

Conditional Law and Occupation Times of Two-Sided Sticky Brownian Motion^{1,2}

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Abstract

Sticky Brownian motion on the real line can be obtained as a weak solution of a system of stochastic differential equations. We find the conditional distribution of the process given the driving Brownian motion, both at an independent exponential time and at a fixed time $t > 0$. As a classical problem, we find the distribution of the occupation times of a half-line, and at 0, which is the sticky point for the process.

Keywords: sticky Brownian motion, resolvent density, Arcsine law, weak solution

1. Introduction

An important special case of diffusions with boundary conditions is sticky Brownian motion, defined by means of an infinitesimal generator on the half-line with a sticky boundary condition at 0 [5]. Ito and McKean [9] provided a construction of the process as a time-changed reflected Brownian motion. Ikeda and Watanabe [8] studied such diffusions with boundary conditions as solutions to stochastic differential equations. A sticky Brownian motion on the real line, where 0 is not at the boundary, is characterized as a solution of a system of stochastic differential equations. Recently, both versions have attracted further research interest motivated by applications [1, 2, 3, 4, 6, 11, 13]. In this paper, we derive the conditional law of the real-valued sticky Brownian motion at a snapshot in time, given the driving Brownian motion,

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as well as the occupation times at each half-line and 0 based on the results of [4, 13].

The one-sided, or reflecting, sticky Brownian motion is determined as a weak solution of the stochastic differential equation

$$dX_t = 1_{\{X_t > 0\}} dB_t + \theta 1_{\{X_t = 0\}} dt \quad (1)$$

where $\theta > 0$ is the stickiness parameter, with $X_0 \geq 0$ [13, Eq.(1.1)], [4, Eq.(3.3) and Thm.5]. In [13], the conditional distribution of X_t given \mathcal{F}_t , where $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by the driving Brownian motion B , has been found as

$$\mathbb{P}\{X_t < x | \mathcal{F}_t\} = e^{-2\theta(B_t + L_t - x)} \quad (2)$$

where $L_t = \sup_{s \leq t} \{-B_s \wedge 0\}$, $x \in [0, B_t + L_t]$, and initial condition $(X_0, B_0) = (0, 0)$. This remarkable result is verified from the point of view of stochastic flows in [6]. On the other hand, the two-sided, or global, version of sticky Brownian motion is described by a system of stochastic differential equations given by

$$dX_t = 1_{\{X_t \neq 0\}} dB_t \quad (3)$$

$$1_{\{X_t = 0\}} dt = \frac{1}{\theta} dl_t^0(X) \quad (4)$$

where $l_t^0(X)$ denotes the local time of X at 0. We will refer to the two-sided version as *the sticky Brownian motion* in this paper. It has been recently shown in [3] and [4] that there exists no strong solution to equations (3) and (4), but the system admits a weak solution uniquely determining the probability law of X . Moreover, if (X_t, B_t) solves (3)-(4), then $(|X_t|, \hat{B}_t)$ with $\hat{B}_t := \int_0^t \text{sgn}(X_s) dB_s$ solves (1) as shown in the proof of [4, Thm.5, pg.18-19]. As a result, $|X_t|$ is a one-sided sticky Brownian motion with the same parameter θ . Based on this association, the positive and negative parts of X behave similar to the one-sided sticky Brownian and play a crucial role in our results.

Our first result is the characterization of the joint distribution of (X_t, B_t) through their resolvent density as given in Theorem 1. This is then used in Theorem 2 to derive the conditional law of X_T given B_T at an independent exponential time T with parameter λ as

$$\mathbb{P}\{X_T < x | B_T = b\} = \left(\frac{\text{sgn}(b-x)}{4} + \text{sgn}(x) \frac{\gamma}{4\delta} \right) e^{-\delta|b-x| + \gamma(|b| - \text{sgn}(x)x)} + \frac{i_x(b)}{2} \quad (5)$$

where $\gamma^2 = 2\lambda$, $\delta^2 = 2\lambda + 2\theta\gamma$, and $i_x(b) = 1_{\{b < x\}}$. We invert the Laplace transforms involved in (5) to obtain $\mathbb{P}\{X_t < x | B_t = b\}$ in Corollary 1. Our approach is parallel to [13] by relying on Knight's theorem. However, we condition on a single random variable B_t rather than the filtration \mathcal{F}_t .

As another result, we derive the distributions of the occupation times of the sticky Brownian motion in Theorem 3. Our exposure is along the lines of Brownian motion, for which the celebrated Arcsine law holds [12]. The counterparts for the one-sided case are derived based on the results of [13] for finding the distribution of the occupation time at 0. The occupation time at 0 during $[0, 1]$, A_1^0 , is nonzero as the process is sticky at 0, and satisfies

$$\mathbb{P}\{A_1^0 > t\} = \text{Erfc}\left(\frac{\theta t}{\sqrt{2(1-t)}}\right)$$

for $0 < t < 1$. The distribution of the occupation time of the positive axis, A_1^+ , is found as

$$\mathbb{P}\{A_1^+ > t\} = 1 - \mathbb{E} \left[\text{Erfc} \left(\frac{-Z^2 + Z\sqrt{Z^2 + \theta^2(1-t)}}{\theta\sqrt{2t}} \right) \right]$$

which approximates Arcsine law when $\theta \rightarrow \infty$ as expected.

The paper is organized as follows. Section 2 gives the resolvents involved and the conditional law for the sticky Brownian motion. The occupation distributions are found in Section 3.

2. Joint Resolvent and Conditional Distribution

In this section, we first obtain some fundamental identities using Skorohod representations, which will also be useful to derive the occupation time distributions in the next section. Then, we recover the resolvent of the sticky Brownian motion X found as a weak solution of (3)-(4), with $X_0 = 0$. As for the main results, we derive the joint resolvent of X and the driving Brownian motion B , and the conditional law of X_t given B_t for each $t > 0$.

