A Newsvendor Problem with Markup Pricing in the Presence of Within-period Price Fluctuations

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Abstract

We consider a single-item single-period joint inventory management and pricing problem of a retailer selling an item that has selling price uncertainties. Unlike most of the literature on the newsvendor problem, we assume that price-dependent demand arrives randomly according to a stochastic arrival process whose rate depends on the fluctuating market input price process. The retailer's problem is to choose the order quantity and a proportional price markup over the input price to maximize the expected profit. This setting is mostly encountered by retailers that trade in different currencies or have to purchase and convert commodities for seasonal sales. For this setting, we characterize both the optimal inventory and markup levels. We present monotonicity properties of the expected profit function with respect to each decision variable. We also show that more volatile input price processes lead to lower expected profits.

Keywords: inventory, markup pricing, stochastic processes

1 Introduction and Literature Review

Pricing is one of the most effective tools that a firm has to increase its revenues. The impact of a successful pricing strategy lies in its effect on sales as price is one of most critical determinants of customer demand. By effectively controlling the demand, firms have also the potential to enhance revenues by managing the mismatch between supply and demand. Even more value can be created by integrating pricing decisions with inventory, production and distribution decisions. It is well known that integration of pricing and inventory decisions has the potential of thoroughly increasing supply chain effectiveness (Chan et al., 2004).

In classical monopolistic inventory/pricing models, firms are assumed to control the selling prices without any limitations. However, for some industries, firms also have to deal with input price volatilities which constrain and determine the selling price to customers. This is usually the case for exchange rate volatility if critical inputs are imported. Gold and jewelry retailers have to consider price volatilities in making both ordering and pricing decisions as underlying commodity prices are market-determined. Fresh grocery products are another example where customers expect the selling prices to follow the supply market price. An efficient ordering-pricing policy should take the expected future evolution of such volatile prices into account.

In this paper, we examine how a firm that sells a product with randomly fluctuating input prices that directly impact selling prices coordinates its ordering and pricing decisions. In a single-period, single-item setting, we explicitly model the stochastic behavior of the input price as a general continuous price process. In addition, we model the individual customer demand arrival process that is price dependent. This approach allows us to see the impact of volatile market and selling prices on ordering and pricing policies. Rather than determining a fixed selling price that is independent from the market input price, we assume that the retailer determines a proportional markup on the market price to reflect the effect of prevailing input prices. We explicitly characterize the form of optimal ordering quantity and markup rate. This enables us to obtain insights on how market price volatility impacts expected sales revenue and profit.

Our model is within the long-standing literature on coordinated inventory management and pricing research stream which is based on simultaneously finding the optimal ordering and pricing policies. However, different than many pricing models, it also has components that is connected to the literature on inventory models involving stochastic input prices and changing selling prices.

Many models in the coordinated inventory-pricing literature, whether stochastic or deterministic, customer demand is assumed to be a function of the selling price. Whitin (1955) was the first to allow selling price to be set simultaneously with order quantity in the newsvendor model. His method is to first determine the optimal ordering quantity as a function of price and then find the corresponding optimal price. Petruzzi and Dada (1999) present a unified approach for this problem. Yano and Gilbert (2005) present reviews of the early joint inventory/pricing literature. In a related work to this paper, Xiao et al. (2015) consider a similar dual source model to Chen et al. (2013) and investigate the joint replenishment and pricing policy. Gayon et al. (2009) present a model for a production/inventory system that includes a joint pricing component where the purchase prices are fixed but demand rates depend on the state-of-the world. Liu and Yang (2015) investigate a multi-period joint pricing and inventory control model where raw material costs randomly fluctuate in a Markovian fashion and show the optimality of base-stock-list-price type policies.

Focusing on the single-period newsvendor model, Petruzzi and Dada (1999) investigate the joint ordering and pricing decisions for both additive and multiplicative demand models and provide conditions that are sufficient to ensure monotonicity of the profit function. They also characterize some important properties of optimal prices. There are many papers on the newsvendor problem that follow Petruzzi and Dada (1999) to explore different pricing aspects and most of these papers make the standard assumptions that only require using the total demand that arrives in the sales horizon. In our model, the timing of individual arrivals over the horizon is critical because the revenue from sales depends on the continuously fluctuating input price at the given instance. We are aware of only a few papers that account for the timing of demand arrivals over the sales season for a newsvendor. In Grubbström (2010), demand arrives according to a compound renewal process and arrival times are critical since a net present value objective is taken and revenues are discounted. Unlike our paper, the prices are fixed and there is no price optimization component. Hu and Su (2018) consider a purchasing and pricing problem for a newsvendor. The price process fluctuates continuously during the purchasing period but price-dependent demand occurs at a later time in one shot. In our model, the price fluctuations have a direct effect on the process generating the demand (and therefore the revenue) and markup pricing is assumed. Gürel and Güllü (2019) consider a problem that is similar to ours with two classes of customers where one class of customers is priced at a constant markup above the market price and the other class that arrives later is charged a fixed price. They characterize the expected revenue function and characterize the optimal replenishment quantity but do not consider the price dependence and markup optimization aspects for customers who arrive earlier in the season. We explore the joint replenishment and markup optimization problem and present analytical results on the effects of price volatility. Finally, we also note that Canyakmaz et al. (2015) present some preliminary numerical simulation findings that are consistent with the theoretical results that are formally proven in this paper.

Although we focus on a single-period model in this paper, we assume a backorder setting where the firm needs to procure at the random market price at the end of sales period to satisfy backordered customers. This aspect of our model is similar to inventory models with stochastic input prices in the literature. A prominent example in this research stream is Kalymon (1971), who incorporates random purchase prices that are modulated by a Markov process into a multi-period inventory model with fixed costs. He proves that a price-dependent (s, S) ordering policy is optimal. Golabi (1985) considers a single-item deterministic demand inventory system. He assumes that at the beginning of each cycle, the ordering price is determined according to a known distribution function. He derives a policy in which it is optimal to order for the next n periods if the price falls into a certain interval. Assuming a deterministic demand setting with stochastic purchase prices following a geometric Brownian motion process, Li and Kouvelis (1999) investigate optimal purchasing strategies for different supply contract structures that differ depending on the timing and quantity of the purchase. Yang and Xia (2009) consider a Poisson arrival process in a continuous-review inventory model where purchase price is a discrete-state Markov process. They find conditions that yield monotone optimal order-up-to levels in terms of purchase price. More recently, Berling and Martínez-de Albéniz (2011) investigate a Poisson demand system where price is a continuous Markov process unlike Yang and Xia (2009). Specifically, they consider geometric Brownian motion and Ornstein-Uhlenbeck processes for the price and characterize optimal base-stock levels as a series of thresholds and provide an algorithm to calculate them. Berling and Xie (2014) study the same model as in Berling and Martínez-de Albéniz (2011) and propose approximations to the optimal purchasing policy based on the decomposition approach.

In a similar manner, another research stream focuses on a randomly fluctuating spot market and its effect on inventory replenishment decisions in the context of dual sourcing. For instance, Goel and Gutierrez (2012) consider a multi-period stochastic inventory model where a firm may purchase from both spot and future markets by explicitly allowing spot and future prices as stochastic processes. In Inderfurth and Kelle (2011), demand and market prices are independent random variables in each period. Inderfurth et al. (2018) extend this to the case of correlated demand and price processes. Chen et al. (2013) consider a dual source replenishment problem with lost sales where the spot market price process follows a Markovian stochastic process and investigate the optimal policy. Our model differs from these papers by assuming that fluctuations in stochastic input prices pass to customers to a degree that can be controlled by a price markup.

There are also a number of inventory papers where model parameters (including purchase prices) vary according to an external state of the world process which is Markov Chain. Examples include Özekici and Parlar (1999) and Karabağ and Tan (2019). In these papers the optimal inventory replenishment decision depends on the current state-of-the world. Unlike these papers, our model assumes that the price process is a continuous random process which is appropriate for exchange rates, commodities or spot markets.

