KOÇ UNIVERSITY

MATH 102

FIRST MIDTERM November 11, 2015

Duration of Exam: 75 minutes

INSTRUCTIONS:

- **No calculators may be used on the test.**
- **No books, no notes, no questions, and no talking allowed.**
- **You must always explain your answers and SHOW YOUR WORK to receive full credit.**
- **Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!**

SURNAME, Name: ______________________

Student ID no: KEY

Signature: ______________________

(Check One):
(Selda Küçükçifçi – MW 13:00-14:15) : —
(Selda Küçükçifçi – MW 16:00-17:15) : —

PROBLEM	1	2	3	4	BQ	TOTAL
POINTS	30	20	30	20	5	105
SCORE						

Problem 1 (30 pts) Find the following limits.

(a) (10 pts) $\lim_{x\to\frac{1}{2}^-}\left(\frac{2x-1}{|\,2x^3-x^2|}\right)$ $= \lim_{x\to\frac{1}{2}^-} \frac{2x-1}{x^2\,|2x-1|}$

$$= \lim_{x\to\frac{1}{2}^-} \frac{2x-1}{x^2(-2x+1)} = \lim_{x\to\frac{1}{2}^-} \frac{-1}{x^2} = \frac{-1}{\frac{1}{4}} = -4$$

(b) (8 pts) $\lim_{x\to 0^+}\left(\left(1+\sin(\frac{2\pi}{x})\right)\sqrt{x}\right)$ $-1 \le \sin\left(\frac{2\pi}{x}\right) \le 1 \Rightarrow 0 \le 1+\sin\left(\frac{2\pi}{x}\right) \le 2$

$$\Rightarrow 0 \le \left(1+\sin\left(\frac{2\pi}{x}\right)\right)\sqrt{x} \le 2\sqrt{x}$$

Since $\lim_{x\to 0^+} 2\sqrt{x} = 0$ by Sandwich Theorem

$$\lim_{x\to 0^+}\left(1+\sin\left(\frac{2\pi}{x}\right)\right)\sqrt{x} = 0.$$

(c) (12 pts) $\lim_{x\to\infty}(e^x+x)^{1/x}$

$$y = (e^x+x)^{1/x}$$

$$\ln y = \frac{1}{x}\ln(e^x+x)$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(e^x+x)}{x} \underset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{\frac{1}{e^x+x}\cdot(e^x+1)}{1} = \lim_{x\to\infty}\frac{e^x+1}{e^x+x} \quad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty}\frac{e^x}{e^x+1} \underset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{e^x}{e^x} = 1$$

Since $\ln(\lim_{x\to\infty} y) = \lim_{x\to\infty}\ln y = 1 \Rightarrow \lim_{x\to\infty} y = e.$

Problem 2 (20 pts) Using intermediate value theorem show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

Let $f(x) = x^3 - 15x + 1$

f is continuous everywhere since f is a polynomial function. So f is continuous on $[-4, 4]$ too.

$f(-4) = -64 + 60 + 1 = -3 < 0$

$f(0) = 1 > 0$

$f(1) = 1 - 15 + 1 = -13 < 0$

$f(4) = 64 - 60 + 1 = 5 > 0$.

Then by Intermediate Value Theorem

Since $-3 = f(-4) < 0 < f(0) = 1$ there is $c_1 \in (-4, 0)$ such that $f(c_1) = 0$

Since $-13 = f(1) < 0 < f(0)$ there is $c_2 \in (0, 1)$ such that $f(c_2) = 0$

and

Since $-13 = f(1) < 0 < f(4) = 5$ there is $c_3 \in (1, 4)$ such that $f(c_3) = 0$.

Hence $x^3 - 15x + 1 = 0$ has three solutions c_1, c_2, c_3 in $[-4, 4]$.

Problem 3 (30 pts) Find the derivative of the function f in (a) – (b). Simplify your answer, if it is possible.

(a) $f(x) = (e^{\sqrt{x}} + \ln \sqrt[5]{x})^2$

$$f'(x) = 2\left(e^{\sqrt{x}} + \frac{1}{5}\ln x\right) \cdot \left(e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} + \frac{1}{5} \cdot \frac{1}{x}\right)$$

$$= 2\left(e^{\sqrt{x}} + \frac{1}{5}\ln x\right)\left(\frac{e^{\sqrt{x}}}{2\sqrt{x}} + \frac{1}{5x}\right).$$

(b) $f(x) = \tan^{-1}(\frac{1-x}{1+x})$

$$f'(x) = \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \cdot \frac{-1(1+x)-(1-x)}{(1+x)^2} = \frac{\cancel{(1+x)^2}}{(1+x)^2+(1-x)^2} \cdot \frac{-2}{\cancel{(1+x)^2}}$$

$$= \frac{-2}{(1+x)^2+(1-x)^2} = \frac{-2}{1+x^2+2x+1-2x+x^2} = \frac{-2}{2+2x^2} = \frac{-1}{1+x^2}$$

(c) Express $\lim_{x \to \frac{\pi}{3}^-} \left(\dfrac{\cos x - 0.5}{x - \frac{\pi}{3}} \right)$ as a derivative and evaluate it.

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$\lim_{x \to \frac{\pi}{3}^-} \frac{\cos x - 0.5}{x - \frac{\pi}{3}} = f'\left(\frac{\pi}{3}\right), \quad \text{where } f(x) = \cos x$$

$$f\left(\frac{\pi}{3}\right) = 0.5$$

$$f'(x) = -\sin x \qquad f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}.$$

So $$\lim_{x \to \frac{\pi}{3}^-} \frac{\cos x - 0.5}{x - \frac{\pi}{3}} = -\frac{\sqrt{3}}{2}.$$

Problem 4 (20 pts) Using implicit differentiation find an equation of the tangent line to the curve

$$y^2(y^2-4) = x^2(x^2-5)$$

at the point $(0,-2)$.

$$y^4 - 4y^2 = x^4 - 5x^2$$

$$4y^3y' - 8yy' = 4x^3 - 10x$$

$$y'(4y^3 - 8y) = 4x^3 - 10x$$

$$y' = \frac{4x^3 - 10x}{4y^3 - 8y}$$

$$y'\Big|_{(0,2)} = 0.$$

equation of the tangent line at $(0,2)$; $y+2 = 0(x-0)$

$y = -2$.

BONUS QUESTION (5 pts) If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

$$f(f^{-1}(x)) = x \Rightarrow f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1$$

So $\frac{d}{dx} f^{-1}(x) = \dfrac{1}{f'(f^{-1}(x))}$

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \frac{1}{2}.$$

then

Since $f(0) = 4 \Rightarrow f^{-1}(4) = 0$

and $f'(x) = 1 + e^x$