The occupation times of X_t and their right continuous inverses are defined as follows

$$A_t^0 = \int_0^t 1_{\{X_s=0\}} ds \quad \alpha_t^0 = \inf\{u : A_u^0 > t\} \quad (6)$$

$$A_t^+ = \int_0^t 1_{\{X_s>0\}} ds \quad \alpha_t^+ = \inf\{u : A_u^+ > t\} \quad (7)$$

$$A_t^- = \int_0^t 1_{\{X_s<0\}} ds \quad \alpha_t^- = \inf\{u : A_u^- > t\} \quad (8)$$

Note that A_t^0 , A_t^- and A_t^+ are right continuous and non-decreasing processes justifying the definition of the corresponding right inverses.

Let $\mathbb{P}^{(x,a)}$ and $\mathbb{E}^{(x,a)}$ denote the probability law and the expectation, respectively, given the initial condition $(X_0, B_0) = (x, a)$, $x, a \in \mathbb{R}$. We simply use \mathbb{P} and \mathbb{E} when $(X_0, B_0) = (0, 0)$. Let X^+ and X^- denote the positive and negative parts of the real-valued random variable X , respectively. The identities shown in the following lemma will be used in the sequel.

Lemma 1. *There exist independent Brownian motions W_t^+ and W_t^- such that*

$$X_{\alpha_t^+}^+ = W_t^+ + \frac{\theta}{2} A_{\alpha_t^+}^0, \quad X_{\alpha_t^-}^- = W_t^- + \frac{\theta}{2} A_{\alpha_t^-}^0$$

and

$$A_{\alpha_t^+}^0 = \frac{2}{\theta} \sup_{s \leq t} (-W_s^+) , \quad A_{\alpha_t^-}^0 = \frac{2}{\theta} \sup_{s \leq t} (-W_s^-) .$$

Proof. Tanaka formula [12, Thm.VI.1.2] for X satisfying equations (3)-(4) yields

$$\begin{aligned} X_t^+ &= \int_0^t 1_{\{X_s > 0\}} dB_s + \frac{\theta}{2} A_t^0 \\ X_t^- &= - \int_0^t 1_{\{X_s < 0\}} dB_s + \frac{\theta}{2} A_t^0 . \end{aligned} \quad (9)$$

As a result of a time change $t \rightarrow \alpha_t^+$, we get

$$X_{\alpha_t^+}^+ = \int_0^{\alpha_t^+} 1_{\{X_s > 0\}} dB_s + \frac{\theta}{2} A_{\alpha_t^+}^0$$

Define

$$W_t^+ := \int_0^{\alpha_t^+} 1_{\{X_s > 0\}} dB_s \quad \text{and} \quad W_t^- := - \int_0^{\alpha_t^-} 1_{\{X_s < 0\}} dB_s .$$

Then, $\langle W_t^+, W_t^+ \rangle = \langle W_t^-, W_t^- \rangle = t$ by definitions (7) and (8). This implies that W^+ and W^- are Brownian motions by Levy's characterization theorem. Note that $\langle W_{A_t^+}^+, W_{A_t^+}^+ \rangle = A_t^+$, $\langle W_{A_t^-}^-, W_{A_t^-}^- \rangle = A_t^-$ and $\langle W_{A_t^-}^-, W_{A_t^+}^+ \rangle = 0$. Therefore, by Knight's theorem ([12, Thm.V.1.9]) W_t^+ and W_t^- are independent Brownian motions as $A_\infty^+ = A_\infty^- = \infty$. Furthermore, $\frac{\theta}{2} A_{\alpha_t^+}^0$ is a continuous and strictly increasing function of α_t^+ and constant on the set $\{t : X_{\alpha_t^+}^+ > 0\}$. Thus, the result follows by Skorohod's Lemma [12, Lem.VI.2.1]. \square

The following lemma gives the resolvent kernel of the sticky Brownian motion X starting from zero. Although it is available in [7, pg.23], where α^0 and analogous definition for the right continuous inverse of the time spent in $\mathbb{R} \setminus \{0\}$ are considered, we provide a proof referring to the times spent by X in the three regions introduced above inspired by the proof of [13, Prop.13] for one-sided sticky. Our approach emphasizes the two-sided nature of X , also used in Section 4.

Lemma 2. *The resolvent kernel $p_\lambda(0, dy)$ of the sticky Brownian motion X starting from 0 is given as*

$$p_\lambda(0, dy) = \frac{\theta}{\theta\gamma + \lambda} e^{-\gamma|y|} dy + \frac{1}{\theta\gamma + \lambda} \delta_0(dy)$$

where $\gamma^2 = 2\lambda$.

Proof. Take exponentially distributed random times T_1, T_2 and T_3 with rate λ independently from (X, W) , and define

$$T = \alpha_{T_1}^0 \wedge \alpha_{T_2}^+ \wedge \alpha_{T_3}^- . \quad (10)$$

By Lemma 1, $(\theta/2)A_{\alpha_t^+}^0$ and $(\theta/2)A_{\alpha_t^-}^0$ are the running supremums of independent Brownian motions W^+ and W^- , respectively. Therefore, $A_{\alpha_{T_2}^+}^0$ and $A_{\alpha_{T_3}^-}^0$ are exponentially distributed with rate $\frac{\theta\gamma}{2}$, where $\gamma^2 = 2\lambda$. Note that T is also exponentially distributed with rate λ independent of X , similar to the one-sided case given in [13], essentially due to independence of T_i , $i = 1, \dots, 3$. Now, we have $X_T = 0$ if and only if $T = \alpha_{T_1}^0$, which can be true only if $\alpha_{T_1}^0 < \alpha_{T_2}^+ \wedge \alpha_{T_3}^-$. Since A_t^0 is a right continuous and nondecreasing process, if $\alpha_{T_1}^0 < \alpha_{T_2}^+ \wedge \alpha_{T_3}^-$ then $T_1 < A_{\alpha_{T_2}^+}^0 \wedge A_{\alpha_{T_3}^-}^0$. Thus, we get

$$\begin{aligned} \mathbb{P}\{X_T = 0\} &= \mathbb{P}\{T = \alpha_{T_1}^0\} \\ &= \mathbb{P}(T_1 < A_{\alpha_{T_2}^+}^0 \wedge A_{\alpha_{T_3}^-}^0) \\ &= \frac{\lambda}{\lambda + \theta\gamma} . \end{aligned}$$