Few papers consider the effect of changing selling prices on firm's optimal inventory decisions. Hariga (1995) extends economic order quantity (EOQ) model by incorporating the impact of inflation on selling price. Khouja and Park (2003) and Banerjee and Meitei (2009) consider the impact of decreasing selling prices over the life cycle of technological products such as cell-phones. Canyakmaz et al. (2019) analyze a multi-period replenishment problem under a random input price process that also determines the selling prices which are also stochastic. The authors characterize the optimal replenishment policy. In these papers, although selling prices are affected by external factors in a deterministic or stochastic fashion, the firm is assumed to have no control over the selling price. In our paper, we explicitly model the impact of exogenous stochastic input prices as well as firm's endogenous price markup decision on firm's profit and characterize the optimal inventory and markup pricing strategies.

With our model and analysis, we contribute to the literature by incorporating the effect of fluctuating input prices into inventory and pricing decisions. This enables us to capture the effect of price volatility during the sales season. We assume that the firm chooses a proportional markup over the stochastic input price which directly impacts the customer arrival process. We also investigate how the level of fluctuations in market prices affect optimal performance measures. The assumptions are fairly general and many similar cases in the literature can be addressed by looking at special cases such as where input prices are constant or deterministic, price and demand processes are independent etc. Under certain assumptions, we are able to characterize the optimal ordering quantity and markup price and their interaction effect. In addition, using tools from stochastic orderings of stochastic processes, we analytically characterize the effects of price monotonicity and volatility.

The rest of the paper is organized as follows. We specify the basics of our model in §2, and the form of optimal policy with several characterizations in §3. In §4, we analyze the impact of the variability and monotonicity of the random prices on the optimal expected revenues and profits. Finally, we present our concluding remarks in §5.

2 The Model

We consider a retailer that sells a single product which has an intrinsic commodity price value that continuously evolves through time. The retailer determines a single inventory level y at the beginning of a selling season of length T where all orders are received immediately at $t = 0$ (i.e., there is no lead time). We assume that the retailer sets a proportional markup α at $t = 0$ to be applied to the market price of the commodity which together determine the selling price at each time. We employ a general modeling approach for stochastic evolution of the market price process that determines the input price and assume that it evolves according to a stochastic process $\mathcal{P} = \{P_t \; ; t \geq 0\}$ with state space $\mathbb{R}^+ = [0, +\infty)$. There are many examples of such price processes that have been used in the extant literature. Below, we present some examples. (We assume $0 \leq t \leq T$, and $p_0 \geq 0$):

$P_t = p_0 :$ a constant price process

$P_t = e^{-rt} p_0 :$ a discounted price process

$P_t = g(t)$ where $g(t)$ is a deterministic function of time

$P_t = p_i$ if $X_t = i$ where X_t is a continuous-time Markov chain starting at $X_0 = 0$

$P_t = p_0 + \mu t + \sigma B_t$ where B_t is a standard Brownian motion (Wiener process)

$P_t = p_0 e^{\mu t + \sigma B_t}$ where B_t is a standard Brownian motion (Berling and Martínez-de Albéniz, 2011)

More sophisticated two-level processes that are known to be good models of commodity prices

(Schwartz and Smith, 2000, Canyakmaz et al., 2019)

Note that under a fluctuating price process and a proportional markup α, the effective selling price of the product is αP_t at time t. This implies that the selling prices are driven by the purchase price process.

To model the individual customer demand process, we assume that customer arrival rates are modulated by the price process that we consider. More specifically, we assume that individual customers arrive according to a doubly-stochastic Poisson process with a stochastic arrival rate process $\Lambda = \{\Lambda_t = \lambda(\alpha P_t); t \geq 0\}$ where $\lambda(.)$ is a nonnegative deterministic function of random selling price.

We denote the customer arrival process as $\mathcal{N} = \{N_t^{\alpha}; t \geq 0\}$ where superscript α denotes its connection to price markup α. Analogously, we use $\mathcal{T} = \{T_n : n \geq 1\}$ to denote the random arrival times of customers where T_n is arrival time of n^{th} customer. Stochastic processes $\mathcal{N}$ and $\mathcal{T}$ are tied via $P\{T_n \leq t\} = P\{N_t \geq n\}$.

We remark that doubly stochastic Poisson processes are generalizations of non-homogeneous Poisson processes where customer arrival rates change in time in a deterministic fashion. In other words, if $\Lambda_t = \lambda(t)$ is deterministic, then the customer arrival process reduces to a non-homogeneous Poisson process with arrival rate $\lambda(t)$ at time t. Furthermore, the case where $\Lambda_t = \lambda$ gives the ordinary Poisson process with arrival rate λ.

Besides modulating customer arrivals, we assume that stochastic market price process also affects customers' individual demand quantities. In particular, we let $\mathcal{X} = \{X_n; n \geq 1\}$ denote the individual demand process that depends on the selling prices. This dependence is through a price-dependent deterministic mean demand function μ and a random shock ξ with $\mathbb{E}[\xi] = 0$ and distribution function F which is independent of the price process $\mathcal{P}$ and arrival process $\mathcal{N}$. More specifically, we assume that $X_n = \mu(\alpha P_{T_n}) + \xi$ where αP_{T_n} is the selling price at the time of n^{th} customer arrival. We use D_k to denote the cumulative demand by k^{th} arriving customer so that

$$D_k = \sum_{n=1}^{k} X_n.$$

Using this notation, total random demand during the sales season can be denoted as $D_{N_T^{\alpha}}$.

Without any loss of generality, we assume that the purchase price of the item is the initial market price P_0 since one can always reflect any differences in purchase price by shifting the market price process appropriately. We allow backorders and assume that in case of shortage, newly arrived customers are charged at the prevailing selling price and satisfied at time $t = T$. This is a plausible assumption for cases where the retailer sells exclusive products such that arriving customers may be unable to find elsewhere. Jewelry stores, for instance, usually take orders for diamond rings etc. to be supplied later, yet their selling prices are determined considering the current market prices of diamond and gold at the time of customer order, not the market prices at the time of delivery. Justification of this setting through real life examples encourages us to focus on this particular backorder model since it also provides a rather tractable analysis.

We assume that the retailer incurs a penalty cost of b for each unit of unsatisfied demand and needs to replenish the inventory until all backorders are satisfied by purchasing at the market price P_T at the end of sales season. To avoid the uninteresting case where the retailer chooses to backorder all customers, we assume that $b + \mathbb{E}[P_T] - P_0 > 0$ (i.e., unit underage cost is positive). Without loss of generality, we do not assume any physical holding cost or salvage revenue in our analysis, yet by discounting all future cash flow, we are capturing the opportunity costs associated with the retailer's capital investment. We remark here that our results carry over to the case where the repurchase price P_T is independent of the price process and our characterizations simplify in most cases. This case could be especially relevant if the retailer contracted repurchases in advance at a fixed cost or possesses financial securities such as futures or call options contingent on the market price at time T in sufficient amounts.

Next, we construct the total customer demand variable along with its distribution, revenues from sales and ultimately the expected profit function for our subsequent optimization and comparison analyses.

2.1 Distribution of Demand and Expected Sales

As customers arrive according to a doubly stochastic Poisson process, total number of individual customers N_T^α who arrive during the sales season is Poisson with random mean

$$M_T^\alpha = \mathbb{E}\left[N_T^\alpha \mid \mathcal{P}\right] = \int_0^T \lambda(\alpha P_t)dt.$$

Here, M_T^α is the expected number of individual arrivals given all price realizations during the sales season. As each arriving customer demands a random amount of the item, total demand $D_{N_T^\alpha}$ is a modified version of compound Poisson with dependent demand quantities (where the dependence is through the price process $\mathcal{P}$). In the Appendix, we show that expected total demand $d_T(\alpha)$ as a function markup α reduces to

$$d_T(\alpha) = \mathbb{E}\left[D_{N_T^\alpha}\right] = \int_0^T \mathbb{E}\left[\overline{\lambda}(\alpha P_t)\right] dt \tag{1}$$

where $\overline{\lambda}(x) = \lambda(x)\mu(x)$. We call $\overline{\lambda}$ the modified intensity function. Note that it contains both customers' arrival intensity λ as well as their mean demand function μ. An interesting implication of this result is that in expectation, the demand process behaves as if customers arrive according to a doubly stochastic process with the modified intensity process $\overline{\lambda}$ and demand one unit of the item. Moreover, the whole price path impacts the demand distribution and consequently the expected demand. Finally, note that in the backorder case, the total demand is equal to the total sales.

