

# A Long-Range Dependent Workload Model for Packet Data Traffic

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We consider a probabilistic model for workload input into a telecommunication system. It captures the dynamics of packet generation in data traffic as well as accounting for long-range dependence and self-similarity exhibited by real traces. The workload is found by aggregating the number of packets, or their sizes, generated by the arriving sessions. The arrival time, duration, and packet-generation process of a session are all governed by a Poisson random measure. We consider Pareto-distributed session holding times where the packets are generated according to a compound Poisson process. For this particular model, we show that the workload process is long-range dependent and fractional Brownian motion is obtained as a heavy-traffic limit. This yields a fast synthesis algorithm for generating packet data traffic as well as approximating fractional Brownian motion.

*Key words:* infinite source Poisson model; packet data traffic; fractional Brownian motion

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**1. Introduction.** We consider a probabilistic model for workload input into a telecommunication system. It is a tailored version of the general workload models studied in Kurtz (1996) for packet data traffic. More recently, efforts have been directed to such structural models as opposed to black-box models (Willinger et al. 2001). The aim is to relate self-similarity and long-range dependence observed in measured traffic to more elementary properties. In this paper, we study a structural model that captures the dynamics of data packet generation while accounting for the scaling properties of the traffic in high-speed networks.

The model captures the packet dynamics in data traffic as well as accounting for long-range dependence (LRD) and self-similarity exhibited by real traces. The workload is found by aggregating the number of packets, or their sizes, generated by the arriving sessions. The arrival time, duration, and packet-generation process of a session are all governed by a Poisson random measure. The resulting packet traffic has stationary increments as observed in high-speed data networks for time periods as long as an hour. We consider Pareto-distributed session holding times where the packets are generated according to a compound Poisson process. Hence, packets of fixed size—as in ATM networks, or variable size—as in other packet-switched networks can be incorporated. The preliminary version of the present paper, where packet generation is modeled by a Poisson process, has been presented in Çağlar (2001). Empirical evidence from TCP traffic supports the model of a compound Poisson process for packet generation. On the basis of extensive traffic measurements, Cao et al. (2001) report that packet arrivals tend to a Poisson process, and packet sizes become independent as the rate of new TCP connections increases.

Heavy traffic limits exist for the general workload input models of this type, as shown in Kurtz (1996). In particular, our workload process converges to a fractional Brownian motion (FBM) as the number of session arrivals and the number of packet generations grow, while the packet sizes and the lowest possible holding times shrink. However, for most time scales, the correlation structure of the input process is as in FBM even before taking the limit.

The present work is related to the traffic construction from finitely many sources of Taquq et al. (1997), called the on/off model. Because there is no restriction on the number of arrivals in the Poisson case, our model is sometimes called an infinite-source Poisson model.

Fractional Brownian motion is a good representation for data traffic aggregated from many sources (Norros 1995). Both the on/off model and our infinite source model approximate FBM under heavy traffic while capturing the data-generation dynamics. A generalization of the on/off model is the superposition of renewal reward processes (Taqqu and Levy 1986). When the inter-renewal times have infinite variance and the rewards have finite variance, FBM can be obtained as a limit. If the rewards also have infinite variance, then the limit is a stable self-similar process as opposed to an FBM (Levy and Taqqu 2000). According to the scaling chosen, the limit process is either a Lévy motion with stationary and independent increments, or a long-range dependent process characterized through its characteristic function (Pipiras and Taqqu 2000).

Other infinite-source Poisson models include a deterministic traffic-generation process over sessions, in particular, fluid injection to the system. Konstantopoulos and Lin (1998) study deterministic data traffic generation instead of the compound Poisson process of the present study and show that a scaling that yields a Lévy motion in the limit exists. Resnick and van den Berg (2000) prove the same convergence in the Skorohod's  $M_1$  topology. More recently, Mikosch et al. (2002) have investigated the conditions for an FBM or Lévy motion limit in the case of both on/off and infinite-source models. For unification and comparison purposes, the data transmission to the system is assumed to be at a constant, more simply at a unit, rate. They show that if session arrival rates are modest relative to session holding time distribution tails, then stable Lévy motion is a good approximation. Otherwise, if session arrival rates are large, then FBM is the appropriate approximation. These conditions are valid for both on/off and infinite-source models.

Functional central limit theorems for Poisson shot-noise-type processes as in this study have been proved previously in Klüppelberg and Mikosch (1995), where the workload corresponds to cumulative payout of insurance claims. Recently, Klüppelberg et al. (2003) studied the limiting behavior of Poisson shot noise when the limits are infinite-variance stable processes. As applications, they specify several noise processes, corresponding to our packet-generation process together with session duration, and discuss particular limits including Lévy motion. Results in both the Gaussian and the stable case are quite general in the sense that the limit is an unfamiliar self-similar process to be made precise by the choice of the noise. Our choice of the packet-generation process and the accompanying session duration is a special case, but our results do not follow immediately from the mentioned work because one needs to demonstrate the sufficient conditions explicitly.

Empirical studies of data traffic show a multifractal behavior at small time scales, while indicating the presence of long-range dependence at large time scales represented by a single parameter. Maulik and Resnick (2001) model the packet-generation process, or the transmission schedule, and the size of the files to be transmitted such that both small and large time scaling behaviors are captured. Modeling file size could be more natural than the length of transmission, or session duration, as the latter is determined by the file size. A related work is Nuzman et al. (2002), in which a bi-Pareto distribution is fitted to session durations in view of empirical evidence, and hence can account for both small and large time scales. Our present study originates from the ideas in Chandramouli (1997) for TCP/IP traffic. Another related model is the Markov-modulated Poisson process for construction of traffic at the packet level (see, e.g., Heffes and Lucantoni 1986).

The implementation of our model as a traffic generator is also studied. We find that this model lends itself to an accurate and fast synthesis algorithm similar to the micropulses approach considered in Çağlar (2000). The algorithm is  $O(t)$  for the generation of traffic for  $t$  time units with a steady memory requirement throughout. Hence, it is comparable to a wavelet synthesis of FBM, but the approach has superior qualities in terms of representing the real traffic behavior.

We describe the workload input model in §2. In §3, we find the first- and second-order properties of the workload increment process and show that it is long-range dependent. In

§4, we provide the proof of the heavy-traffic limit. Finally, we study the use of this model as a synthesis algorithm for traffic in §5.

**2. Packet traffic model.** General workload models of Kurtz (1996) represent data traffic from a potentially large number of sources. The idea is similar to an  $M/G/\infty$  queue, but with a better abstraction and approximation of real traffic. At the highest level, one can consider session arrivals originating from several sources. Then, each source remains active for a duration that has a heavy-tailed distribution and leaves the system as in an  $M/G/\infty$  queue. When the session is active, an associated cumulative input process generates data which are then aggregated to form the workload input. Here, we specify the input process to be compound Poisson in order to represent packet data traffic.