For $y \neq 0$, we consider only $y > 0$ since on negative axis the calculations are similar. For $X_T > 0$, we should have $T = \alpha_{T_2}^+$ and this can be possible only if $A_{\alpha_{T_2}^+}^0 < T_1 \wedge A_{\alpha_{T_3}^-}^0$. Then, note that $X_{\alpha_{T_2}^+}^+$ is a reflected Brownian motion and its probability law for $y \neq 0$ is independent from the level of

the running supremum of $-W^+$ (or negative of the running infimum of W^+), which is equal to $A_{\alpha_{T_2}^+}^0$ by Lemma 1. This is due to the fact that the reflected Brownian motion is formed by the excursions of W^+ from its infimum and the excursion measure does not depend on the level of the last infimum (see e.g. [12, Sec.XII.2]). Moreover, $X_{\alpha_{T_2}^+}^+$ is independent from $A_{\alpha_{T_3}^-}^0$ again by Lemma 1. Therefore, $X_{\alpha_{T_2}^+}^+$ is independent from the event $\{A_{\alpha_{T_2}^+}^0 < T_1 \wedge A_{\alpha_{T_3}^-}^0\}$. Then, for $y > 0$, we have

$$\begin{aligned} \mathbb{P}\{X_T \in dy\} &= \mathbb{P}\{X_{\alpha_{T_2}^+}^+ \in dy, A_{\alpha_{T_2}^+}^0 < T_1 \wedge A_{\alpha_{T_3}^-}^0\} \\ &= \mathbb{P}\{X_{\alpha_{T_2}^+}^+ \in dy\} \mathbb{P}\{A_{\alpha_{T_2}^+}^0 < T_1 \wedge A_{\alpha_{T_3}^-}^0\} \\ &= \gamma e^{-\gamma y} \frac{\frac{\theta\gamma}{2}}{\lambda + \theta\gamma} dy = \frac{\theta\lambda}{\lambda + \theta\gamma} e^{-\gamma|y|} dy \end{aligned} \quad (11)$$

where we have used the fact that the distribution of the reflected Brownian motion at T_2 is exponential with parameter $\gamma = \sqrt{2\lambda}$. For $y < 0$, we have the same expression due to symmetry. The result follows as $p_\lambda(0, dy) = (1/\lambda)\mathbb{P}(X_T \in dy)$. \square

We are ready to find the joint resolvent \mathcal{V}_λ of the process (X, B) , defined by

$$\mathcal{V}_\lambda f(x, a) = (1/\lambda)\mathbb{E}^{(x,a)}[f(X_T, B_T)]$$

for bounded measurable functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, where T is an independent exponential time with parameter λ .

Theorem 1. *Let (X_t, B_t) be the solution to stochastic differential equations (3) - (4) with $X_0 = 0$. Then, there exists a Brownian motion W_t^0 such that*

$$X_t + W_{A_t^0}^0 = B_t \quad (12)$$

and the joint resolvent \mathcal{V}_λ of the process (X_t, B_t) is given by

$$\begin{aligned} \mathcal{V}_\lambda f(x, a) &= \frac{1}{\theta} \int_{-\infty}^{\infty} \nu_\lambda(x, a, 0, b) f(0, b) db + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nu_\lambda(x, a, y, b) f(y, b) dy db \\ &\quad + \int_{\{a+y=b+x\}} r_\lambda^-(x, y) f(y, b) dy \end{aligned}$$

where

$$\nu_\lambda(x, a, y, b) = \frac{\theta}{2\delta} e^{-\gamma(|y|+|x|)-\delta|b-y-a+x|}$$

with $\gamma^2 = 2\lambda$, $\delta^2 = 2\lambda + 2\theta\gamma$, and

$$r_\lambda^-(x, y) = \gamma^{-1}(e^{-\gamma|y-x|} - e^{-\gamma|y+x|})$$

which is the resolvent density of Brownian motion killed at 0.

Proof. When $X_0 = 0$, (3) - (4) imply

$$X_t + \int_0^t 1_{\{X_s=0\}} dB_s = B_t.$$

Define W_t^0 as

$$W_t^0 = \int_0^{\alpha_t^0} 1_{\{X_s=0\}} dB_s.$$

Then, Levy's characterization and definition of α_t^0 yield W_t^0 is a Brownian motion and so decomposition (12) is shown.

The resolvent of (X, B) can be derived by similar methods used for the resolvent kernel of X as in Lemma 2. Recall the definition of T in (10) as $T = \alpha_{T_1}^0 \wedge \alpha_{T_2}^+ \wedge \alpha_{T_3}^-$. Now, we have

$$A_T^0 = T_1 \wedge A_{\alpha_{T_2}^+}^0 \wedge A_{\alpha_{T_3}^-}^0 \quad (13)$$

where $A_{\alpha_{T_2}^+}^0$, $A_{\alpha_{T_3}^-}^0$ and T_1 are independent exponential random variables with rates $\frac{\theta\gamma}{2}$, $\frac{\theta\gamma}{2}$ and λ , respectively. The distributions of $A_{\alpha_{T_2}^+}^0$ and $A_{\alpha_{T_3}^-}^0$, together with their independence follow from Lemma 1. Therefore, A_T^0 is exponentially distributed with rate $\lambda + \theta\gamma$. Moreover, $A_{\alpha_{T_2}^+}^0$, $A_{\alpha_{T_3}^-}^0$, and W^0 are all independent because W_t^+ and W_t^- , and W_t^0 are independent Brownian motions again by Knight's theorem. It follows that the W^0 is independent of A_T^0 given in (13), and $W_{A_T^0}^0$ is double exponentially distributed with parameter $\delta := \sqrt{2(\lambda + \theta\gamma)}$, with the probability density function $f(x) = (\delta/2)e^{-\delta|x|}$ for $x \in \mathbb{R}$.