2.2 Expected Revenue from Sales

Similar to total demand, we let R_T^α denote the total discounted revenue until time T and generate it by summing all individual discounted revenues collected from arriving customers. More specifically,

$$R_T^\alpha = \sum_{n=1}^{N_T^\alpha} e^{-rT_n} \alpha P_{T_n} X_n$$

where T_n denotes the arrival time of n^{th} customer and, as stated before, N_T^α is the total number of individual customers arrived by time T when markup is α. Also, $\alpha P_{T_n} X_n$ is the random revenue obtained from the n^{th} customer. Note that both selling price and demand amount are random at each customer arrival. We also discount all individual revenues to time 0 by factor e^{-rT_n} where r is the interest rate per unit time. Risk-neutral retailer is concerned with the expected total revenue by time T which we denote as $r_T(\alpha) = \mathbb{E}[R_T^\alpha]$ as a function of markup α. Note that total revenue from sales is independent of the stocking decision y in the backorder setting. We show in the Appendix that r_T^α can be written in a compact form as

$$r_T(\alpha) = \int_0^T e^{-rt} \mathbb{E}\left[\alpha P_t \overline{\lambda}\left(\alpha P_t\right)\right] dt. \tag{2}$$

As in the case of expected sales function in (1), expected revenue depends on the whole price path between $[0, T]$ and is modulated by the modified intensity rate $\overline{\lambda}$.

The way we form customer demand and revenues by explicitly modeling arrival times is not common in inventory literature. Most existing models take demand as a random variable to be realized at the end of the sales period. We specifically use this particular setup to investigate the effect of price fluctuations at the time of customer arrivals. A similar approach with fixed prices can be seen in Grubbström (2010) who investigates a different version of the newsvendor problem where demand is generated by a compound renewal customer arrival process with no fixed selling horizon and sales continue until the retailer runs out of inventory.

2.3 Expected Total Profit

Assuming that there is no initial inventory, we can write the expected total profit as a function of markup α and order-up-to level y as

$$g(y, \alpha) = -P_0 y + r_T(\alpha) - \mathbb{E}\left[(b + P_T)\left(D_{N_T^\alpha} - y\right)^+\right] \tag{3}$$

where $x^+ \equiv \max\{x, 0\}$. Here the first term denotes the total purchase cost, the second term denotes the expected total discounted revenue and the last term denotes the backorder and repurchase costs. The objective of the decision maker is to solve

$$\max_{\alpha \in \mathbb{R}^+, y \in \mathbb{R}^+} g(y, \alpha)$$

by choosing a proportional markup $\alpha \in \mathbb{R}^+$ and an order-up-to level $y \in \mathbb{R}^+$. In the next section, we present a characterization of the form of the optimal inventory-markup pricing policy.

3 Optimal Order Quantity & Markup Price

In this section, we analyze the behavior of the expected profit function $g(y, \alpha)$ with respect to y and α and corresponding optimal inventory and markup pricing strategies. We begin by analyzing the optimal inventory policy for a fixed markup decision α.

3.1 Optimal Order Quantity for a Given Markup

In the following, we will use $P\left\{D_{N_T^\alpha} = y\right\}$ to denote the probability distribution function of $D_{N_T^\alpha}$ evaluated at y.

Theorem 1 *Given markup α, $g(y, \alpha)$ is concave in y and an order-up-to (base-stock policy) is optimal, i.e., it is optimal to order up to the optimal base-stock level*

$$y^*(\alpha) = \inf\left\{y : \mathbb{E}\left[(b + P_T)\, 1_{\left\{D_{N_T^\alpha} \leq y\right\}}\right] \geq b + \mathbb{E}[P_T] - P_0\right\} \tag{4}$$

if initial inventory is less than $y^*(\alpha)$; otherwise, it is optimal not to order.

Proof. The first and second derivatives of $g(y, a)$ with respect to y are given by

$$g_y(y, \alpha) = -P_0 + \mathbb{E}\left[(b + P_T)\, 1_{\left\{D_{N_T^\alpha} \geq y\right\}}\right] = -P_0 + b + \mathbb{E}[P_T] - \mathbb{E}\left[(b + P_T)\, 1_{\left\{D_{N_T^\alpha} < y\right\}}\right]$$

and

$$\begin{aligned} g_{yy}(y, \alpha) &= -\mathbb{E}\left[(b + P_T)\, \frac{\partial}{\partial y}\mathbb{E}\left[1_{\left\{D_{N_T^\alpha} < y\right\}} \mid \mathcal{P}\right]\right] = -\mathbb{E}\left[(b + P_T)\, \frac{\partial}{\partial y} P\left\{D_{N_T^\alpha} < y \mid \mathcal{P}\right\}\right] \\ &= -\mathbb{E}\left[(b + P_T)\, P\left\{D_{N_T^\alpha} = y \mid \mathcal{P}\right\}\right]. \end{aligned} \tag{5}$$

As (5) is negative for all y for given markup, expected profit function is concave and a base-stock policy is optimal. For each markup level α, the optimal base-stock level is the maximizer of $g(y, \alpha)$ which is found by,

$$\begin{aligned} y^*(\alpha) &= \inf\left\{y : -P_0 + b + \mathbb{E}[P_T] - \mathbb{E}\left[(b + P_T)\, 1_{\left\{D_{N_T^\alpha} < y\right\}}\right] \leq 0\right\} \\ &= \inf\left\{y : \mathbb{E}\left[(b + P_T)\, 1_{\left\{D_{N_T^\alpha} \leq y\right\}}\right] \geq b + \mathbb{E}[P_T] - P_0\right\}. \end{aligned}$$

■

The concavity of the objective function ensures that an order-up-to inventory policy is optimal given markup α and the optimal base-stock level is given by (4). Note that this is a different version of the newsvendor critical fractile solution where the cumulative distribution of demand cannot be separated from the random purchase costs at time T since total demand $D_{N_T^\alpha}$ and final price P_T are dependent. If one can take alternative assumptions that break this dependence (such as demand being independent from price), the critical fractile would appear clearly.

The structure of the optimal order-up-to level allows us to see the impact of customer demand process. First, the larger the intensity function λ for a given price process P, the larger the optimal order quantity. This is due to the fact that the total demand increases in a stochastic sense for a given price process and inventory level y. Hence, the left hand side of the inequality in (4) decreases and the smallest y that satisfies it increases. The same is true for customers' mean demand function μ. That is, the more customers demand (in expectation) at each arrival, the larger inventory the retailer should hold.

3.2 Optimal Markup for a Given Inventory Level

In this section, we analyze the behavior of the expected profit function and find the optimal markup policy for a given inventory level. First, we make some reasonable assumptions and prove a series of results that will lead to our main characterization. Analogues of these assumptions in models without price volatilities are quite common in pricing literature (see Ziya et al., 2004).

Assumption 1 *The revenue rate $x\overline{\lambda}(x)$ is concave.*

Assumption 1 is standard in price optimization literature which ensures that the revenue function is well-behaved. In our case, we assume that it holds for the revenue rate function. This assumption supports one of the most important models: the linear case where $\overline{\lambda}(p) = \Lambda(1 - bp)^+$. It also supports many other plausible models.

Assumption 2 *$\overline{\lambda}(x)$ is non-increasing and convex.*

The first part of Assumption 2 states that the demand arrival rate is non-increasing in price which is the standard starting assumption in price optimization literature. For the convexity of the demand arrival rate, many standard models starting with the linear case and the exponential case ($\lambda(p) = \Lambda e^{-bp}$) also satisfy this assumption. This also would be the case if the price response function is estimated from data by a linear regression or a log-regression.

The next two lemmas establish that the total expected revenue is concave and expected total sales is convex in markup for a given inventory level.

Lemma 1 *Expected total discounted revenue $r_T(\alpha)$ is concave in markup α.*

Proof. Since $x\overline{\lambda}(x)$ is concave, $\alpha P_t \lambda(\alpha P_t)$ is concave in α for each P_t which makes $e^{-rt}\mathbb{E}\left[\alpha P_t \lambda(\alpha P_t)\right]$, hence $r_T(\alpha)$ given in (2) concave in α. ■

Lemma 1 asserts that the expected total discounted revenue has decreasing marginal returns in α implying that the markup price optimization should have a reasonable solution.