Let  $(\Omega, \mathcal{H}, \mathbb{P})$  be a probability space. Let  $E = D(\mathbb{R} \rightarrow \mathbb{R}_+)$  be the space of right continuous functions on  $\mathbb{R}$  taking values in  $\mathbb{R}_+$  with left limits where  $\mathbb{R}_+ = [0, \infty)$ . Let  $N$  be a Poisson random measure (Prm) on  $(\mathbb{R}_+ \times [b, \infty) \times E, \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{B}_E)$ ,  $0 < b < 1$ , with mean measure

$$(1) \quad \eta(ds, dr, du) = \lambda db^\delta r^{-\delta-1} ds dr \mu(du),$$

where  $1 < \delta < 2$  and  $\mu$  is the distribution of a compound Poisson process  $U$  in  $\mathbb{R}_+$  given by

$$(2) \quad U(t) = \sum_{i=1}^{M(t)} B_i, \quad t \in \mathbb{R}_+$$

for positive-valued i.i.d. random variables  $B_1, B_2, \dots$  and a Poisson process  $M$  with rate  $\alpha$ . Each atom  $(S_i, R_i, U_i)$  represents a building block of the input traffic; the arrival time  $S_i$  of a session, its duration, or holding time  $R_i$  and data-generation process  $U_i$ . Then, the total workload input to the system in  $[0, t]$  is given by

$$Y(t) = \int_0^t \int_b^\infty \int_E N(ds, dr, du) u(r \wedge (t-s)) \equiv \sum_{S_i \leq t} U_i(R_i \wedge (t-S_i)).$$

The sessions arrive in a Poisson fashion and stay alive for a period with an infinite-variance Pareto distribution. Over an active session, the packets are generated according to a compound Poisson process. Then, the work  $Y$  denotes the cumulative amount of traffic input to the link under consideration. When the packets have a fixed size, as in ATM networks, one can choose  $U$  to be a Poisson process by setting  $B = 1$  and have  $Y$  count the number of packets. Otherwise, we assume that the packet sizes are independent and identically distributed.

In this paper, we assume that the link in consideration has been receiving traffic forever. That is, we let  $N$  to be defined on  $(-\infty, \infty) \times [b, \infty) \times E$ . Then,

$$Y(t) = \int_{-\infty}^0 \int_b^\infty \int_E N(ds, dr, du) [u(r \wedge (t-s)) - u(r \wedge (-s))] \\ + \int_0^t \int_b^\infty \int_E N(ds, dr, du) u(r \wedge (t-s)).$$

In this case, the traffic has stationary increments because the distribution of  $\{Y(t+s) - Y(t) : s \in \mathbb{R}\}$  does not depend on  $t$  for each  $t \in \mathbb{R}$ .

**3. Means, variances, and characteristic function.** In this section, we compute the first- and second-order properties of the process  $Y$ . We consider the *workload increment*

process over unit intervals in  $\mathbb{R}_+$  defined by  $X_k = Y(k) - Y(k-1)$ ,  $k = 1, 2, \dots$ . That is,

$$X_k = \int_{-\infty}^{k-1} \int_b^{\infty} \int_E N(ds, dr, du)[u(r \wedge (k-s)) - u(r \wedge (k-1-s))] \\ + \int_{k-1}^k \int_b^{\infty} \int_E N(ds, dr, du)u(r \wedge (k-s)).$$

Defining  $u(s) = 0$  for  $s < 0$ , we can simplify the notation above. Then, the expected number of packets in  $[k-1, k]$ ,  $k = 1, 2, \dots$ , is given by

$$\mathbb{E}X_k = \mathbb{E} \int_{-\infty}^k \int_b^{\infty} \int_E N(ds, dr, du)[u(r \wedge (k-s)) - u(r \wedge (k-1-s))] \\ = \lambda \int_{-\infty}^k ds \int_b^{\infty} dr \delta b^\delta r^{-\delta-1} \int_E \mu(du)[u(r \wedge (k-s)) - u(r \wedge (k-1-s))] \\ = \lambda \int_{-\infty}^1 ds \int_b^{\infty} dr \delta b^\delta r^{-\delta-1} \int_E \mu(du)[u(r \wedge (1-s)) - u(r \wedge (-s))],$$

where we made a change of variable  $s - k + 1$  to  $s$ . Now consider  $\mu$  to be the distribution of a compound Poisson process as in (2). The integrals with respect to  $\mu$  above are expectations of the increments of this process given by  $\mathbb{E}U(t) = \alpha t \mathbb{E}B$  over  $[0, t]$ . Recalling  $u(s) = 0$  for  $s < 0$ , we get

$$(3) \quad \mathbb{E}X_k = \lambda \delta b^\delta \alpha \mathbb{E}B \left\{ \int_{-\infty}^{-b} ds \int_{-s}^{1-s} dr r^{-\delta-1} (r+s) + \int_{-\infty}^0 ds \int_{1-s}^{\infty} dr r^{-\delta-1} \right. \\ \left. + \int_0^{1-b} ds \int_b^{1-s} dr r^{-\delta} + \int_0^{1-b} ds \int_{1-s}^{\infty} dr r^{-\delta-1} (1-s) \right. \\ \left. + \int_{1-b}^1 ds \int_b^{\infty} dr r^{-\delta-1} (1-s) + \int_{-b}^0 ds \int_b^{1-s} dr r^{-\delta-1} (r+s) \right\} \\ = \lambda \alpha \frac{\delta b}{\delta - 1} \mathbb{E}B,$$

which does not depend on  $k$  because the increments are stationary. Note that the expected increment of traffic per time unit is the product of the arrival rate of sessions, the arrival rate of packets over a session, the mean duration of a session, and the mean packet size.

In the variance and covariance calculations, the following formula will be useful (Kallenberg 1983). If  $N$  is a Prm on  $D$  with mean measure  $\nu$ , then

$$(4) \quad \mathbb{E} \int_D \tilde{N}(dx) \int_D \tilde{N}(dx') f(x)g(x) = \int_D \nu(dx) f(x)g(x'),$$

where  $\tilde{N} = N - \nu$ .

The variance of the incremental workload in each time unit is given by

$$(5) \quad \text{Var}(X_k) = \mathbb{E} \left\{ \int_{-\infty}^k \int_b^{\infty} \int_E \tilde{N}(ds, dr, du)[u(r \wedge (k-s)) - u(r \wedge (k-1-s))] \right\}^2 \\ = \lambda \delta b^\delta \int_{-\infty}^k ds \int_b^{\infty} dr r^{-\delta-1} \int_E \mu(du)[u(r \wedge (k-s)) - u(r \wedge (k-1-s))]^2$$

in view of (4). Taking  $\mu$  to be the distribution of a compound Poisson process as in (2), we get

$$(6) \quad \text{Var}(X_k) = \frac{2\lambda\alpha^2 b^\delta (\mathbb{E}B)^2}{(\delta-1)(2-\delta)(3-\delta)} + \lambda\alpha^2 b^2 \delta (\mathbb{E}B)^2 \left( \frac{b}{3(3-\delta)} - \frac{1}{2-\delta} \right) + \lambda\alpha \frac{\delta b}{\delta-1} \mathbb{E}B^2.$$

The details of this calculation, being similar to the expectation calculation in (3), are omitted. Here, the second moment of the increment of a compound Poisson process as needed in (5) is found through its expectation and variance given, respectively, by  $\mathbb{E}U(t) = \alpha t\mathbb{E}B$  and  $\text{Var}(U(t)) = \alpha t\mathbb{E}B^2$ . We assume the packet size  $B$  has a finite second moment.