To find the joint resolvent, we first consider a starting point of the form $(X_0, B_0) = (0, a)$. Then, (12) becomes

$$X_t + a + W_{A_t^0}^0 = B_t$$

We will consider the events $X_T = 0$ and $X_T \neq 0$ separately. The first event corresponds to the event $T = \alpha_{T_1}^0$, whereas the second one occurs only if

$T = \alpha_{T_2}^+$ or $\alpha_{T_3}^-$. Since $T = \alpha_{T_1}^0$ is equivalent to $A_T^0 = T_1$, we have

$$\begin{aligned} \mathbb{P}^{(0,a)}\{X_T = 0, B_T \in db\} &= \mathbb{P}^{(0,a)}\{X_T = 0, W_{A_T^0}^0 \in d(b-a)\} \\ &= \mathbb{P}^{(0,a)}\{A_T^0 = T_1, W_{A_T^0}^0 \in d(b-a)\} \\ &= \frac{\lambda}{2\delta} e^{-\delta|b-a|} db \end{aligned}$$

by independence of A_T^0 and W^0 as explained after (13) above, and T_1 being the minimum of three independent exponential random variables. On the other hand, when $X_T \neq 0$, it is implicit that $T = \alpha_{T_2}^+$ or $\alpha_{T_3}^-$. Then, for $y > 0$ ($y < 0$), we have

$$\begin{aligned} \mathbb{P}^{(0,a)}\{X_T \in dy, B_T \in db\} &= \mathbb{P}^{(0,a)}\{X_T \in dy, W_{A_T^0}^0 \in d(b-y-a)\} \\ &= \mathbb{P}^{(0,a)}\{X_T \in dy\} \mathbb{P}^{(0,a)}\{W_{A_T^0}^0 \in d(b-y-a)\} \\ &= \frac{\lambda}{2\delta} e^{-\gamma|y| - \delta|b-y-a|} dy db \end{aligned}$$

where we have used (11), and the independence of X_T and $W_{A_T^0}^0$ when $T = \alpha_{T_2}^+$ ($\alpha_{T_3}^-$). To see the independence, recall that (W_t^+, W_t^-, W_t^0) is a 3-dimensional Brownian motion by Knight's theorem. Therefore, $X_{\alpha_{\alpha^+}^+}^+$, $X_{\alpha_{\alpha^-}^-}^-$ and W^0 are independent processes by Lemma 1. Moreover, $X_{\alpha_{T_2}^+}^+$ is independent from the event $\{\alpha_{\alpha_{T_2}^+}^0 < T_1 \wedge \alpha_{\alpha_{T_3}^-}^0\}$ as explained in the proof of Lemma 1 before (11), and these together characterize the distribution of X_T . It follows that X_T is independent from $W_{A_T^0}^0$, analogous to the one-sided case given in [13, pg.12].

Now, as $\mathcal{V}_\lambda f(x, a) = (1/\lambda)\mathbb{E}^{(x,a)}[f(X_T, B_T)]$, we get

$$\begin{aligned} \mathcal{V}_\lambda f(0, a) &= \int \int \mathbb{P}^{(0,a)}\{X_T \in dy, B_T \in db\} f(y, b) dy db \\ &\quad + \int \mathbb{P}^{(0,a)}\{X_T = 0, B_T \in db\} f(0, b) db \\ &= \frac{1}{\theta} \int \int \nu_\lambda(0, a, y, b) f(y, b) dy db + \int \nu_\lambda(0, a, 0, b) f(0, b) db \end{aligned}$$

where $\nu_\lambda(x, a, y, b) := \frac{\theta}{2\delta} e^{-\gamma(|y|+|x|) - \delta|b-y-a+x|}$ is used for brevity of notation.

To find the resolvent of (X_t, B_t) starting from (x, a) , $x \neq 0$, let us define the time H_0 at which X_t hits zero for the first time, and use the strong Markov property of X_t at H_0 , the hitting time of X to 0. Since X_t is in

lockstep with B_t for $t \in [0, H_0]$, having the same trajectory according to (3) - (4), the law of the process X during $t \in [0, H_0]$ is just that of a Brownian motion started at x and killed at 0. If we define \mathcal{R}_λ^- as the resolvent of Brownian motion killed at 0, with $\psi_\lambda(x) = \mathbb{E}^{(x,a)}[e^{-\lambda H_0}] = e^{-\gamma|x|}$, where $\gamma = \sqrt{2\lambda}$, and $f_{x,a}(y) = f(y, a + y - x)$, we get

$$\mathcal{V}_\lambda f(x, a) = \mathcal{R}_\lambda^- f_{x,a}(x) + \psi_\lambda(x) \mathcal{V}_\lambda f(0, a - x)$$

which follows from the strong Markov property at the hitting time to 0; see e.g. [7, pg.24] for more details in the one-sided case. Denoting the density of \mathcal{R}_λ^- with respect to Lebesgue measure by r_λ^- yields the desired result. \square

Next, we use the joint resolvent of the process (X_t, B_t) derived in Theorem 1 to obtain the conditional distribution at an independent exponential time.

Theorem 2. *Let (X_t, B_t) satisfy (3)-(4) with initial condition $(0, 0)$ and let T be exponentially distributed with rate λ independent from (X, B) . Then, we have*

$$\mathbb{P}\{X_T < x | B_T = b\} = \left(\frac{\text{sgn}(b-x)}{4} + \text{sgn}(x) \frac{\gamma}{4\delta} \right) e^{-\delta|b-x| + \gamma(|b| - \text{sgn}(x)x)} + \frac{i_x(b)}{2}$$

where $\gamma^2 = 2\lambda$, $\delta^2 = 2\lambda + 2\theta\gamma$ and $i_x(b) = 1_{\{b < x\}}$.