Lemma 2 *Expected total demand (sales) $d_T(\alpha)$ is non-increasing and convex in markup α.*

Proof. Since $\overline{\lambda}(.)$ is non-increasing and convex, $\overline{\lambda}(\alpha P_t)$ is also non-increasing and convex for each P_t which makes $\mathbb{E}\left[\overline{\lambda}(\alpha P_t)\right]$, hence $d_T(\alpha)$ given in (1) non-increasing and convex in α. ■

Lemma 2 presents an analogous result to Lemma 1, this time in terms of the expected total demand. The two lemmas establish that the assumptions on the demand (modified intensity) and the revenue rates propagate to expected demand and expected revenue.

From now on, we use the notation $\triangle\mathbb{E}\left[(y-D_k)^+\right] = \mathbb{E}\left[(y-D_{k+1})^+ - (y-D_k)^+\right]$ and $\triangle^2\mathbb{E}\left[(y-D_k)^+\right] = \mathbb{E}\left[(y-D_{k+2})^+ - 2(y-D_{k+1})^+ + (y-D_k)^+\right]$ to denote the first and second order forward differences of the expected leftover inventory function with respect to the number of customer arrivals. We will also make use of the following lemma in our subsequent analysis.

Lemma 3 $\mathbb{E}\left[(y-D_k)^+\right]$ *and* $\mathbb{E}\left[(D_k-y)^+\right]$ *are integer convex in k.*

Proof. For any discrete function to be integer convex, second order forward differences should be positive. Note that,

$$\triangle\mathbb{E}\left[(y-D_k)^+\right] = \mathbb{E}\left[(y-D_{k+1})^+ - (y-D_k)^+\right] = \mathbb{E}\left[(y-D_k-X_{k+1})^+ - (y-D_k)^+\right].$$

Using $(a-b)^{+}=a-\min\{a,b\}$ for any $a,b\in\mathbb{R}$, we can write

$$\triangle\mathbb{E}\left[(y-D_k)^{+}\right]=-\mathbb{E}\left[\min\left\{X_{k+1},(y-D_k)^{+}\right\}\right]. \tag{6}$$

As k increases, $-\min\left\{X_{k+1},(y-D_k)^{+}\right\}$ increases so that the second order difference with respect to k is nonnegative, i.e., $\triangle^2\mathbb{E}\left[(y-D_k)^{+}\right]\geq 0$. Similarly, $\mathbb{E}\left[(D_k-y)^{+}\right]=\mathbb{E}\left[D_k-y+(y-D_k)^{+}\right]$ is also integer convex in k. ■

In the following theorem, we show the concavity of the expected profit function with respect to markup variable α using these two lemmas. For this, we assume that mean demand function is constant at μ. This implies that demand quantity X is independent of the price with $\mathbb{E}[X_n]=\mu$ for every $n\geq 1$. In the following characterizations, $(M_T^{\alpha})'=\frac{\partial}{\partial\alpha}M_T^{\alpha}$, $(M_T^{\alpha})''=\frac{\partial^2}{\partial\alpha^2}M_T^{\alpha}$, $r_T'(\alpha)=\frac{\partial}{\partial\alpha}r_T^{\alpha}$ and $r_T''(\alpha)=\frac{\partial^2}{\partial\alpha^2}r_T^{\alpha}$ denote first and second order derivatives of M_T^{α} and r_T^{α}, respectively.

Theorem 2 *For any fixed order-up-to level y, $g(y,a)$ is concave in α and optimal markup $\alpha^*(y)$ is found by solving*

$$r_T'(\alpha)-\mu\mathbb{E}\left[(b+P_T)(M_T^{\alpha})'\right]+\mathbb{E}\left[(b+P_T)(M_T^{\alpha})'\min\left\{\left(y-D_{N_T^{\alpha}}\right)^{+},X_{N_T^{\alpha}+1}\right\}\right]=0. \tag{7}$$

Proof. It is shown in the Appendix that the first and second order partial derivatives of $g(y,a)$ with respect to α are given by

$$g_{\alpha}(y,\alpha)=r_T'(\alpha)-\mu\mathbb{E}\left[(b+P_T)(M_T^{\alpha})'\right]+\mathbb{E}\left[(b+P_T)(M_T^{\alpha})'\min\left\{\left(y-D_{N_T^{\alpha}}\right)^{+},X_{N_T^{\alpha}+1}\right\}\right]$$

and

$$\begin{aligned} g_{\alpha\alpha}(y,\alpha) &= r_T''(\alpha)-\mu\mathbb{E}\left[(b+P_T)(M_T^{\alpha})''\right]-\mathbb{E}\left[(b+P_T)(M_T^{\alpha})''\,\mathbb{E}\left[\triangle\left(y-D_{N_T^{\alpha}}\right)^{+}\mid\mathcal{P}\right]\right]\\ &\quad -\mathbb{E}\left[(b+P_T)\left((M_T^{\alpha})'\right)^2\mathbb{E}\left[\triangle^2\left(y-D_{N_T^{\alpha}}\right)^{+}\mid\mathcal{P}\right]\right],\end{aligned}$$

respectively. By Lemma 1 and Lemma 2, $r_T''(\alpha)\leq 0$ and $(M_T^{\alpha})''\geq 0$. Additionally, by Lemma 3, the last term is also negative. Observe also that,

$$\mathbb{E}\left[\triangle\left(y-D_{N_T^{\alpha}}\right)^{+}\mid\mathcal{P}\right]=-\mathbb{E}\left[\min\left\{X_{N_T^{\alpha}+1},\left(y-D_{N_T^{\alpha}}\right)^{+}\right\}\mid\mathcal{P}\right]\geq-\mathbb{E}\left[X_{N_T^{\alpha}+1}\right]=-\mu.$$

Then the following inequality holds:

$$\begin{aligned} g_{\alpha\alpha}(y,\alpha) &\leq r_T''(\alpha) - \mu\mathbb{E}\left[(b+P_T)(M_T^\alpha)''\right] + \mu\mathbb{E}\left[(b+P_T)(M_T^\alpha)''\right] \\ &\quad -\mathbb{E}\left[(b+P_T)\left((M_T^\alpha)'\right)^2\mathbb{E}\left[\triangle^2\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \\ &= r_T''(\alpha) - \mathbb{E}\left[(b+P_T)\left((M_T^\alpha)'\right)^2\mathbb{E}\left[\triangle^2\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \leq 0. \end{aligned}$$

The last inequality is due to Lemma 3 and $r_T''(\alpha) \leq 0$. Since $g_{\alpha\alpha}(y,\alpha) \leq 0$, expected total profit is concave in markup α for each inventory level y. We can find the optimal markup by setting the first partial derivative with respect to α equal to zero, i.e., $g_\alpha(y,\alpha) = 0$, which is given in (7) . ■

Establishing concavity (in terms of α) is not a trivial issue in pricing and Theorem 2 proves that this can be done in fairly general terms based solely on some standard pricing assumptions on the demand arrival rate function in the case where demand quantities are independent of the price. Unfortunately, concavity is not guaranteed in the fully general case where both customers' arrival rate and demand depend on the continuous price process and when the markup α is assumed to be continuous. We should however note that if the markup is chosen from a finite discrete set of options (i.e 5%, 10% or 20%), the markup optimization problem for the general model where the demand quantity also depends on the price can be solved by evaluating each of these markup options for the corresponding optimal order quantity using Theorem 1.

By Theorems 1 and 2, we characterize the optimality equations for the base-stock level y and markup level α. An interesting issue is to understand how the two decisions interact. The next proposition proves that the expected profit function is submodular in the two decision variables which leads to useful monotonicity properties for the optimal decisions.