Let  $r$  denote the autocovariance function of  $X$ ; that is,

$$r(j) = \text{Cov}(X_k, X_{k+j}) \quad j = 0, 1, 2, \dots, k = 1, 2, \dots$$

We calculate the autocovariance function  $r$  of  $X$  for larger time lags, namely  $j \geq 2$ . The computation and the result are relatively simple in this case because the cutoff value  $b$  does not appear compared with  $j < 2$ . For convenience of this computation, we have imposed the restriction  $b < 1$  in the definition of  $N$ . Besides, the large time lags are relevant for long-range dependence. Then, for  $j \geq 2$

$$\begin{aligned}
 (7) \quad r(j) &= \mathbb{E} \int_{-\infty}^1 \int_b^{\infty} \int_E \tilde{N}(ds, du, dr) [u(r \wedge (1-s)) - u(r \wedge (-s))] \\
 &\quad \cdot \int_{-\infty}^{j+1} \int_b^{\infty} \int_E \tilde{N}(ds, du, dr) [u(r \wedge (j+1-s)) - u(r \wedge (j-s))] \\
 &= \int_{-\infty}^1 \int_b^{\infty} \int_E \eta(ds, dr, du) [u(r \wedge (1-s)) - u(r \wedge (-s))] [u(r \wedge (j+1-s)) \\
 &\quad - u(r \wedge (j-s))] \\
 &= \lambda \delta b^\delta \left\{ \int_{-\infty}^0 ds \int_{j-s}^{j+1-s} dr r^{-\delta-1} \int \mu(du) [u(1-s) - u(-s)] [u(r) - u(j-s)] \right. \\
 &\quad + \int_{-\infty}^0 ds \int_{j+1-s}^{\infty} dr r^{-\delta-1} \int \mu(du) [u(1-s) - u(-s)] \\
 &\quad \quad \cdot [u(j+1-s) - u(j-s)] \\
 &\quad + \int_0^1 ds \int_{j-s}^{j+1-s} dr r^{-\delta-1} \int \mu(du) u(1-s) [u(r) - u(j-s)] \\
 &\quad \left. + \int_0^1 ds \int_{j+1-s}^{\infty} dr r^{-\delta-1} \int \mu(du) u(1-s) [u(j+1-s) - u(j-s)] \right\},
 \end{aligned}$$

where we used (4) and the independence of a Prm on nonoverlapping regions.

Recall that fractional Brownian motion (FBM) is a mean zero Gaussian process  $Z$  on  $\mathbb{R}$  with  $Z(0) = 0$  and covariance

$$\text{Cov}(Z(t_1), Z(t_2)) = \frac{c}{2}(t_1^{2H} + t_2^{2H} - |t_2 - t_1|^{2H})$$

for some  $c > 0$  and Hurst parameter  $0 < H < 1$  (Samorodnitsky and Taqqu 1994). The increment process, defined as  $\{Z(k+1) - Z(k) : k = \dots, -1, 0, 1, \dots\}$  and called fractional Gaussian noise (FGN), is a stationary Gaussian process. Its covariance at time lag  $j$ ,  $j \in \mathbf{Z}$  is given by

$$(8) \quad \text{Cov}(Z(k+1) - Z(k), Z(k+j+1) - Z(k+j)) = \frac{c}{2}(|j+1|^{2H} + |j-1|^{2H} - 2j^{2H})$$

for all  $k \in \mathbf{Z}$ . In the following proposition we show that the autocovariance of  $X$  is not far from that of an FGN.

**PROPOSITION 1.** *Let  $\mu$  be the distribution of a compound Poisson process with arrival rate  $\alpha$  and square integrable random variable  $B$ . Then, the autocovariance function of the increment process  $X$  coincides with the autocovariance function of an FGN for lags  $j \geq 2$ , given by*

$$r(j) = \frac{\lambda \alpha^2 b^\delta (\mathbb{E}B)^2}{(\delta - 1)(2 - \delta)(3 - \delta)} [(j+1)^{3-\delta} + (j-1)^{3-\delta} - 2j^{3-\delta}].$$

PROOF. From (7), we get

$$(9) \quad r(j) = \lambda \delta b^\delta \alpha^2 (\mathbb{E}B)^2 \left\{ \int_{-\infty}^0 ds \int_{j-s}^{j+1-s} dr r^{-\delta-1} (r-j+s) + \int_0^1 ds \int_{j+1-s}^{\infty} dr r^{-\delta-1} (1-s) \right. \\ \left. + \int_{-\infty}^0 ds \int_{j+1-s}^{\infty} dr r^{-\delta-1} + \int_0^1 ds \int_{j-s}^{j+1-s} dr r^{-\delta-1} (1-s)(r-j+s) \right\} \\ = \frac{\lambda \alpha^2 b^\delta (\mathbb{E}B)^2}{(\delta-1)(2-\delta)(3-\delta)} [(j+1)^{3-\delta} + (j-1)^{3-\delta} - 2j^{3-\delta}],$$

which is the autocovariance function of a fractional Gaussian noise as given in (8) with  $H = (3 - \delta)/2$ , for  $j \geq 2$ .  $\square$

Due to the form of  $r$  in (9), the process  $X$  is long-range dependent according to Beran (1994, Definition 2.1). To see this, apply the binomial series to relevant terms in  $r$  for large  $j$  to get

$$r(j) = \frac{\lambda \alpha^2 b^\delta (\mathbb{E}B)^2}{\delta - 1} j^{1-\delta} + O(j^{-\delta}).$$

That is,  $\lim_{j \rightarrow \infty} \rho(j)/(c_\rho j^{1-\delta}) = 1$  where  $\rho$  is the autocorrelation function of  $X$  and  $c_\rho$  is a constant determined by those in Equations (6) and (9). Also note that the variance of  $X_1$ , given in (6), is equal to  $r$  in (9) evaluated at 0 plus some extra terms due to cutoff  $b$ . Because  $1 < \delta < 2$ , the covariance  $r(j)$  tends to 0 as  $j \rightarrow \infty$ . Still, this convergence is so slow that  $\sum_{j=0}^{\infty} r(j)$  diverges; see the discussion in Samorodnitsky and Taqqu (1994, p. 335) for FGN. For more general definitions in terms of slowly varying functions and other characterizations of LRD, see Beran (1994).

Finally, the characteristic function for the distribution of  $X_k$ ,  $k = 1, 2, \dots$ , the number of packets arriving in each time unit, is given by

$$\mathbb{E}e^{i\xi X_k} = \exp \lambda \int_{-\infty}^1 ds \int_b^{\infty} dr \delta b^\delta r^{-\delta-1} \int_E \mu(du) [e^{i\xi[u(r \wedge (1-s)) - u(r \wedge (-s))]} - 1] \\ = \exp \lambda \int_{-\infty}^0 ds \int_b^{\infty} dr \delta b^\delta r^{-\delta-1} [e^{\alpha[r \wedge (1-s) - r \wedge (-s)][\phi_B(\xi) - 1]} - 1] \\ + \exp \lambda \int_0^1 ds \int_b^{\infty} dr \delta b^\delta r^{-\delta-1} [e^{\alpha[r \wedge (1-s)][\phi_B(\xi) - 1]} - 1],$$

where  $\xi \in \mathbb{R}$  and  $\phi_B$  is the characteristic function of  $B$ .