Proof. For a bounded Borel function g on \mathbb{R}^2 , the resolvent \mathcal{V}_λ of (X, B) satisfies $\mathcal{V}_\lambda g(0, 0) = \mathbb{E}^{(0,0)} g(X_T, B_T)$. In particular, consider

$$g(y, b) = 1_{\{y \leq x\}} f(b) =: i_x(y) f(b)$$

for a bounded Borel function f on \mathbb{R} . On the other hand, the projection property of conditional expectation implies

$$\mathbb{E}[g(X_T, B_T)] = \mathbb{E}[1_{\{X_T \leq x\}} f(B_T)] = \mathbb{E}[\mathbb{P}\{X_T \leq x | B_T\} f(B_T)].$$

Therefore, if we show that

$$\mathcal{V}_\lambda g(0, 0) = [\mathcal{V}_\lambda(i_x f)](0, 0) = [\mathcal{R}_\lambda(e_x f)](0)$$

for some $e_x : \mathbb{R} \rightarrow \mathbb{R}$, where \mathcal{R}_λ is the resolvent of Brownian motion, we can conclude that $e_x(b) = \mathbb{P}\{X_T < x | B_T = b\}$ with $(X_0, B_0) = (0, 0)$. Now, by

Theorem 1, we get

$$\begin{aligned}
[\mathcal{V}_\lambda(i_x f)](0, 0) &= \frac{1}{\theta} \int_{-\infty}^{\infty} \nu_\lambda(0, 0, 0, b) i_x(0) f(b) db \\
&\quad + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nu_\lambda(0, 0, y, b) i_x(y) f(b) dy db \\
&= i_x(0) \frac{1}{2\delta} \int_{-\infty}^{\infty} e^{-\delta|b|} f(b) db \\
&\quad + \frac{\theta}{2\delta} \int_{-\infty}^{\infty} \int_{-\infty}^x e^{-\gamma|y| - \delta|b-y|} f(b) dy db
\end{aligned}$$

where $\nu_\lambda(x, a, y, b) = \frac{\theta}{2\delta} e^{-\gamma(|y|+|x|) - \delta|b-y-a+x|}$. Assume $x > 0$, then

$$\begin{aligned}
[\mathcal{V}_\lambda(i_x f)](0, 0) &= \frac{1}{2\delta} \int_{-\infty}^{\infty} e^{-\delta|b|} f(b) db + \frac{\theta}{2\delta} \int_{-\infty}^{\infty} f(b) \int_{-\infty}^x e^{-\gamma|y| - \delta|b-y|} dy db \\
&= \frac{1}{2\delta} \int_{-\infty}^{\infty} e^{-\delta|b|} f(b) db + \frac{\theta}{2\delta} \int_{-\infty}^{\infty} f(b) \int_{-\infty}^x e^{-\gamma|y| - \delta|b-y|} dy db \\
&= \frac{1}{2\delta} \int_{-\infty}^{\infty} e^{-\delta|b|} f(b) db \\
&\quad + \frac{\theta}{2\delta} \int_x^{\infty} f(b) \left(\int_0^x e^{-\gamma y - \delta(b-y)} dy + \int_{-\infty}^0 e^{\gamma y - \delta(b-y)} dy \right) db \\
&\quad + \frac{\theta}{2\delta} \int_0^x f(b) \left(\int_b^x e^{-\gamma y - \delta(y-b)} dy + \int_0^b e^{-\gamma y - \delta(b-y)} dy \right. \\
&\quad \left. + \int_{-\infty}^b e^{\gamma y - \delta(b-y)} dy \right) db \\
&\quad + \frac{\theta}{2\delta} \int_{-\infty}^0 f(b) \left(\int_0^x e^{-\gamma y - \delta(y-b)} dy + \int_b^0 e^{\gamma y - \delta(y-b)} dy \right. \\
&\quad \left. + \int_{-\infty}^b e^{\gamma y - \delta(b-y)} dy \right) db \\
&=: \int_{-\infty}^{\infty} \frac{1}{\gamma} e^{-\gamma|b|} e_x^{(1)}(b) f(b) db = [\mathcal{R}_\lambda(e_x^{(1)} f)](0)
\end{aligned}$$

where \mathcal{R}_λ is the resolvent of Brownian motion and explicitly

$$e_x^{(1)}(b) = e^{-\delta|b-x| + \gamma(|b|-x)} \left(\frac{\operatorname{sgn}(b-x)}{4} + \frac{\gamma}{4\delta} \right) + \frac{i_x(b)}{2}. \quad (14)$$

For the case $x \leq 0$, we have

$$\begin{aligned}
[V_\lambda(i_x f)](0, 0) &= \frac{\theta}{2\delta} \int_{-\infty}^{\infty} f(b) \int_{-\infty}^x e^{-\gamma|y| - \delta|b-y|} dy db \\
&= \frac{\theta}{2\delta} \int_x^{\infty} f(b) \int_{-\infty}^x e^{\gamma y - \delta(b-y)} dy db \\
&\quad + \frac{\theta}{2\delta} \int_{-\infty}^x f(b) \left(\int_b^x e^{\gamma y - \delta(y-b)} dy + \int_{-\infty}^b e^{\gamma y - \delta(b-y)} dy \right) db \\
&= \int_{-\infty}^{\infty} \frac{1}{\gamma} e^{-\gamma|b|} e_x^{(2)}(b) f(b) db \\
&= [\mathcal{R}_\lambda(e_x^{(2)} f)](0)
\end{aligned}$$

where

$$e_x^{(2)}(b) = e^{-\delta|b-x| + \gamma(|b|+x)} \left(\frac{\operatorname{sgn}(b-x)}{4} - \frac{\gamma}{4\delta} \right) + \frac{i_x(b)}{2}.$$

Letting $e_x(b) = 1_{\{x>0\}} e_x^{(1)}(b) + 1_{\{x \leq 0\}} e_x^{(2)}(b)$, we obtain

$$\mathbb{E}[1_{\{X_T < x\}} f(B_T)] = \mathbb{E}[e_x(B_T) f(B_T)]$$

which completes the proof. \square

Theorem 2 allows us to calculate the conditional distribution of X_t given B_t , for $t > 0$, next.