Proposition 1 *$g(y,a)$ is submodular. Moreover, optimal inventory level $y^*(\alpha)$ is decreasing in α and optimal markup $\alpha^*(y)$ is decreasing in y.*

Proof. We show the submodularity of $g(y,\alpha)$ by showing that $g_{y\alpha}(y,\alpha) < 0$. Note that

$$g_{y\alpha}(y,\alpha) = -\mathbb{E}\left[(b+P_T)(M_T^\alpha)'\left(P\left\{D_{N_T^\alpha}+1 < y \mid \mathcal{P}\right\} - P\left\{D_{N_T^\alpha} < y \mid \mathcal{P}\right\}\right)\right]$$

as shown in the Appendix. Since $(M_T^\alpha)' \leq 0$ and $P\left\{D_{N_T^\alpha+1} \leq y \mid \mathcal{P}\right\} \leq P\left\{D_{N_T^\alpha} \leq y \mid \mathcal{P}\right\}$, derivative of the expected profit function with respect to each variable is negative, i.e., $g_{y\alpha}(y,\alpha) < 0$.

Moreover, in (4), as α increases $1_{\left\{D_{N_T^\alpha}<y\right\}}$ increases for fixed y which results in lower $y^*(\alpha)$. Similarly, since $g_{y\alpha}(y,\alpha)<0$, $g_\alpha(y,\alpha)$ is lower for higher values of y that is $g_\alpha(y,\alpha)$ is decreasing in y. Therefore, $\alpha^*(y)$ is lower for higher values of y. ■

Proposition 1 establishes that the optimal base-stock level is decreasing in the markup and the optimal markup is decreasing in the base-stock level. Both properties are known to be true for simpler inventory/pricing models (Petruzzi and Dada, 1999). Here, we show that they continue to hold for the more complicated case of fluctuating prices.

This section focused on the optimal ordering and markup decisions for a given price process. In the next section, we analyze the effect of volatile market prices on the expected revenues, sales and profits.

4 The Impact of Price Process on Performance: Monotonicity and Variability

In our model, we use a general stochastic price process which is assumed to have non-negative price paths. We now analyze how different performance metrics of the inventory system change with respect to the monotonicity as well as the variability of the price process. Our goal is to obtain general results with few specifications. To this end, to make comparisons, we use some tools from the stochastic and convex ordering of random variables and stochastic processes. The following definitions were taken from Müller and Stoyan (2002).

Definition 1 *Let X and Y denote two generic random variables. X is said to precede Y in convex (stochastic, increasing convex) order if $\mathbb{E}[f(X)]\leq\mathbb{E}[f(Y)]$ for all convex (increasing, increasing convex) functions f, i.e.,*

$$X \underset{cx(st,icx)}{\leq} Y \Leftrightarrow \mathbb{E}[f(X)]\leq\mathbb{E}[f(Y)]$$

for all convex (increasing, increasing convex) functions f.

Similar to convex (stochastic) ordering of random variables, two stochastic processes are said to be convexly (or stochastically) ordered if random values at each time are convexly (stochastically) ordered as the next definition states.

Definition 2 *Let $X = \{X_t; t \geq 0\}$ and $Y = \{Y_t; t \geq 0\}$ denote two stochastic processes. Then,*

$$X \underset{cx(st,icx)}{\leq} Y \Leftrightarrow \mathbb{E}\left[f\left(X_t\right)\right] \leq \mathbb{E}\left[f\left(Y_t\right)\right]$$

for all $t \geq 0$ and for all convex (increasing, increasing convex) functions f.

A practical implication of convex orders is that if a random variable Y is greater than X in terms of convex order, Y has a higher variance then X while having the same expected value. This is particularly important for our purposes in investigating the impact of price volatility on the inventory system. In particular, in our analysis, we will use two market price processes, which we denote as $\mathcal{P}^{(1)} = \left\{P_t^{(1)}; t \geq 0\right\}$ and $\mathcal{P}^{(2)} = \left\{P_t^{(2)}; t \geq 0\right\}$, to compare the expected revenues, profits and sales. Assuming that one price process is larger in terms of convex order ensures that the prices for both processes are equal in expectation at each time point so that effects due to the drift of prices are isolated and we can focus solely on the impact of price variability. This is especially important as the entire price path during the sales season modulates customers' arrival frequency as well as their demand amount.

The convex ordering condition in Definition 2 is generally straightforward to check for many widely used stochastic price processes. In many cases, it is sufficient to compare the respective process parameters. For instance, consider two Brownian motions $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ where $P_t^{(i)} = p_0^{(i)} + \mu^{(i)} t + \sigma^{(i)} B_t^{(i)}$ with drift $\mu^{(i)}$ and volatility $\sigma^{(i)}$ for $i = \{1, 2\}$. $B_t^{(i)}$ is a standard Brownian motion (Wiener) process with $\mathbb{E}[B_t^{(i)}] = 0$ and $Var(B_t^{(i)}) = t$. As $P_t^{(i)}$ is Normally distributed with mean $p_0^{(i)} + \mu t$ and variance $(\sigma^{(i)})^2 t$, $P_t^{(1)} \underset{cx}{\leq} P_t^{(2)}$ if $p_0^{(1)} = p_0^{(2)}$, $\mu^{(1)} = \mu^{(2)}$ and $\sigma^{(1)} \leq \sigma^{(2)}$ (Müller and Stoyan, 2002). Since this is true for any $t \geq 0$, $\mathcal{P}^{(1)} \underset{cx}{\leq} \mathcal{P}^{(2)}$ by Definition 2. In other words, if two Brownian motions have the same initial price and drift, then the one with the larger volatility parameter is also larger in terms of convex order (i.e., more variable). This comparison technique is also valid for the geometric Brownian motion process where prices at each time point are Lognormally distributed.

To make comparisons across several performance measures based on price processes $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$, we denote the corresponding arrival rate (intensity) processes as $\Lambda^{(i)}$ where $\Lambda^{(i)} = \{\Lambda_t^{(i)} = \lambda(\alpha P_t^{(i)}); t \geq 0\}$ and corresponding individual customer arrival processes as $\mathcal{N}^{(i)} = \{N_t^{(i)}; t \geq 0\}$ for $i = \{1, 2\}$. We also use $r_T^{(i)}(\alpha)$, $d_T^{(i)}(\alpha)$ and $g^{(i)}(y, \alpha)$ to denote the expected revenue, expected sales and expect profit functions under market price process $\mathcal{P}^{(i)}$.

4.1 The Effect of Price Process Monotonicity

We start our analysis with the impact of price monotonicity and establish first that stochastically higher prices lead to lower expected sales for a fixed markup α.

Proposition 2 *If $\mathcal{P}^{(1)} \underset{st}{\leq} \mathcal{P}^{(2)}$, then $d_T^{(1)}(\alpha) \geq d_T^{(2)}(\alpha)$ for each $\alpha \in \mathbb{R}^+$.*

Proof. As $\overline{\lambda}$ is non-increasing, by Definition 1 we have $\mathbb{E}\left[\overline{\lambda}\left(\alpha P_t^{(1)}\right)\right] \geq \mathbb{E}\left[\overline{\lambda}\left(\alpha P_t^{(2)}\right)\right]$ for each $t \geq 0$. Then it follows that

$$d_T^{(1)}(\alpha) = \int_0^T \mathbb{E}\left[\overline{\lambda}\left(\alpha P_t^{(1)}\right)\right] dt \geq d_T^{(2)}(\alpha) = \int_0^T \mathbb{E}\left[\overline{\lambda}\left(\alpha P_t^{(2)}\right)\right] dt$$

■

The basic intuition behind Proposition 2 is that as market prices are more likely to be higher during the sales season, total customer demand will more likely be lower as the modified intensity rate $\overline{\lambda}$ is non-increasing in price. However, this does not necessarily mean that customers will arrive less frequently as prices rise. Note that other than nonnegativity, we do not make any other specific assumptions on the arrival intensity function λ and the mean demand function μ. An interesting implication of this is that, Proposition 2 may still hold if customers arrive more (less) frequently but individually demand less (more) in expectation when prices are high. The only required condition is the modified arrival rate $(\lambda\mu)$ being non-increasing in price.

Although stochastically higher input prices lead to lower expected demand (hence sales), this does not necessarily translate to lower optimal order quantities for the retailer. This is due to stochastically higher repurchase cost at the end of the sales season which induces the retailer to order more. The conflicting effects of prices on the total demand and the repurchase cost can be seen in the optimal order quantity condition given in (4). It is not clear which impact is dominating in this general setting where demand is influenced by the entire price path during the sales season. On the other hand, in the special case where the repurchase cost is fixed and independent of the price process, a stochastically larger price process leads to lower optimal order quantities. This could be especially relevant if the retailer has made an agreement (such as an option contract) with the supplier to repurchase items at a fixed price at the end of sales season.