**4. Heavy-traffic limit.** We consider the limit of the workload input process as the number of session arrivals and the number of packet generations increase while the packet sizes and the lowest possible holding times decrease. The limit is fractional Brownian motion (FBM), which is a good fit for data traffic aggregated from many sources (Norros 1995). Although it has no physical interpretation, it can be considered as the fluctuations of traffic after the mean is subtracted. The characterization of FBM essentially by a single parameter, namely the Hurst parameter  $H$ , makes it a parsimonious choice.

**4.1. Fractional Brownian motion as a limit.** It is noted in Kurtz (1996) that the following scaling is sufficient for an FBM limit. In this subsection, we supply a detailed proof of this fact in the case of compound Poisson packet generation. Let  $n \in \mathbb{Z}_+$ . Let the arrival rate of the sessions be  $\lambda_n = n^{\delta+1} \lambda$  and the mean measure of  $N_n$  be

$$(10) \quad \eta_n(ds, dr, du) = \lambda_n \delta \left(\frac{b}{n}\right)^\delta r^{-\delta-1} ds dr \mu_n(du) \quad r \geq b/n,$$

where  $\mu_n$  is the distribution of  $(1/n)U(n \cdot)$ . Note that the mean measure (10) is a product of  $\lambda_n$ , the Lebesgue measure, and the distribution of  $((1/n)U(n \cdot), R/n)$ . That is, both time

and workload measurements are rescaled. Note that  $(1/n)U(n\cdot)$  is a compound Poisson process with rate  $n\alpha$  and packet size  $B/n$ . Then, at each session more and more packets arrive in unit time, but with smaller sizes as  $n \rightarrow \infty$ . Indeed, there is empirical evidence that the traffic consists of many small-size packets under heavy traffic (Heyman 2001). In addition, more and more sessions arrive, possibly with very small durations as  $n \rightarrow \infty$  due to the form of  $\eta_n$ .

Let  $Y_n(t)$  be the workload process generated by  $N_n$ ; that is,

$$(11) \quad Y_n(t) = \int_{-\infty}^t \int_{b/n}^{\infty} \int_E N_n(ds, dr, du)[u(r \wedge (t-s)) - u(r \wedge (-s))],$$

with the convention  $u(s) = 0$  for  $s < 0$ . The following lemma shows that the covariance function of  $Y_n$  converges to that of an FBM when properly scaled. A sufficient condition for this would be a uniform integrability condition. This is indicated in Kurtz (1996, Theorem 3.1). We choose the direct approach here.

LEMMA 1. For  $t_j, t_k \in \mathbb{R}_+$ , we have

$$\lim_{n \rightarrow \infty} \frac{\text{Cov}(Y_n(t_j), Y_n(t_k))}{n} = \frac{\lambda \alpha^2 b^\delta (\mathbb{E}B)^2}{(\delta-1)(2-\delta)(3-\delta)} (t_j^{3-\delta} + t_k^{3-\delta} + |t_j - t_k|^{3-\delta}).$$

PROOF. Fix  $t_j < t_k$ . We can assume  $t_j > b/n$  and  $t_k - t_j > b/n$  for sufficiently large  $n$ . To find covariances, we subtract their respective means from  $Y_n(t_j)$  and  $Y_n(t_k)$ , and take expectation. Using Formula (4), we get

$$\begin{aligned} \text{Cov}(Y_n(t_j), Y_n(t_k)) &= n\lambda\delta b^\delta \int_{-\infty}^t ds \int_{b/n}^{\infty} dr r^{-\delta-1} \int_E \mu_n(du)[u(r \wedge (t_j-s)) - u(r \wedge (-s))] \\ &\quad \cdot [u(r \wedge (t_k-s)) - u(r \wedge (-s))]. \end{aligned}$$

The integral with respect to  $\mu_n$  involves covariances of the compound Poisson process  $U(n\cdot)$ . Substituting these covariances and considering the proper limits of integration, we get

$$\begin{aligned} \frac{\text{Cov}(Y_n(t_j), Y_n(t_k))}{n} &= \lambda\delta b^\delta \left\{ \int_{-\infty}^0 ds \int_{\max(b/n, -s)}^{t_j-s} dr r^{-\delta-1} \left[ \frac{\alpha(r+s)}{n} \mathbb{E}B^2 + \alpha^2(r+s)^2 (\mathbb{E}B)^2 \right] \right. \\ &\quad + \int_{-\infty}^0 ds \int_{t_j-s}^{t_k-s} dr r^{-\delta-1} \left[ \frac{\alpha t_j}{n} \mathbb{E}B^2 + \alpha^2(r+s)t_j (\mathbb{E}B)^2 \right] \\ &\quad + \int_{-\infty}^0 ds \int_{t_k-s}^{\infty} dr r^{-\delta-1} \left[ \frac{\alpha t_j}{n} \mathbb{E}B^2 + \alpha^2 t_j t_k (\mathbb{E}B)^2 \right] \\ &\quad + \int_0^{t_j} ds \int_{\max(b/n, t_j-s)}^{t_k-s} dr r^{-\delta-1} \left[ \frac{\alpha(t_j-s)}{n} \mathbb{E}B^2 + \alpha^2 r(t_j-s) (\mathbb{E}B)^2 \right] \\ &\quad + \int_0^{t_j-b/n} ds \int_{b/n}^{t_j-s} dr r^{-\delta-1} \left[ \frac{\alpha r}{n} \mathbb{E}B^2 + \alpha^2 r^2 (\mathbb{E}B)^2 \right] \\ &\quad \left. + \int_0^{t_j} ds \int_{t_k-s}^{\infty} dr r^{-\delta-1} \left[ \frac{\alpha(t_j-s)}{n} \mathbb{E}B^2 + \alpha^2(t_j-s)(t_k-s) (\mathbb{E}B)^2 \right] \right\}. \end{aligned}$$

We split and evaluate the integrals that involve the factor  $\mathbb{E}B^2/n$  in their integrands. All of these tend to 0 as  $n \rightarrow \infty$  with rate  $n^{\delta-2}$ ,  $1 < \delta < 2$ . The remaining terms, which do not involve the factor  $1/n$ , are all bounded by

$$\begin{aligned} &\int_{-\infty}^0 ds \int_{-s}^{t_j-s} dr r^{-\delta-1} (r+s)^2 + \int_{-\infty}^0 ds \int_{t_j-s}^{t_k-s} dr r^{-\delta-1} (r+s)t_j + \int_{-\infty}^0 ds \int_{t_k-s}^{\infty} dr r^{-\delta-1} t_j t_k \\ &\quad + \int_0^{t_j} ds \int_{t_j-s}^{t_k-s} dr r^{-\delta} (t_j-s) + \int_0^{t_j} ds \int_0^{t_j-s} dr r^{-\delta+1} + \int_0^{t_j} ds \int_{t_k-s}^{\infty} dr r^{-\delta-1} (t_j-s)(t_k-s) \\ &= \frac{t_j^{3-\delta} + t_k^{3-\delta} + |t_j - t_k|^{3-\delta}}{\delta(\delta-1)(2-\delta)(3-\delta)} \end{aligned}$$

after  $\alpha^2(\mathbb{E}B)^2$  is factored out. The result follows by bounded convergence theorem.  $\square$