Corollary 1. *The conditional distribution of sticky Brownian motion X_t , given the driving Brownian motion B_t , both starting from zero, is given by*

$$\begin{aligned}
\mathbb{P}\{X_t \leq x | B_t = b\} &= \sqrt{2\pi t} e^{\frac{b^2}{2t}} \left[\frac{\operatorname{sgn}(b-x)}{4} f_1(x, b, t) * f_2(x, b, t) \right. \\
&\quad \left. + \frac{\operatorname{sgn}(x)}{4} f_3(x, b, t) * f_4(x, b, t) + \frac{i_x(b)}{2} f_2(x, b, t) \right]
\end{aligned}$$

where $*$ is the convolution operator, and

$$\begin{aligned}
f_1(x, b, t) &= \frac{|b-x|}{\sqrt{2\pi t^3}} e^{-\frac{2|b-x|^2}{4t} - \theta|b-x|} \\
&\quad + \frac{\theta}{2\sqrt{2\pi t^3}} \int_{\sqrt{2}|b-x|}^{\infty} \xi e^{-\frac{\xi^2}{4t} - \frac{\theta\xi}{\sqrt{2}}} I_1\left(\theta\sqrt{\frac{\xi}{2} - |b-x|^2}\right) d\xi \\
f_2(x, b, t) &= \frac{1}{\sqrt{\pi t}} e^{-\frac{|x|^2}{2t}} \\
f_3(x, b, t) &= \frac{1}{2\sqrt{2\pi t^3}} \int_0^{\infty} \xi e^{-\frac{\xi^2}{4t} - \frac{\theta\xi}{\sqrt{2}}} d\xi \\
f_4(x, b, t) &= \frac{|x|}{\sqrt{2\pi t^3}} e^{-\frac{\sqrt{2}|x|^2}{4t}}
\end{aligned}$$

Proof. Under the initial condition $(X_0, B_0) = (0, 0)$, we can write the joint distribution as

$$\begin{aligned}
\mathbb{P}\{X_T \leq x, B_T \in db\} &= \mathbb{P}\{X_T < x | B_T = b\} \mathbb{P}\{B_T \in db\} \\
&= \left[\frac{\gamma}{2} \left(\frac{\operatorname{sgn}(b-x)}{4} + \operatorname{sgn}(x) \frac{\gamma}{4\delta} \right) e^{-\delta|b-x| - \gamma|x|} + \frac{i_x(b)}{2} \frac{\gamma}{2} e^{-\gamma|b|} \right] db \quad (15)
\end{aligned}$$

where we have used the fact that B_T is double exponentially distributed with parameter $\gamma = \sqrt{2\lambda}$ as T is exponential with parameter λ . By conditioning on T , we have

$$\begin{aligned}
\mathbb{P}\{X_T < x, B_T \in db\} &= \int_0^{\infty} \mathbb{P}\{X_T < x, B_T \in db | T = t\} \lambda e^{-\lambda t} dt \\
&= \int_0^{\infty} \mathbb{P}\{X_t < x, B_t \in db\} \lambda e^{-\lambda t} dt
\end{aligned}$$

Thus, we can obtain $\mathbb{P}\{X_t < x, B_t \in db\}$ by Laplace inversion of (15).

Denoting the Laplace transform operator by \mathcal{L}_λ , we get

$$\begin{aligned}
\mathbb{P}\{X_t < x, B_t \in db\} &= \mathcal{L}_\lambda^{-1}\left(\frac{\text{sgn}(b-x)}{4\gamma} \exp\{-\delta|b-x| - \gamma|x|\}\right) db \\
&+ \mathcal{L}_\lambda^{-1}\left(\frac{1}{4\delta} \text{sgn}(x) \exp\{-\delta|b-x| - \gamma|x|\}\right) db \\
&+ \mathcal{L}_\lambda^{-1}\left(\frac{i_x(b)}{2\gamma} e^{-\gamma(|b|)}\right) db \\
&= \frac{\text{sgn}(b-x)}{4} \mathcal{L}_\lambda^{-1}\left(\exp\{-\delta|b-x|\} \frac{1}{\gamma} \exp\{-\gamma|x|\}\right) db \\
&+ \frac{\text{sgn}(x)}{4} \mathcal{L}_\lambda^{-1}\left(\frac{1}{\delta} \exp\{-\delta|b-x|\} \exp\{-\gamma|x|\}\right) db \\
&+ \frac{i_x(b)}{2} \mathcal{L}_\lambda^{-1}\left(\frac{1}{\gamma} e^{-\gamma(|b|)}\right) db
\end{aligned}$$

Let the inverse Laplace transforms $f_i : \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}$ be assigned as

$$\begin{aligned}
f_1(x, b, t) &= \mathcal{L}_\lambda^{-1}\left(\exp\{-\delta|b-x|\}\right)[t], & f_2(x, b, t) &= \mathcal{L}_\lambda^{-1}\left(\frac{1}{\gamma} \exp\{-\gamma|x|\}\right)[t] \\
f_3(x, b, t) &= \mathcal{L}_\lambda^{-1}\left(\frac{1}{\delta} \exp\{-\delta|b-x|\}\right)[t], & f_4(x, b, t) &= \mathcal{L}_\lambda^{-1}\left(\exp\{-\gamma|x|\}\right)[t]
\end{aligned}$$

Then, the inverse Laplace formula in [10, pp. 978-979] yields the result. \square

3. Distribution of Occupation Times

In this section, we find the distributions of occupation times of positive and negative axes by the two-sided sticky Brownian motion. We also find the distribution of the time spent at 0, which we show to be the same as that of the one-sided case. The desired distributions will be calculated via the relation between the two-sided and the one-sided sticky Brownian motions.