4.2 The Effect of Price Process Variability

We now analyze the impact of price variability on different performance measures. We start with expected revenues and expected sales.

Proposition 3 *If $\mathcal{P}^{(1)} \underset{cx}{\leq} \mathcal{P}^{(2)}$, then $d_T^{(1)}(\alpha) \leq d_T^{(2)}(\alpha)$ and $r_T^{(1)}(\alpha) \geq r_T^{(2)}(\alpha)$ for each $\alpha \in \mathbb{R}^+$.*

Proof. Since $\overline{\lambda}(\alpha P_t)$ is convex in P_t by Assumption 2, for any $t \geq 0$, $P_t^{(1)} \underset{cx}{\leq} P_t^{(2)}$ implies $\mathbb{E}\left[\overline{\lambda}\left(\alpha P_t^{(1)}\right)\right] \leq \mathbb{E}\left[\overline{\lambda}\left(\alpha P_t^{(2)}\right)\right]$. Then, it follows that

$$d_T^{(1)}(\alpha) = \int_0^T \mathbb{E}\left[\overline{\lambda}\left(\alpha P_t^{(1)}\right)\right] dt \leq \int_0^T \mathbb{E}\left[\overline{\lambda}\left(\alpha P_t^{(2)}\right)\right] dt = d_T^{(2)}(\alpha).$$

Similarly, since $\alpha P_t \overline{\lambda}(\alpha P_t)$ is concave in P_t by Assumption 1, $P_t^{(1)} \underset{cx}{\leq} P_t^{(2)}$ implies $\mathbb{E}\left[\alpha P_t^{(1)} \overline{\lambda}\left(\alpha P_t^{(1)}\right)\right] \geq \mathbb{E}\left[\alpha P_t^{(2)} \overline{\lambda}\left(\alpha P_t^{(2)}\right)\right]$ for any $t \in [0, T]$. Then it follows that

$$r_T^{(1)}(\alpha) = \int_0^T e^{-rt} \mathbb{E}\left[\alpha P_t^{(1)} \overline{\lambda}\left(\alpha P_t^{(1)}\right)\right] dt \geq \int_0^T e^{-rt} \mathbb{E}\left[\alpha P_t^{(2)} \overline{\lambda}\left(\alpha P_t^{(2)}\right)\right] dt = r_T^{(2)}(\alpha).$$

■

Proposition 3 establishes an interesting result that a more volatile price process leads to lower (weakly) expected revenues and higher (weakly) expected total sales for a given markup level. The intuition is directly related to how revenue and sales functions react to price variations at different price levels. More specifically, marginal changes in prices when they are low have a larger impact on sales compared to the case where prices are high. This is since the sales function is convex and non-increasing in markup level (hence in price for a given markup level). As prices at both ends are more likely to be realized for a more volatile price process (compared to a less volatile price process), what is gained on the lower end of prices in terms of sales outweighs what is lost on the higher end of prices. This leads to higher sales in expectation. A similar intuition holds for the expected revenues, but the effects are reversed. As revenue is concave in prices, larger price variations lead to lower revenues in expectation.

In the following, we prove a series of results that, along with Proposition 3, enable us to state our main theorem in this section. The next lemma proves that if two market price processes are convexly ordered, their corresponding rate processes are also convexly ordered, however only directionally. For this part, we assume that rate function λ is convex.

Lemma 4 *If* $\mathcal{P}^{(1)} \underset{cx}{\leq} \mathcal{P}^{(2)}$, *then* $\Lambda^{(1)} \underset{icx}{\leq} \Lambda^{(2)}$.

Proof. Let $t \geq 0$ be fixed and f be an increasing convex function. Then, $f(\lambda)$ is convex since $f(\lambda)'' = \lambda'' f'(\lambda) + (\lambda')^2 f''(\lambda) \geq 0$ as λ is convex and $f' \geq 0$ by assumption. Since $f(\lambda)$ is convex,

$$\mathbb{E}\left[f\left(\Lambda_t^{(1)}\right)\right] = \mathbb{E}\left[f\left(\lambda\left(\alpha P_t^{(1)}\right)\right)\right] \leq \mathbb{E}\left[f\left(\lambda\left(\alpha P_t^{(2)}\right)\right)\right] = \mathbb{E}\left[f\left(\Lambda_t^{(2)}\right)\right].$$

Since this is true for any convex increasing f and $t \geq 0$, $\Lambda^{(1)} \underset{icx}{\leq} \Lambda^{(2)}$. ■

Lemma 4 connects the volatility of the price processes to that of arrival rate functions. In particular, it states that a more variable price process leads to a more variable customer arrival rate process with a (weakly) higher expected value at each time point (by Definition 2)[1]. Similar to the result in Proposition 2, this is mainly due to the convexity of the rate function.

We next use the following proposition from Błaszczyszyn and Yogeshwaran (2009) that links the convex ordering of intensity measure Λ of a doubly-stochastic Poisson process to the counting measure $\mathcal{N}$.

Proposition 4 $\Lambda^{(1)} \underset{cx,icx}{\leq} \Lambda^{(2)}$ *implies* $\mathcal{N}^{(1)} \underset{cx,icx}{\leq} \mathcal{N}^{(2)}$.

Proposition 4 states that a more variable *rate* process yields a more variable arrival process in terms of the convex order. A direct result of this together with Lemma 4 is that a more variable *price* process leads to a more variable arrival process which is also weakly faster in expectation. This is stated in the following immediate corollary.

Corollary 1 $\mathcal{P}^{(1)} \underset{cx}{\leq} \mathcal{P}^{(2)}$ *implies* $\mathcal{N}^{(1)} \underset{icx}{\leq} \mathcal{N}^{(2)}$.

Corollary 1 ties the convex ordering of price process to a convex order of arrival counts. This is critical to extend the comparisons to expected profits where we need to handle the newsvendor mismatch cost functions. This is addressed in the following lemma.

[1]To see this, take $f(x) = x$. Then $E[\lambda(\alpha P_t^{(1)})] \leq E[\lambda(\alpha P_t^{(2)})]$ for all $t \geq 0$ since $\Lambda^{(1)} \underset{icx}{\leq} \Lambda^{(2)}$.

Lemma 5 *$\mathcal{P}^{(1)} \underset{cx}{\leq} P^{(2)}$ implies* $\mathbb{E}\left[P_T^{(1)}(D_{N_T^{\alpha(1)}} - y)^+\right] \leq \mathbb{E}\left[P_T^{(2)}(D_{N_T^{\alpha(2)}} - y)^+\right]$.

Proof. Assume that $\mathcal{P}^{(1)} \underset{cx}{\leq} \mathcal{P}^{(2)}$. Then by Corollary 1, $\mathcal{N}^{(1)} \underset{icx}{\leq} \mathcal{N}^{(2)}$. Together with this relationship, we will use the relationship between convex ordering of conditional random variables and convex ordering of their unconditional counterparts. It is known that if two random variables are convex ordered, so their conditional counterparts, Leskelä et al. (2017). Then, we can write

$$\mathcal{P}^{(1)} \underset{cx}{\leq} \mathcal{P}^{(2)} \Rightarrow \left\{P_t^{(1)}; t \in [0,T)\right\} \mid P_T \underset{cx}{\leq} \left\{P_t^{(2)}; t \in [0,T)\right\} \mid P_T \Rightarrow N_T^{(1)} \mid P_T \underset{icx}{\leq} N_T^{(2)} \mid P_T.$$

Now assume that P_T is given. Then,

$$\begin{aligned} \mathbb{E}\left[(D_{N_T^{\alpha(1)}} - y)^+ \mid P_T\right] &= \mathbb{E}\left[\mathbb{E}\left[(D_{N_T^{\alpha(1)}} - y)^+ \mid N_T^{\alpha(1)}\right] \mid P_T\right] \\ &\leq \mathbb{E}\left[\mathbb{E}\left[(D_{N_T^{\alpha(2)}} - y)^+ \mid N_T^{\alpha(2)}\right] \mid P_T\right] = \mathbb{E}\left[(D_{N_T^{\alpha(2)}} - y)^+ \mid P_T\right]. \end{aligned}$$

This is due to the fact that $\mathbb{E}\left[(D_{N_T^\alpha} - y)^+ \mid N_T^\alpha\right]$ is an increasing convex function of N_T^α by Lemma 3. Then it follows that

$$\begin{aligned} \mathbb{E}\left[P_T^{(1)}(D_{N_T^{\alpha(1)}} - y)^+\right] &= \mathbb{E}\left[P_T^{(1)}\mathbb{E}\left[(D_{N_T^{\alpha(1)}} - y)^+ \mid P_T^{(1)}\right]\right] \\ &\leq \mathbb{E}\left[P_T^{(2)}\mathbb{E}\left[(D_{N_T^{\alpha(2)}} - y)^+ \mid P_T^{(2)}\right]\right] = \mathbb{E}\left[P_T^{(2)}(D_{N_T^{\alpha(2)}} - y)^+\right]. \end{aligned}$$

■

Lemma 5 is a major result proving that a more volatile price process in terms of convex order leads to higher expected mismatch costs between supply and demand for the same markup and starting inventory. This is since the mismatch cost is convex in customer demand and a more variable price process yields a more variable customer arrival process.