LEMMA 2. Suppose  $\mathbb{E}\varphi(B^2) < \infty$  for some nonnegative and convex function  $\varphi$  on  $(0, \infty)$  with  $\varphi(0) = 0$  and  $\lim_{x \rightarrow \infty} \varphi(x)/x = \infty$ . Then, for each  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^t \int_0^\infty \int_E \eta_n(ds, dr, du) \frac{[u(r \wedge (t-s)) - u(r \wedge (-s))]^2}{n} 1_{[u(r \wedge (t-s)) - u(r \wedge (-s))] > \sqrt{n}\epsilon} = 0.$$

PROOF. We sketch the proof. Using the hypothesis, we can show that

$$\sup_n \int_{-\infty}^t \int_0^\infty \int_E \eta_n(ds, dr, du) \varphi\left(\frac{[u(r \wedge (t-s)) - u(r \wedge (-s))]^2}{n}\right) < \infty$$

by evaluating the integrals as in Lemma 1. This implies the uniform integrability condition

$$\lim_{a \rightarrow \infty} \sup_n \int_{-\infty}^t \int_0^\infty \int_E \eta_n(ds, dr, du) \frac{[u(r \wedge (t-s)) - u(r \wedge (-s))]^2}{n} 1_{[u(r \wedge (t-s)) - u(r \wedge (-s))] > a} = 0.$$

Then, the result (being the Lindeberg condition) follows immediately as in Resnick (1992, p. 522).  $\square$

The Lindeberg condition is essential for proving the central limit theorem (below). An example of  $\varphi$  is  $\varphi(x) = x^{3/2}$  so that one could require  $B$  to have a finite third moment.

We are ready to prove the heavy-traffic limit. In the following theorem, we provide a specific representation for the limiting FBM. A different representation in terms of a white noise on  $(\mathbb{R} \times \mathbb{R}_+, \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}_+})$  could be provided. Instead, we give a representation in Theorem 1 that is based on a white noise in a generalized sense, to make the convergence of the mean measure  $\eta_n$  more visible. We elaborate on this convergence and the specific scaling we have chosen subsequently.

DEFINITION 1. Let  $\zeta$  be a Borel measure on  $\mathbb{R} \times \mathbb{R}_+$ , which is absolutely continuous with respect to the Lebesgue measure  $m$ . A random signed measure  $W$  on  $(\mathbb{R} \times \mathbb{R}_+, \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}_+})$  is called a white noise with variance measure  $\zeta$  if

$$W([a, b] \times [c, d]) = \int_a^b \int_c^d B(ds, dr) \sqrt{f(s, r)} \quad a, b \in \mathbb{R}, c, d \in \mathbb{R}_+,$$

where  $B$  is a Brownian sheet on  $\mathbb{R} \times \mathbb{R}_+$  and  $f$  is the Radon-Nikodym derivative of  $\zeta$  with respect to  $m$ .

Note that we obtain white noise in the usual sense when  $f$  is the identity function, as a special case. Also,

$$\mathbb{E}[W([a, b] \times [c, d])]^2 = \zeta([a, b] \times [c, d]) = \int_a^b ds \int_c^d dr f(s, r)$$

and  $W(A)$  and  $W(C)$  are independent mean zero Gaussian random variables when  $A, C \in \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}_+}$  are disjoint. Let

$$V_n(t) = \frac{Y_n(t) - \mathbb{E}Y_n(t)}{\sqrt{n}} = \frac{1}{\sqrt{n}} \int_{-\infty}^t \int_{b/n}^\infty \int_E \tilde{N}_n(ds, dr, du) [u(r \wedge (t-s)) - u(r \wedge (-s))]$$

be the scaled and centered traffic counts.

THEOREM 1. Let  $\mu$  be the distribution of a compound Poisson process with arrival rate  $\alpha$  and packet size  $B$ . Suppose  $\mathbb{E}\varphi(B^2) < \infty$  for some nonnegative and convex function  $\varphi$  on  $(0, \infty)$  with  $\varphi(0) = 0$  and  $\lim_{x \rightarrow \infty} \varphi(x)/x = \infty$ . Then, as  $n \rightarrow \infty$ , the process  $V_n(t)$  converges to a fractional Brownian motion with representation

$$V(t) = \alpha(\mathbb{E}B) \left\{ \int_{-\infty}^0 \int_0^\infty W(ds, dr) [r \wedge (t-s) - r \wedge (-s)] + \int_0^t \int_0^\infty W(ds, dr) [r \wedge (t-s)] \right\}$$

in the Skorohod topology on  $D(\mathbb{R} \rightarrow \mathbb{R}_+)$ , where  $W$  is a white noise on  $(\mathbb{R} \times \mathbb{R}_+, \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}_+})$  with variance measure

$$\zeta(ds, dr) = \lambda \delta b^\delta r^{-\delta-1} ds dr.$$



PROOF. The Hurst parameter  $H$  and the variance parameter  $c$  of the fractional Brownian motion  $V$  are given by

$$H = \frac{3 - \delta}{2}, \quad c = \frac{2\lambda\alpha^2 b^\delta (\mathbb{E}B)^2}{(\delta - 1)(2 - \delta)(3 - \delta)},$$

with  $1/2 < H < 1$  by the following computation of its covariance. We have

$$\begin{aligned} & \text{Cov}(V(t_j), V(t_k)) \\ &= \alpha^2 (\mathbb{E}B)^2 \lambda \delta b^\delta \left\{ \int_{-\infty}^0 ds \int_0^\infty dr r^{-\delta-1} [r \wedge (t_j - s) - r \wedge (-s)] [r \wedge (t_k - s) - r \wedge (-s)] \right. \\ & \quad \left. + \int_0^{t_j} ds \int_0^\infty dr r^{-\delta-1} [r \wedge (t_j - s)] [r \wedge (t_k - s)] \right\} \\ &= \frac{\lambda \alpha^2 b^\delta (\mathbb{E}B)^2}{(\delta - 1)(2 - \delta)(3 - \delta)} (t_j^{3-\delta} + t_k^{3-\delta} + |t_j - t_k|^{3-\delta}), \end{aligned}$$

where we assumed  $t_j < t_k$ . Note that  $\text{Cov}(V_n(t_j), V_n(t_k)) = \text{Cov}(Y_n(t_j), Y_n(t_k))/n$ . Hence, the covariances of  $V_n$  converge to those of  $V$  by Lemma 1. The Lindeberg condition holds by Lemma 2. Then, convergence of the finite-dimensional distributions of  $V_n$  to  $V$  follows from the Lindeberg-Feller central limit theorem (see, for example, Kurtz 1996, Theorem 6.1, or Jacod and Shiryaev 1987, Theorem VII.5.2). For each  $h > 0$  and for all  $n \geq n_0$  such that  $b/n_0 < h$ , we have