We first recover the distribution of the time spent at 0 by the one-sided sticky Brownian motion, which satisfies Equation (1). This result is cited in [13], but not given explicitly. For the sake of completeness, we derive the distribution of both occupation times in the following lemma.

Lemma 3. *Let \bar{X}_t denote the one-sided sticky Brownian motion satisfying (1), and let*

$$\bar{A}_t^0 = \int_0^t 1_{\{\bar{X}_s=0\}} ds \quad \text{and} \quad \bar{A}_t^+ = \int_0^t 1_{\{\bar{X}_s>0\}} ds.$$

Then, we have

$$\begin{aligned} \text{i) } \mathbb{P}\{\bar{A}_s^0 < t\} &= 1 - \text{Erfc}\left(\frac{\theta t}{\sqrt{2(s-t)}}\right) \\ \text{ii) } \mathbb{P}\{\bar{A}_s^+ < t\} &= \text{Erfc}\left(\frac{\theta(s-t)}{\sqrt{2t}}\right) \end{aligned}$$

for $t \leq s$, where $\text{Erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-\xi^2} d\xi$.

Proof. Let $\bar{\alpha}_t^+ = \inf\{s : \bar{A}_s^+ \geq t\}$. It has been shown in [13] that $\bar{A}_{\bar{\alpha}_t^+}^0$ is exponentially distributed with rate $\theta\gamma = \theta\sqrt{2\lambda}$. Formally, we can write

$$\begin{aligned} \theta\sqrt{2\lambda} e^{-\theta\sqrt{2\lambda}x} dx = \mathbb{P}\{\bar{A}_{\bar{\alpha}_t^+}^0 \in dx\} &= \int_0^\infty \mathbb{P}\{\bar{A}_{\bar{\alpha}_t^+}^0 \in dx | T = t\} \mathbb{P}\{T \in dt\} \\ &= \int_0^\infty \mathbb{P}\{\bar{A}_{\bar{\alpha}_t^+}^0 \in dx\} \lambda e^{-\lambda t} dt. \end{aligned}$$

That is, $(1/\lambda)\mathbb{P}\{\bar{A}_{\bar{\alpha}_t^+}^0 \in dx\} = \mathcal{L}_\lambda(f(\cdot, x))$ where $f(t, x) dx = \mathbb{P}\{\bar{A}_{\bar{\alpha}_t^+}^0 \in dx\}$. The formula [10, (10) p.964] yields

$$\mathbb{P}\{\bar{A}_{\bar{\alpha}_t^+}^0 \in dx\} = \frac{\theta\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\theta^2 x^2}{2t}} dx \quad (16)$$

for $x \geq 0$, which is half normal distribution with parameter $\sigma = \sqrt{t}/\theta$. Recall that $t = \bar{A}_t^0 + \bar{A}_t^+$ for each $t > 0$. So, the distribution of $\bar{\alpha}_t^+$ can be calculated as

$$\begin{aligned} \mathbb{P}\{\bar{\alpha}_t^+ < x\} &= \mathbb{P}\{t + \bar{A}_{\bar{\alpha}_t^+}^0 < x\} = \mathbb{P}\{\bar{A}_{\bar{\alpha}_t^+}^0 < x - t\} \\ &= \int_0^{x-t} \frac{\theta\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\theta^2 y^2}{2t}} dy = \int_t^x \frac{\theta\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\theta^2 (y-t)^2}{2t}} dy \end{aligned}$$

in view of (16). Then, the density is given by

$$\mathbb{P}\{\bar{\alpha}_t^+ \in dy\} = \frac{\theta\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\theta^2 (y-t)^2}{2t}} dy \quad \text{for } y \geq t$$

which is also half Normal, but shifted to t . The result i) follows from the definition of $\bar{\alpha}_t^+$ since

$$\mathbb{P}\{\bar{A}_s^+ < t\} = \mathbb{P}\{s < \bar{\alpha}_t^+\} \quad (17)$$

Then, ii) follows from the fact that $\mathbb{P}\{\bar{A}_s^0 < t\} = \mathbb{P}\{s - \bar{A}_s^+ < t\} = 1 - \mathbb{P}\{\bar{A}_s^+ \leq (s - t)\}$. \square

Recall that the occupation times of the sticky Brownian motion have been defined in equations (6) to (8). Clearly, A^+ and A^- have the same distribution, which is given in the following main result of this section. The distribution deviates from the Arcsine law corresponding to Brownian motion [12, Thm.VI.2.7]. Although our method of proof is borrowed from that for Brownian motion, we take into account that A_t^0 is positive with positive probability in the sticky case.

Theorem 3. *Let X be the sticky Brownian motion satisfying (3)-(4) with $X_0 = 0$. For $0 < t < 1$, the occupation times of X satisfy*

$$\begin{aligned} i) \mathbb{P}\{A_1^0 > t\} &= \operatorname{Erfc}\left(\frac{\theta t}{\sqrt{2(1-t)}}\right) \\ ii) \mathbb{P}\{A_1^+ > t\} &= 1 - \mathbb{E}\left[\operatorname{Erfc}\left(\frac{-Z^2 + Z\sqrt{Z^2 + \theta^2(1-t)}}{\theta\sqrt{2t}}\right)\right] \end{aligned}$$

for $0 < t < 1$, where Z is a standard Gaussian random variable.