We are now ready to state the following theorem that uses the previous lemmas and is the main result of this section.

Theorem 3 *If $\mathcal{P}^{(1)} \underset{cx}{\leq} \mathcal{P}^{(2)}$, then $g^{(1)}(y,\alpha) \geq g^{(2)}(y,\alpha)$ for all $y \geq 0$ and $\alpha \geq 0$.*

Proof. The result follows by Proposition 3 and Lemma 5 considering the expected profit function in (3) for a given inventory level y and markup α. ■

Theorem 3 establishes that a more volatile price process leads to lower expected profits for a given markup and starting inventory under general conditions. Please note that no specification of the price process is necessary for the result (apart from the fact that the price processes have to be comparable). Next, we can extend the result to optimal expected profits using first principles of optimization.

Corollary 2 *Let* $g^{(i)*} = \max_{(\alpha,y)\in\mathbb{R}^+\times\mathbb{R}^+} g^{(i)}(y;\alpha)$ *denote the optimal expected profit under market price process* $\mathcal{P}^{(i)}$*. If* $\mathcal{P}^{(1)} \underset{cx}{\leq} \mathcal{P}^{(2)}$*, then* $g^{(1)*} \geq g^{(2)*}$*.*

Proof. Let $y^{*(i)}$ and $\alpha^{*(i)}$ denote the optimal order-up-to level and markup under price process $\mathcal{P}^{(i)}$, $i = \{1, 2\}$. Since $\mathcal{P}^{(1)} \underset{cx}{\leq} \mathcal{P}^{(2)}$, we have $g^{(1)}(y^{*(2)}, \alpha^{*(2)}) \geq g^{(2)}(y^{*(2)}, \alpha^{*(2)})$ by Theorem 3. By optimality, $g^{(1)}(y^{*(1)}, \alpha^{*(1)}) \geq g^{(1)}(y^{*(2)}, \alpha^{*(2)})$ which gives $g^{(1)}(y^{*(1)}, \alpha^{*(1)}) \geq g^{(2)}(y^{*(2)}, \alpha^{*(2)})$ ■

We conclude that more volatile price processes lead to lower optimal expected profits under joint markup pricing and ordering. This is certainly of interest for the model in this paper and its special cases and we hope that the toolbox we use to establish the comparisons based on convex ordering of stochastic processes may be useful to other research that investigates inventory problems with random prices.

5 Conclusion

In this paper, we investigate a single-period joint inventory/pricing problem of a retailer that faces stochastic price volatilities. Unlike classical single-period models, selling prices, customer arrival rates and demand amounts are affected by a continuous-time stochastic price process. Our contributions can be summarized as follows:

- We extend the single-period inventory modeling framework to take into account situations where the sales revenue depends on the arrival time of customers during the season. In addition, the demand arrival rates and amounts are modulated by a continuous external process.

- We establish that the optimal ordering policy is an order-up-to policy for a fixed markup level and optimal markup level can be characterized for a fixed inventory level. In addition, we prove that as the inventory level increases, the optimal markup decreases and as the markup increases, the optimal base-stock level decreases.

- By using stochastic orders for processes, we establish comparison results for two ordered price processes. In particular, we show that as the price process becomes more volatile (variable) the expected revenues and profits decrease while the expected sales increase. To our knowledge, the use of this class of stochastic orders is new for joint inventory/pricing models.

Since our modeling assumptions are quiet general, many other relevant cases can be covered with some modifications. For instance, if the price process is constant and demand is not price dependent, our model becomes an alternate version of Grubbström (2010) but we use a probabilistic analysis not requiring Laplace transforms. If, in addition to the constant price assumption, demand is price dependent, then markup pricing is equivalent to regular static pricing. The resulting model would be a version of Grubbström (2010) with joint pricing. With additional modifications, we can also capture demand dependence on other external processes than the price. For instance, the demand arrival rate could depend on the random weather conditions (temperature, rainfall etc.).

Some other interesting extensions of the model requiring additional work are to consider the case with lost sales (instead of backorders) and a multi-period version of the joint pricing and ordering problem. Both cases present non-trivial technical challenges. Other potential ideas for future work include dynamic markup pricing in the presence of input price fluctuations and downside risk management issues.

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6 Appendix

Let $\mathcal{P} = \{P_t : t \in [0,T]\}$. Then

$$\begin{aligned} r_T(\alpha) &= \mathbb{E}\left[\sum_{n=1}^{N_T^\alpha} e^{-rT_n}\alpha P_{T_n}X_n\right] = \mathbb{E}\left[\sum_{n=1}^{N_T^\alpha} e^{-rT_n}\alpha P_{T_n}\mu\left(\alpha P_{T_n}\right)\right] \\ &= \mathbb{E}_{\mathcal{P}}\left[\mathbb{E}\left[\sum_{n=1}^{N_T^\alpha} e^{-rT_n}\alpha P_{T_n}\mu\left(\alpha P_{T_n}\right)|\mathcal{P}\right]\right] \\ &= \mathbb{E}_{\mathcal{P}}\left[\mathbb{E}_{N_T^\alpha|\mathcal{P}}\left[\sum_{k\geq 0} P\left\{N_T^\alpha = k|\mathcal{P}\right\}\mathbb{E}\left[\sum_{n=1}^{k} e^{-rT_n}\alpha P_{T_n}\mu\left(\alpha P_{T_n}\right)|\mathcal{P}, N_T^\alpha = k\right]\right]\right] \end{aligned}$$

where the second equality is due to $E[\xi] = 0$. Note that conditioned on $N_T^\alpha = k$ and $\mathcal{P}$, T_n is the order statistics of k i.i.d. random variables on $[0,T]$ with cumulative distribution function

$$\Phi(t) = \frac{\int_0^t \lambda(\alpha P_u)du}{\int_0^T \lambda(\alpha P_u)du} \tag{8}$$

and probability distribution function

$$\phi(t) = \Phi'(t) = \frac{\lambda(\alpha P_t)}{\int_0^T \lambda(\alpha P_u)du}.$$

Then

$$r_T(\alpha) = \mathbb{E}_{\mathcal{P}}\left[\mathbb{E}\left[\sum_{k\geq 0} P\left\{N_T^\alpha = k|\mathcal{P}\right\}k\mathbb{E}\left[e^{-r\widetilde{T}}\alpha P_{\widetilde{T}}\mu\left(\alpha P_{\widetilde{T}}\right)|\mathcal{P}\right]\right]\right]$$

where $\widetilde{T}$ is a r.v. with distribution Φ given in (8). Since

$$\sum_{k\geq 0} P\left\{N_T^\alpha = k|\mathcal{P}\right\}k = \mathbb{E}\left[N_T^\alpha|\mathcal{P}\right] = \int_0^T \lambda(\alpha P_u)du$$

we have,

$$
\begin{aligned}
r_T(\alpha) &= \mathbb{E}_{\mathcal{P}}\left[\mathbb{E}\left[N_T^\alpha|\mathcal{P}\right]\mathbb{E}\left[e^{-r\widetilde{T}}\alpha P_{\widetilde{T}}\mu\left(\alpha P_{\widetilde{T}}\right)|\mathcal{P}\right]\right] \\
&= \mathbb{E}_{\mathcal{P}}\left[\mathbb{E}\left[N_T^\alpha|\mathcal{P}\right]\frac{\int_0^T e^{-rt}\alpha P_t\mu\left(\alpha P_t\right)\lambda\left(\alpha P_t\right)dt}{\int_0^T\lambda\left(\alpha P_u\right)du}\right] \\
&= \mathbb{E}\left[\int_0^T e^{-rt}\alpha P_t\mu\left(\alpha P_t\right)\lambda\left(\alpha P_t\right)dt\right] = \int_0^T e^{-rt}E[\alpha P_t\overline{\lambda}\left(\alpha P_t\right)]dt
\end{aligned}
$$

where $\overline{\lambda} = \lambda\mu$. In a similar manner, expected total demand during the sales season is

$$
d_T(\alpha) = \int_0^T e^{-rt}\mathbb{E}\left[\overline{\lambda}\left(\alpha P_t\right)\right]dt.
$$