$$\begin{aligned} (12) \quad \mathbb{E}(V_n(h))^2 &= \frac{2\lambda\alpha^2 b^\delta (\mathbb{E}B)^2 h^{3-\delta}}{(\delta - 1)(2 - \delta)(3 - \delta)} + \frac{\lambda\alpha^2 \delta (\mathbb{E}B)^2 b^3}{3n^{3-\delta}(3 - \delta)} \\ & \quad - \frac{\lambda\alpha^2 \delta (\mathbb{E}B)^2 b^2 h}{n^{2-\delta}(2 - \delta)} + \frac{\lambda\alpha \delta (\mathbb{E}B^2) b h}{n^{2-\delta}(\delta - 1)} \\ &< \frac{2\lambda\alpha^2 b^\delta (\mathbb{E}B)^2 h^{3-\delta}}{(\delta - 1)(2 - \delta)(3 - \delta)} \\ & \quad + \left( \frac{\alpha^2 (\mathbb{E}B)^2 b^\delta}{3(3 - \delta)} - \frac{\alpha^2 (\mathbb{E}B)^2 b^\delta}{2 - \delta} + \frac{\alpha (\mathbb{E}B^2) b^{\delta-1}}{\delta - 1} \right) \lambda \delta h^{3-\delta} \\ &=: Ch^{3-\delta}, \end{aligned}$$

where we used  $b/n < h$ . It follows that  $\lim_{h \rightarrow 0} \sup_n \mathbb{E}(V_n(h))^2 = 0$ , and hence for each  $\epsilon > 0$ , we get

$$(13) \quad \lim_{h \rightarrow 0} \sup_n \mathbb{P}\{|V_n(h)| > \epsilon\} = 0$$

by Chebyshev's inequality. On the other hand, for  $0 \leq h \leq 1$ , we have

$$(14) \quad \mathbb{E}(V_n(t+h) - V_n(t))^2 (V_n(t) - V_n(t-h))^2 = \mathbb{E}(V_n(t+h) - V_n(t))^2 \mathbb{E}(V_n(t) - V_n(t-h))^2,$$

as the compound Poisson process  $U$  has independent increments. By the stationarity of  $V_n$  and Equation (12), we get

$$\mathbb{E}(V_n(t+h) - V_n(t))^2 (V_n(t) - V_n(t-h))^2 \leq C^2 h^{2(3-\delta)},$$

where  $3 - \delta > 1$  because  $1 < \delta < 2$ . In view of (13) and (14), convergence in distribution in the Skorohod topology follows from Kurtz (1996, Theorem 6.2) or Ethier and Kurtz (1986, Theorems 3.8.6 and 3.8.8), characterizing relative compactness.  $\square$

This limit arises as a result of heavy-tailed distributions for session holding times and the ergodicity of the compound Poisson process in charge of packet generation. For the latter, we have  $(1/n)U(nt) \rightarrow \alpha t$  almost surely and in  $L^2$ . Therefore, we get

$$(15) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \frac{\lambda_n}{n} \int_0^\infty dr \delta\left(\frac{b}{n}\right) r^{-\delta-1} 1_{\{r > b/n\}} \int_E \mu_n(du) h(u, r) \\ & = \lim_{n \rightarrow \infty} \lambda n^\delta \mathbb{E} \left[ h\left(\frac{1}{n}U(n\cdot), \frac{R}{n}\right) \right] = \lambda \delta b^\delta \int_0^\infty dr r^{-\delta-1} h(\alpha \cdot, r) \end{aligned}$$

for all bounded continuous functions  $h$  on  $E \times \mathbb{R}_+$  for which there exists an  $r_0 > 0$  such that  $h(u, r) = 0$  for all  $r < r_0$  and  $u \in E$ . Hence,  $\eta_n$  converges as a measure. This is the essence of convergence of the covariances of  $Y_n$  as  $n \rightarrow \infty$ , as given in Lemma 1.

REMARK 1. In general, let  $V_n(t) = \sigma_n^{-1}[Y_n(t) - \mathbb{E}Y_n(t)]$ . We have taken  $\sigma_n = \sqrt{n}$  in addition to the particular scaling of  $\eta_n$  in (10). Other scalings are also possible for Theorem 1. Equation (15) will hold as long as  $\sigma_n^{-2}\lambda_n = n^\delta\lambda$ . For instance, one can choose  $\lambda_n = n^\gamma\lambda$  with  $\gamma > \delta$  and  $\sigma_n = n^{(\gamma-\delta)/2}$ .

**4.2. Comparison with Lévy motion limits.** A similar formulation is considered in Mikosch et al. (2002), where the data generation occurs at a constant rate and the session durations are heavy tailed. By providing Lévy motion and FBM limit theorems, the authors recover many previous results for both the on/off model and infinite-source Poisson model. For the session arrival rate, if a slow-growth condition holds, then the limit is a Lévy motion; if a fast-growth condition holds, then the limit is an FBM.

For comparison purposes, we will simplify the assumptions in Mikosch et al. (2002). In general, the session durations have the complementary distribution function  $\mathbb{P}\{R > r\} = r^{-\delta}L(r)$ , where  $L$  is a slowly varying function. Let us take  $L$  to be a constant by assuming a Pareto distribution for  $R$ . The arrival rate  $\lambda_n$  is specified as a nondecreasing function of  $n$ ; let us take  $\lambda_n = n^{\gamma-1}\lambda$ , where  $\gamma > 1$ . Let  $N_n$  be a Prm with mean measure  $\lambda_n\beta(dr)ds$ , where  $\beta$  is the Pareto distribution. The total accumulated work  $A$  is considered at large times  $nt$ , then centered and scaled to obtain an FBM limit. Explicitly,

$$A(nt) = \int_{-\infty}^0 \int_0^\infty N_n(ds, dr)[r \wedge (nt - s) - r \wedge (-s)] + \int_0^{nt} \int_0^\infty N_n(ds, dr)[r \wedge (nt - s)].$$