Proof. i) The relation between the two-sided and one-sided sticky Brownian motion has been given in [4]. Through the specific construction of (X_t, B_t) given in [4, Sec.2] satisfying (3)-(4), the pair $(|X_t|, \hat{B}_t)$ satisfies Equation (1) for the one-sided sticky Brownian motion and equivalently the system

$$dX_t = 1_{\{X_t \neq 0\}} dB_t + \frac{1}{2} 1_{\{X_t > 0\}} dB_t \quad (18)$$

$$1_{\{X_t=0\}} dt = \frac{1}{2\theta} dl_t^0(X) \quad (19)$$

by [4, Thm.5]. Note that local time of two-sided sticky Brownian motion X_t satisfies

$$\begin{aligned} l_t^0(X) &= \mathbb{P}\text{-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I(-\varepsilon \leq X_s \leq \varepsilon) d\langle X, X \rangle_s \\ &= \mathbb{P}\text{-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I(0 \leq |X_s| \leq \varepsilon) d\langle X, X \rangle_s, \end{aligned}$$

and local time of one-sided sticky Brownian motion $|X_t|$ is defined as $l_t^0(|X|) = \mathbb{P}\text{-}\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t I(0 \leq |X_s| \leq \varepsilon) d\langle X_s, X_s \rangle$ which implies $l_t^0(X) = \frac{1}{2} l_t^0(|X|)$. In view of SDE's (1) (equivalently (18)-(19)) and (3)-(4), we obtain

$$A_t^0 = \frac{1}{\theta} l_t^0(X) = \frac{1}{2\theta} l_t^0(|X|) = \bar{A}_t^0,$$

as in [4, Eq.3.8], where \bar{A}_t^0 is as defined in Lemma 3. This implies that A_t^0 and \bar{A}_t^0 have the same distribution that is also given in Lemma 3.

ii) Note that $t = A_t^0 + A_t^- + A_t^+$, which implies

$$\alpha_t^+ = t + A_{\alpha_t^+}^0 + A_{\alpha_t^+}^- . \quad (20)$$

Since $\mathbb{P}\{A_1^+ > t\} = \mathbb{P}\{1 > \alpha_t^+\}$, we will study α_t^+ to get the distribution of A_1^+ . Recall from Lemma 1 that $\frac{\theta}{2}A_{\alpha_t^+}^0$ is the running supremum of the Brownian motion $-W^+$. Hence, we have

$$A_{\alpha_t^+}^0 = \frac{2}{\theta} \sup_{s \leq t} (-W_s^+) =: \frac{2}{\theta} S_t^+ \quad (21)$$

at time t , where S_t^+ has the law of a reflected Brownian motion. Now, for studying the term $A_{\alpha_t^+}^-$ in (20), define the stopping times

$$\tau_t = \inf\{s > 0 : l_s^0(X) > t\} \quad \text{and} \quad T^{W^-}(a) = \inf\{t : W_t^- > a\}$$

for $a > 0$. Note that (9) implies

$$0 = X_{\tau_t}^- = -W_{A_{\tau_t}^-}^- + \frac{\theta}{2}A_{\tau_t}^0 = -W_{A_{\tau_t}^-}^- + \frac{l_{\tau_t}^0(X)}{2} = -W_{A_{\tau_t}^-}^- + \frac{t}{2}$$

in view of (4). This yields $W_{A_{\tau_t}^-}^- = \frac{t}{2}$. Then, we have

$$A_{\tau_t}^- = T^{W^-}(t/2)$$

for each $t \geq 0$, which follows along the same lines of proof for Brownian motion in [12, Thm.VI.2.7]. Replacing t with $l_{\alpha_t^+}^0$ above yields

$$A_{\alpha_t^+}^- = T^{W^-}(l_{\alpha_t^+}^0/2) = T^{W^-}\left(\frac{\theta}{2}A_{\alpha_t^+}^0\right) = T^{W^-}(S_t^+) \quad (22)$$

where the last equality follows from (21).

We are now ready to compute $\mathbb{P}\{A_1^+ > t\}$. Using (20), (21) and (22), we

get

$$\begin{aligned}
\mathbb{P}\{A_1^+ > t\} &= \mathbb{P}\{1 > \alpha_t^+\} = \mathbb{P}\{0 > t - 1 + A_{\alpha_t^+}^0 + A_{\alpha_t^+}^-\} \\
&= \mathbb{P}\{0 > t - 1 + \frac{2}{\theta}S_t^+ + T^{W^-}(S_t^+)\} \\
&= \mathbb{E}[\mathbb{P}\{0 > t - 1 + \frac{2}{\theta}S_t^+ + T^{W^-}(S_t^+) \mid S_t^+\}] \\
&= \mathbb{E}[\mathbb{P}\{0 > t - 1 + \frac{2}{\theta}S_t^+ + \frac{(S_t^+)^2}{(W_1^-)^2} \mid S_t^+\}] \quad (23) \\
&= \mathbb{P}\left\{0 > t - 1 + \frac{2}{\theta}S_t^+ + \frac{(S_t^+)^2}{(W_1^-)^2}\right\}
\end{aligned}$$

where (23) follows by the independence of S^+ and W^- from [12, III.Prop.3.10], which states that

$$T^{W^-}(a) \stackrel{(d)}{=} \frac{a^2}{(W_1^-)^2} \quad a > 0.$$

As S^+ and W^- are independent, we will only need the roots of the polynomial $P(x) = t - 1 + \frac{2}{\theta}x + \frac{x^2}{w^2}$. The positive root is given by $(w^2/\theta)(-1 + \sqrt{1 + \theta^2(1-t)/w^2})$. As a result, we get

$$\begin{aligned}
\mathbb{P}\{A_1^+ > t\} &= \int_{-\infty}^{\infty} \left(\int_0^{(x^2/\theta)(-1+\sqrt{1+\theta^2(1-t)/x^2})} \frac{2e^{-y^2/(2t)}}{\sqrt{2\pi t}} dy \right) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
&= \int_{-\infty}^{\infty} \left(\int_0^{\frac{x^2}{\theta\sqrt{t}}(-1+\sqrt{1+\theta^2(1-t)/x^2})} \frac{2e^{-y^2/2}}{\sqrt{2\pi}} dy \right) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx
\end{aligned}$$

since S^+ has the law of a reflected Brownian motion. \square

As a final remark, we observe from (23) that the distribution of A_1^+ converges to the Arcsine law as $\theta \rightarrow \infty$, by similar computations that would follow as in the proof of [12, Thm.VI.2.7] for Brownian motion. This is consistent with the fact that the two-sided sticky Brownian motion approximates a standard Brownian motion in this case.

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