}{2}} \frac{1}{4} + \cos^2 x = 1 \Rightarrow \cos^2 x = \frac{3}{4}$

$\cos x = \pm \frac{\sqrt{3}}{2}$

So $\tan x = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}$ or $\tan x = \frac{-1/2}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$

Question 3 (15 Points):

Let $f'(x) = \dfrac{d}{dx} f(x)$ be the derivative of $f(x)$. Find

(a) $f'(x)$ for $f(x) = \sqrt[3]{\sin(x^2)}$

(b) The slope of the tangent at (1,–1) to the circle $x^2 + y^2 = 2$

(c) The function $f(x)$ is continuous in the interval (–5, 3). Find all local extrema of $f(x)$ in the interval (–5, 3) if $f'(1)$ does not exist and

x	(–5, –2)	–2	(–2, –1)	–1	(–1, 0)	0	(0, 1)	(1, 3)
$f'(x)$	–	0	+	0	+	0	–	+

a) $f'(x) = (\sin(x^2))' \cdot \frac{1}{3} \cdot \frac{1}{\sqrt[3]{(\sin x^2)^2}} = \frac{2}{3} x \cos x^2 \cdot \frac{1}{\sqrt[3]{(\sin x^2)^2}}$

b) $x^2 + y^2 = 2$

$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx} 2 \quad \Rightarrow \quad 2x + 2y \cdot \frac{dy}{dx} = 0$

$\Rightarrow \quad 2 \cdot 1 + 2 \cdot (-1) \cdot \frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = 1$ = slope of the tangent line at $(1,-1)$

c)

		–2		–1		0		1	
$f'(x)$	–		+		+		–		+
	↘		↗		↗		↘		↗

f has local maximum at $x = 0$

f has local minimum at $x = -2$ and $x = 1$

Question 4 (10 Points):

(a) Find the $\frac{d}{dx}\int_{\sqrt{x}}^{3x} t^2\,dt$ using the Fundamental Theorem of Calculus.

(b) Find $\frac{d}{dx}\int_{\sqrt{x}}^{3x} t^2\,dt$ by first finding $\int_{\sqrt{x}}^{3x} t^2\,dt$, and then taking the derivative of the result.

(c) Find $\int_{1}^{e} (2(\ln(x)+1))dx$ given that the derivative of $x^2\ln(x)$ is $2(\ln(x)+1)$.

a) $$\frac{d}{dx}\int_{\sqrt{x}}^{3x} t^2\,dt = \frac{d}{dx}\left(\int_{\sqrt{x}}^{0} t^2\,dt + \int_{0}^{3x} t^2\,dt\right)$$

$$= \frac{d}{dx}\left(-\int_{0}^{\sqrt{x}} t^2\,dt\right) + \frac{d}{dx}\int_{0}^{3x} t^2\,dt$$

$$= -\frac{1}{2\sqrt{x}}(\sqrt{x})^2 + 3(3x)^2 = -\frac{1}{2}\sqrt{x} + 27x^2$$

b) $$\int_{\sqrt{x}}^{3x} t^2\,dt = \frac{1}{3}t^3 + C\Big|_{\sqrt{x}}^{3x} = \frac{1}{3}\cdot 27x^3 - \frac{1}{3}\sqrt{x^3} = 9x^3 - \frac{1}{3}\sqrt{x^3}$$

$$\frac{d}{dx}\left(9x^3 - \frac{1}{3}\sqrt{x^3}\right) = 27x^3 - \frac{1}{3}\cdot\frac{3}{2}\cdot\sqrt{x} = 27x^3 - \frac{1}{2}\sqrt{x}$$

c) $$\int_{1}^{e} 2(\ln x + 1)dx = x^2\ln|x| + C\Big|_{1}^{e} = e^2\ln|e| - 1^2\ln|1|$$

$$= e^2$$

Question 5 (20 Points):

(a) Evaluate

$$\int_{0,5}^{1} \frac{x^2+13}{x^2+1}\,dx$$

(b) Find the area between the curve $y = 2x\sqrt{x^2+1}$, $0 \le x \le \sqrt{3}$, and the x-axis

a) $\int_{0,5}^{1} \frac{x^2+13}{x^2+1}\,dx = \int_{0,5}^{1}\left(1+\frac{12}{x^2+1}\right)dx = \int_{0,5}^{1} 1.dx + \int_{0,5}^{1} \frac{12}{1+x^2}\,dx$

$= x + c_1 \Big|_{0,5}^{1} + (12 \arctan x + c_2)\Big|_{0,5}^{1}$

$= 1 - 0,5 + 12 \arctan 1 - 12 \arctan 0,5$

$= \frac{1}{2} + 12 (\arctan 1 - \arctan 0,5)$

b) $A = \int_{0}^{\sqrt{3}} 2x \sqrt{x^2+1}\,dx$

Let $x^2 + 1 = u$ for $x = 0$, $u = 1$

$x = \sqrt{3}$, $u = 4$

$2x\,dx = du$

So $A = \int_{1}^{4} u^{1/2}\,du = \frac{2}{3} u^{3/2} + c \Big|_{1}^{4}$

$= \frac{2}{3}(4)^{3/2} - \frac{2}{3}(1)^{3/2}$

$= \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$

Question 6 (10 Points):

Determine whether the improper integral $\int_0^\infty e^{-x}dx$ is convergent or divergent. If the improper integral is convergent, evaluate it.

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$

$$\int e^{-x}\,dx = -e^{-x}+C$$

So $\lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}(-e^{-x}+C)\Big|_0^b$

$$= \lim_{b\to\infty} -e^{-b}-(-e^{-0})$$

$$= \lim_{b\to\infty} -e^{-b}+1$$

$$= \lim_{b\to\infty} -e^{-b} + \lim_{b\to\infty} 1$$

$$= 0+1$$
$$= 1$$

Hence $\int_0^\infty e^{-x}\,dx$ ~~converges~~ is convergent and $\int_0^\infty e^{-x}\,dx = 1.$

Question 7 (10 Points):

Determine whether the following sequence is convergent or divergent. If the sequence is convergent, find its limit.

(a) $a_n = \dfrac{(-1)^n n}{n+1}$

(b) $a_n = \dfrac{\ln(n+1)}{\sqrt{n}}$

Question 8 (10 Points):

For each of the following series, write the first 2 terms and determine whether the series is convergent or divergent. If the series converges, find its sum.

(a) $\sum_{n=1}^{\infty} (-1)^n$

(b) $\sum_{n=0}^{\infty} \frac{2^{2n}}{3^{n+1} 5^n}$