By change of variables  $s$  to  $s' = s/n$  and  $r$  to  $r' = r/n$ , we obtain a process  $A'$  such that  $nA'(\cdot)$  has the same probability law as  $A(n\cdot)$  above. We have

$$nA'(t) := n \int_{-\infty}^0 \int_0^\infty N'_n(ds', dr')[r' \wedge (t - s') - r' \wedge (-s')] + n \int_0^t \int_0^\infty N'_n(ds', dr')[r' \wedge (t - s')],$$

where  $N'_n$  is a Prm with mean measure  $n\lambda_n\beta_n(dr)ds = n^\gamma\lambda\beta_n(dr)ds$  and  $\beta_n$  is the distribution of  $R/n$ . As a result, the scaled and centered workload is equal in distribution to

$$(16) \quad \frac{nA'(t) - \mathbb{E}nA'(t)}{(\lambda n^{\gamma-1} n^3 n^{-\delta})^{1/2}},$$

which is shown to converge to an FBM in Mikosch et al. (2002, Theorem 3), where we omitted some of the constants. After simplification, the expression (16) is equal to  $\sigma_n^{-1}[A'(t) - \mathbb{E}A'(t)]$  with  $\sigma_n = n^{(\gamma-\delta)/2}$ . What is more, the fast-growth condition sufficient for FBM limit reduces to  $\gamma > \delta$  (Mikosch et al. 2002, Lemma 1). As a result, the scalings of  $A'$  and  $Y_n$  are equivalent in view of Remark 1. This is as expected because in both cases the packet-generation or data-transmission process essentially remains the same. In our case, we have  $(1/n)U(n\cdot) \rightarrow \alpha\cdot$ , where  $\alpha$  corresponds to a constant transmission rate

in the limit. Also, the session durations are scaled as  $R/n$ , the session arrival rate is scaled as  $n^\gamma \lambda$ , and the scaling coefficient  $\sigma_n$  is the same. On the other hand, if a slow-growth condition equivalent to  $\gamma < \delta$  holds, the scaled and centered process

$$(17) \quad \frac{nA'(t) - \mathbb{E}nA'(t)}{\lambda^{1/\delta} n^{(\gamma-1)/\delta} n^{1/\delta}}$$

converges to a  $\delta$ -stable Lévy motion (Mikosch et al. 2002, Theorem 1). Note that the denominator is proportional to  $n^{(\gamma-\delta)/\delta}$  after simplification. Analogously, one would expect a similar scaling to yield a Lévy motion in the case of compound Poisson packet generation of the present study.

A similar construction is given in Maulik and Resnick (2001), where the cumulative traffic has the form

$$\sum_{S_i \leq t} U_i(t - S_i) \wedge R_i$$

and  $R$  represents the size of the files to be transferred. The form of  $U$  is left quite general but with several assumptions, such as being multifractal with stationary increments and having a certain limit behavior after scaling. A scaling of the form (17) is shown to yield a Lévy motion. For Poisson shot-noise processes in general, Klüppelberg et al. (2003) show a sufficient condition, which they call regular variation in the mean, for convergence to an infinite-variance stable process. A process  $X$  is called a Poisson shot-noise process if  $X(t) = \sum_{S_i \leq t} Y_i(t - S_i)$ , where  $S_i$ ,  $i = 1, 2, \dots$  are arrival times of a Poisson process. Several special cases for  $Y$  are considered as applications in Klüppelberg et al. (2003). For example, if the shots  $Y_i$  are compound Poisson processes given by  $\sum_{i=1}^{M_i(t)} B_i$ , and  $B_i$  are strictly stable random variables, then there exists a scaling that yields a stable limit. The authors also demonstrate their sufficient condition for convergence to a stable limit in the case of the teletraffic construction of Mikosch et al. (2002). As a result, although a similar result is expected, the details need to be worked out to establish the conditions for a Lévy motion or a stable limit in general for our model.

**4.3. Application in queuing.** An important application of Theorem 1 is in queuing. Suppose traffic modeled by (11) arrives at a switch with an infinite buffer and the contents of the buffer is released continuously, as in a fluid queue. Then, the queue-length process, or the content of the queue, is given by

$$L_n(t) = Y_n(t) - c_n t - \bigwedge_{s=0}^t (Y_n(s) - c_n s),$$

where  $c_n$  is a release rate. Resnick and van den Berg (2000) show that the queue content converges to that of the limiting process for their model. Similarly, the queue content in our case behaves like the content process of FBM input for large  $n$ , as given in Corollary 1. Note that  $\mathbb{E}Y_n(t)$  is linear in  $t$  and is given by  $\mathbb{E}Y_n(t) = \lambda \alpha \delta (\delta - 1)^{-1} b \mathbb{E}B n^\delta t$  in view of (3). Let  $\xrightarrow{d}$  denote convergence in distribution.

COROLLARY 1. *Suppose that*

$$\lim_{n \rightarrow \infty} \frac{c_n - \lambda \alpha \delta (\delta - 1)^{-1} b \mathbb{E}B n^\delta}{\sqrt{n}} = 0.$$

Then,

$$\frac{L_n(t)}{\sqrt{n}} \xrightarrow{d} V(t) - \bigwedge_{s=0}^t V(s) \quad \text{as } n \rightarrow \infty$$

in the Skorohod topology on  $D(\mathbb{R} \rightarrow \mathbb{R}_+)$ .

The proof follows along the same lines as in Resnick and van den Berg (2000, §5), with Skorohod  $M_1$  topology replaced with  $J_1$  in our case. Basically, because the reflection mapping  $x(t) \rightarrow x(t) - \bigwedge_{s=0}^t x(s)$  is continuous, the continuous mapping theorem (Billingsley 1999, Theorem 2.7) applies.

As a matter of fact, the tail of the queue-length distribution is found to behave like a Weibull distribution with FBM input (Norros 1995). With the on/off model, Heath et al. (1998) show that the content process has heavy tails. The tail of the queue length in the case of Lévy input is much heavier than a Weibull-like tail corresponding to the FBM case. Also, the marginal distributions are quite different, being Gaussian in FBM and heavy right tailed in Lévy motion. From an application point of view, both the empirical queue length and the marginal distributions determine which regime is valid.

**5. A fast-traffic generator.** Analytical queuing results in the presence of self-similar traffic, as modeled in this paper, are few in comparison to traditional Markovian traffic models. Therefore, simulation studies are crucial for further performance analysis. In this section, we show that a fast-traffic-generation algorithm can be implemented with our traffic model. The approach is very similar to the synthesis of FBM traffic, with micropulses given by Cioczek-Georges and Mandelbrot (1996) and considered in Çağlar (2000). The steps consist of simulation of Poisson arrivals, association of a Pareto-distributed session with each arrival, generation of a Poisson process over each session, and keeping track of all these as long as the session is active at the current time. The method is fast and accurate with a steady requirement of memory.

One technical issue is the representation of the infinite past, which guarantees stationarity. It can be accounted for by replacing  $-\infty$  by a sufficiently small  $T < 0$ . Let  $Y_T$  denote the traffic counts in this case. For traffic generated on  $[0, t]$ ,  $t \geq 1$ , the expected difference due to truncation of the infinite past is given by

$$\begin{aligned}
 (18) \quad \mathbb{E}|Y(t) - Y_T(t)| &= \lambda \delta b^\delta \alpha \left\{ \int_{-\infty}^T ds \int_{-s}^{t-s} dr r^{-\delta-1} (r+s) + t \int_{-\infty}^T ds \int_{t-s}^{\infty} dr r^{-\delta-1} \right\} \\
 &= \frac{\lambda b^\delta \alpha [t^2(t-T)^{-\delta} - 2tT(t-T)^{-\delta} + T^2(t-T)^{-\delta} - (-T)^{2-\delta}]}{(\delta-1)(2-\delta)} \\
 &= \frac{\lambda b^\delta \alpha (2t-\delta)(-T)^{1-\delta}}{(\delta-1)(2-\delta)} + O((-T)^{-\delta}).
 \end{aligned}$$

