KOÇ UNIVERSITY

MATH 102

FINAL EXAM December 28, 2019

Duration of Exam: 100 minutes

INSTRUCTIONS:

- No calculators may be used on the test.
- No books, no notes, no questions, and no talking allowed.
- You must always explain your answers and SHOW YOUR WORK to receive full credit.
- Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

SURNAME, Name: ________________

Student ID no: KEY

Signature: ________________

(Check One):

(Selda Küçükçifçi – TTh 8:30-9:45) : —
(Selda Küçükçifçi – TTh 13:00-14:15) : —
(E. Şule Yazıcı – TTh 16:00-17:15) : —

PROBLEM	POINTS	SCORE
1	10	
2	10	
3	12	
4	12	
5	10	
6	42	
7	12	
TOTAL	108	

Problem 1 (10 points) Determine the following limit, if it exists.

$$\lim_{x\to 0} \frac{\tan(2x)}{\sin 8x} = \lim_{x\to 0} \frac{\sin(2x)}{2x} \cdot \frac{1}{\cos(2x)} \cdot \frac{8x}{\sin(8x)} \cdot \frac{2}{8}$$

$$= 1 \cdot 1 \cdot 1 \cdot \frac{2}{8} = \frac{1}{4}$$

Problem 2 (10 points) Let $f(x) = \int_1^{\ln x} e^{t^2}\, dt$. Find $f'(e^2)$.

$$f'(x) = e^{(\ln x)^2} \cdot \frac{1}{x}$$

$$= e^{(\ln e^2)^2} \cdot \frac{1}{e^2}$$

$$= \frac{e^4}{e^2} = e^2.$$

Problem 3 (12 points) Suppose that x years after its founding in 2000, a sport club membership number is given by the function $f(x) = 100(2x^3 - 45x^2 + 264x)$. At what time between 2000 and 2005 was the membership of the association largest? What was the membership at that time?

Maximize $f(x) = 100(2x^3 - 45x^2 + 264x)$

$0 \le x \le 5$.

$$f'(x) = 100(6x^2 - 90x + 264)$$

$$600(x^2 - 15x + 44) = 0$$

$$x_{1,2} = \frac{15 \pm \sqrt{225 - 4 \cdot 1 \cdot 44}}{2} = \frac{15 \pm 7}{2}$$

$$x_1 = 11 \notin [0,5] \quad x_2 = 4$$

$f(4) = 46400$

$f(0) = 0$

$f(5) = 44500$.

So in 2004 the membership was the largest.

it was 46400.

Problem 4 (12 points) Show that the equation $x^3 + 15x - 5 = 0$ has exactly one root.

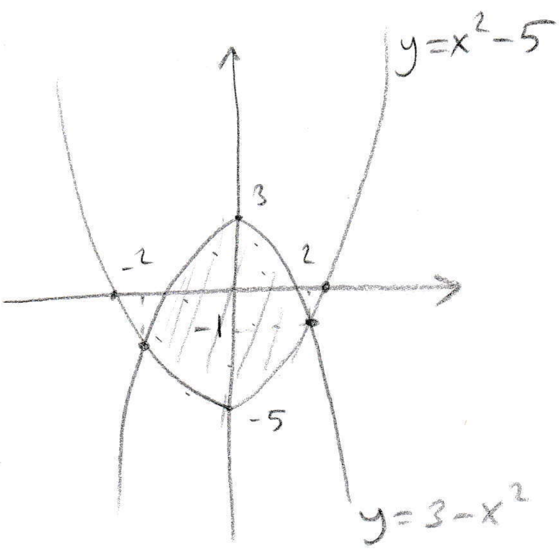
Let $f(x) = x^3 + 15x - 5$. f is continuous everywhere.

$f(0) = -5 < 0$ and $f(2) = 8 + 30 - 5 = 33 > 0$.

So since $f(0) < 0 < f(2)$ there exists $c \in (0,2)$ such that $f(c) = 0$, that is $x^3 + 15x - 5 = 0$ has at least one root.

If $f(x) = 0$ has at least two roots, say a & b, $f(a) = f(b) = 0$. But then there is $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a} = 0$ by Mean Value Theorem. But $f'(x) = 3x^2 + 15$ is always positive. Hence $x^3 + 15x - 5 = 0$ has exactly one root.

Problem 5 (10 points) Find the area of the region enclosed by the curves $y = x^2 - 5$ and $y = 3 - x^2$.



$$x^2 - 5 = 3 - x^2$$

$$2x^2 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$$

$$\int_{-2}^{2} (3 - x^2 - x^2 + 5)\,dx = \int_{-2}^{2} (-2x^2 + 8)\,dx$$

$$= \left[\frac{-2}{3} x^3 + 8x \right]_{-2}^{2}$$

$$= \frac{-16}{3} + 16 - \left(+ \frac{16}{3} - 16 \right)$$

$$= 32 - \frac{32}{3} = \frac{2 \times 32}{3} = \frac{64}{3}.$$

Problem 6 Evaluate the following integrals.

(a) (15 points) $\int \frac{\ln x}{x^2}\, dx$

$u = \ln x$ $\qquad v = \frac{x^{-1}}{-1} = \frac{-1}{x}$

$du = \frac{1}{x} dx$ $\qquad dv = x^{-2} dx$

$$\int \frac{\ln x}{x^2} dx = -\frac{1}{x}\ln x + \int x^{-2} dx = -\frac{1}{x}\ln x - \frac{1}{x} + C.$$

(b) (15 points) $\int \frac{2x+4}{(x^2+1)(x+1)}\, dx$

$$\frac{2x+4}{(x^2+1)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$2x+4 = A(x^2+1) + (Bx+C)(x+1)$$

$$2x+4 = (A+B)x^2 + (B+C)x + (A+C)$$

$$\begin{cases} A+B=0 \\ B+C=2 \\ A+C=4 \end{cases} \Rightarrow \begin{matrix} A=1 \\ B=-1 \\ C=3 \end{matrix}$$

$$\int \frac{2x+4}{(x^2+1)(x+1)} dx = \int \frac{dx}{x+1} - \int \frac{x\,dx}{x^2+1} + 3\int \frac{dx}{x^2+1}$$

$$= \ln|x+1| - \frac{1}{2}\ln(x^2+1) + 3\arctan x + C.$$

(c) (12 points) $\int_{\pi^2}^{4\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ $= \int_{\pi}^{2\pi} 2\sin u\,du = 2\left[-\cos u\right]_{\pi}^{2\pi}$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$= 2(-1-(1)) = -4.$

Problem 7 (12 points) Determine whether the following integral is convergent or divergent. If it is convergent, then evaluate it.

$$\int_2^\infty \frac{dx}{\sqrt[3]{x^2 - x}}$$

$x^2 - x < x^2$ since $x > 2$

So $\sqrt[3]{x^2 - x} < \sqrt[3]{x^2}$

$\Rightarrow \frac{1}{\sqrt[3]{x^2-x}} > \frac{1}{x^{2/3}}$

But $\int_2^\infty \frac{dx}{x^{2/3}}$ is divergent, hence $\int_2^\infty \frac{dx}{\sqrt[3]{x^2-x}}$ is divergent, too.