The error increases linearly with  $t$  for fixed  $T$  as in the micropulses-generation method. On the other hand, the algorithm is not adversely affected by the large magnitude of  $T$ . The number of sessions that last until time 0 or longer is small. Consider the discretization of the interval  $[T, -1]$  by intervals of length  $\Delta t > 0$ . The number of arrivals on  $(T + (k-1)\Delta t, T + k\Delta t]$  that have widths greater than  $a_k \equiv -(T + k\Delta t)$  is a Poisson random variable with mean  $\Delta t \lambda \mathbb{P}\{R > a_k\} = \Delta t \lambda b^\delta a_k^{-\delta}$ . The computational cost for the method is proportional to the total number of sessions generated. For each  $\Delta t > 0$ , the expected number of all sessions generated on  $[T, -1]$  is given by

$$l \equiv \Delta t \lambda b^\delta \sum_{k=1}^m (-(T + k\Delta t))^{-\delta},$$

where  $m = (-1 - T)/\Delta t$  is assumed to be an integer through the choice of  $\Delta t$ . We can approximate  $l$  by

$$l \approx \lambda b^\delta \int_T^{-1} (-x)^{-\delta} dx = \frac{\lambda b^\delta}{(\delta-1)} (1 - (-T)^{1-\delta})$$

as  $\Delta t \rightarrow 0$ . The upper bound in (18) suggests the choice of  $T \approx -t^{\frac{1}{\delta-1}}$  when we evaluate  $Y$  on  $[0, t]$ . That is,  $l \approx O(1)$ . Therefore, the total number of sessions generated is  $O(t)$  when the sessions generated over  $[-1, t]$  are also included. The expected number of packets generated is just a multiple of this requirement with  $\alpha$ .

For an FBM limit, the value of  $b$  is scaled with  $1/n$ , which results in a higher number of short sessions and more packet generations in each session. Nevertheless, the resulting computational burden is of  $O(t)$  because the mean of a Poisson process is linear in  $t$ . Therefore, our packet traffic model lends itself to a fast synthesis algorithm for FBM, as in the case of micropulses approach. The two models are similar in mathematical construction. However, the current model provides a better abstraction of packet traffic. For synthesizing an FBM, it is as fast as wavelet synthesis like the micropulses algorithm and can be made as accurate as needed with the modification of the scaling parameter  $n$  and the truncation parameter  $T$ .

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## References

- Beran, J. 1994. *Statistics of Long Memory Processes*. Chapman and Hall, New York.
- Billingsley, P. 1999. *Convergence of Probability Measures*. Wiley, New York.
- Çağlar, M. 2000. Simulation of fractional Brownian motion with micropulses. *Adv. Performance Anal.* **3** 43–69.
- Çağlar, M. 2001. A workload input model for packet data traffic. *INFORMS Appl. Probab. Conf.*, New York. INFORMS, Baltimore, MD.
- Cao, J., W. S. Cleveland, D. Lin, D. X. Sun. 2001. On the nonstationarity of Internet traffic. *Proc. ACM Sigmetrics*. ACM Press, New York.
- Chandramouli, Y. 1997. A model for IP traffic generation. Preprint, Bellcore, Navesink, NJ.
- Cioczek-Georges, R., B. B. Mandelbrot. 1996. Alternative micropulses and fractional Brownian motion. *Stochastic Proc. Appl.* **64** 143–152.
- Ethier, S. N., T. G. Kurtz. 1986. *Markov Processes*. John Wiley and Sons, New York.
- Heath, D., S. Resnick, G. Samorodnitsky. 1998. Heavy tails and long range dependence in on/off processes and associated fluid models. *Math. Oper. Res.* **23** 145–165.
- Heffes, H., D. M. Lucantoni. 1986. A Markov modulated characterization of packetized voice and data traffic and related statistical multiplexer performance. *IEEE J. Selected Areas in Comm.: Special Issue on Network Performance Evaluation* **6** 856–868.
- Heyman, D. P. 2001. Estimation of Markov modulated Poisson process models of IP traffic. *INFORMS Appl. Probab. Conf.*, New York. INFORMS, Baltimore, MD.
- Jacod, J., A. Shiryaev. 1987. *Limit Theorems for Stochastic Processes*. Springer-Verlag, Berlin, Germany.
- Kallenberg, O. 1983. *Random Measures*. Academic Press, New York.
- Klüppelberg, C., T. Mikosch. 1995. Explosive Poisson shot noise processes with applications to risk reserves. *Bernoulli* **1** 125–147.
- Klüppelberg, C., T. Mikosch, A. Schärf. 2003. Regular variation in the mean and stable limits for Poisson shot noise. *Bernoulli* **9** 467–496.
- Konstantopoulos, T., S. Lin. 1998. Macroscopic models for long-range dependent network traffic. *Queueing Systems* **28** 215–243.
- Kurtz, T. 1996. Limit theorems for workload input models. F. P. Kelly, S. Zachary, I. Ziedins, eds. *Stochastic Networks: Theory and Applications*. Clarendon Press, Oxford, U.K.
- Levy, J. B., M. S. Taquq. 2000. Renewal reward processes with heavy-tailed interrenewal times and heavy-tailed rewards. *Bernoulli* **6** 23–44.
- Maulik, K., S. Resnick. 2001. Small and large time scale analysis of a network traffic model. Technical report, Cornell University, Ithaca, NY.
- Mikosch, T., S. Resnick, H. Rootzén, A. Stegeman. 2002. Is network traffic approximated by stable Lévy motion or fractional Brownian motion? *Ann. Appl. Probab.* **12** 23–68.
- Norros, I. 1995. On the use of fractional Brownian motion in the theory of connectionless networks. *IEEE J. Selected Areas Comm.* **13** 953–962.
- Nuzman, C., I. Saniee, W. Sweldens, A. Weiss. 2002. A compound model for TCP connection arrivals for LAN and WAN applications. *Comput. Networks* **40** 319–337.
- Pipiras, V., M. Taquq. 2000. The limit of a renewal reward process is not a linear fractional stable motion. *Bernoulli* **6** 607–614.

- Resnick, S. 1992. *Adventures in Stochastic Processes*. Birkhauser, Boston, MA.
- Resnick, S., E. van den Berg. 2000. Weak convergence of high-speed network traffic models. *J. Appl. Probab.* **37** 575–597.
- Samorodnitsky, G., M. S. Taqqu. 1994. *Stable Non-Gaussian Random Processes*. Chapman and Hall, New York.
- Taqqu, M. S., J. Levy. 1986. Using renewal processes to generate long-range dependence and high variability. E. Eberlein, M. S. Taqqu, eds. *Dependence in Probability and Statistics*. Birkhauser, Boston, MA, 73–89.
- Taqqu, M. S., W. Willinger, R. Sherman. 1997. Proof of a fundamental result in self-similar traffic modeling. *Comput. Comm. Rev.* **27** 5–23.
- Willinger, W., V. Paxson, R. Reidi, M. Taqqu. 2001. Long-range dependence and data network traffic. P. Doukhan, G. Oppenheim, M. S. Taqqu, eds. *Long-Range Dependence: Theory and Applications*. Birkhauser, Boston, MA.