Derivation of Partial Derivatives of $g(y,\alpha)$

The first order derivative of $g(y,\alpha)$ with respect to α is

$$
g_\alpha(y,\alpha) = \frac{\partial}{\partial\alpha}\mathbb{E}\left[R_T^\alpha - (b+P_T)D_{N_T^\alpha} - (b+P_T)(y-D_{N_T^\alpha})^+\right].
$$

Assume $\mu(x) = \mu$. Then, we find

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}\mathbb{E}\left[(b+P_T)D_{N_T^\alpha}\right] &= \frac{\partial}{\partial\alpha}\mathbb{E}\left[\mathbb{E}\left[(b+P_T)D_{N_T^\alpha}\mid\mathcal{P}\right]\right] = \mathbb{E}\left[(b+P_T)\frac{\partial}{\partial\alpha}\mathbb{E}\left[D_{N_T^\alpha}\mid\mathcal{P}\right]\right] \\
&= \mu\mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)'\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}\mathbb{E}\left[(b+P_T)(y-D_{N_T^\alpha})^+\right] &= \frac{\partial}{\partial\alpha}E\left[\mathbb{E}\left[(b+P_T)(y-D_{N_T^\alpha})^+\mid\mathcal{P}\right]\right] \\
&= \mathbb{E}\left[(b+P_T)\frac{\partial}{\partial\alpha}\mathbb{E}\left[(y-D_{N_T^\alpha})^+\mid\mathcal{P}\right]\right] \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}\mathbb{E}\left[(y-D_{N_T^\alpha})^+ \mid \mathcal{P}\right] &= \frac{\partial}{\partial\alpha}\sum_{k=0}^{\infty} P\left\{N_T^\alpha = k \mid \mathcal{P}\right\}\mathbb{E}\left[(y-D_k)^+\right] \\
&= \frac{\partial}{\partial\alpha}\sum_{k=0}^{\infty}\left(\frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^k}{k!}\mathbb{E}\left[(y-D_k)^+\right]\right) \\
&= \sum_{k=0}^{\infty}\left((M_T^\alpha)'\left(-\frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^k}{k!}+\frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^{k-1}}{(k-1)!}\right)\mathbb{E}\left[(y-D_k)^+\right]\right) \\
&= -\left(M_T^\alpha\right)'\sum_{k=0}^{\infty} P\left\{N_T^\alpha = k \mid \mathcal{P}\right\}\mathbb{E}\left[(y-D_k)^+\right] \\
&\quad +\left(M_T^\alpha\right)'\sum_{k=0}^{\infty} P\left\{N_T^\alpha = k \mid \mathcal{P}\right\}\mathbb{E}\left[(y-D_{k+1})^+\right].
\end{aligned}
$$

We defined earlier that $\triangle_k (y-D_k)^+ = (y-D_{k+1})^+ - (y-D_k)^+$. This makes

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}\mathbb{E}\left[(y-D_{N_T^\alpha})^+ \mid \mathcal{P}\right] &= (M_T^\alpha)'\sum_{k=0}^{\infty} P\left\{N_T^\alpha = k \mid \mathcal{P}\right\}\mathbb{E}\left[\triangle_k (y-D_k)^+\right] \\
&= (M_T^\alpha)'\,\mathbb{E}\left[\triangle\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] \qquad (10)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
g_\alpha(y,\alpha) &= r_T'(\alpha) - \mu\mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)'\right] - \mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)'\mathbb{E}\left[\triangle\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \\
&= r_T'(\alpha) - \mu\mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)'\right] - \mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)'\triangle\left(y-D_{N_T^\alpha}\right)^+\right] \\
&= r_T'(\alpha) - \mu\mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)'\right] + \mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)'\min\left\{\left(y-D_{N_T^\alpha}\right)^+, X_{N_T^\alpha+1}\right\}\right].
\end{aligned}
$$

By (9), second order derivative with respect to α is

$$
g_{\alpha\alpha}(y,\alpha) = r_T''(\alpha) - \mu\mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)''\right] - \mathbb{E}\left[(b+P_T)\frac{\partial^2}{\partial\alpha^2}\mathbb{E}\left[(y-D_{N_T^\alpha})^+ \mid \mathcal{P}\right]\right]
$$

where we can write by (10) that

$$
\frac{\partial^2}{\partial\alpha^2}\mathbb{E}\left[(y-D_{N_T^\alpha})^+ \mid \mathcal{P}\right] = \left(M_T^\alpha\right)''\mathbb{E}\left[\triangle\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] + \left(M_T^\alpha\right)'\frac{\partial}{\partial\alpha}\mathbb{E}\left[\triangle\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right].
$$

Performing a similar analysis for finding (10), one can write

$$\frac{\partial}{\partial \alpha}\mathbb{E}\left[\triangle\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] = \left(M_T^\alpha\right)' \mathbb{E}\left[\triangle^2\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]$$

which leads to

$$\frac{\partial^2}{\partial \alpha^2}\mathbb{E}\left[(y - D_{N_T^\alpha})^+ \mid \mathcal{P}\right] = \left(M_T^\alpha\right)'' \mathbb{E}\left[\triangle\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] + \left(\left(M_T^\alpha\right)'\right)^2 \mathbb{E}\left[\triangle^2\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]$$

and

$$\begin{aligned}
g_{\alpha\alpha}(y,\alpha) &= r_T''(\alpha) - \mu\mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)''\right] - \mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)''\mathbb{E}\left[\triangle\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \\
&\quad -\mathbb{E}\left[(b+P_T)\left(\left(M_T^\alpha\right)'\right)^2\mathbb{E}\left[\triangle^2\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \\
&= r_T''(\alpha) - \mu\mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)''\right] - \mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)''\triangle\left(y - D_{N_T^\alpha}\right)^+\right] \\
&\quad -\mathbb{E}\left[(b+P_T)\left(\left(M_T^\alpha\right)'\right)^2\triangle^2\left(y - D_{N_T^\alpha}\right)^+\right].
\end{aligned}$$

Finally, partial derivative with respect to each variable is

$$g_y(y,\alpha) = -P_0 + \mathbb{E}\left[(b+P_T)\,1_{\left\{D_{N_T^\alpha}\geq y\right\}}\right] = -P_0 + b + \mathbb{E}\left[P_T\right] - \mathbb{E}\left[(b+P_T)\,1_{\left\{D_{N_T^\alpha}< y\right\}}\right]$$

$$\begin{aligned}
\frac{\partial}{\partial \alpha}g_y(y,\alpha) &= -\frac{\partial}{\partial \alpha}\mathbb{E}\left[(b+P_T)\,1_{\left\{D_{N_T^\alpha}< y\right\}}\right] \\
&= -\mathbb{E}\left[(b+P_T)\frac{\partial}{\partial \alpha}\mathbb{E}\left[1_{\left\{D_{N_T^\alpha}< y\right\}} \mid \mathcal{P}\right]\right] \\
&= -\mathbb{E}\left[(b+P_T)\frac{\partial}{\partial \alpha}P\left\{D_{N_T^\alpha} < y \mid \mathcal{P}\right\}\right] \\
&= -\mathbb{E}\left[(b+P_T)\frac{\partial}{\partial \alpha}\sum_{k=0}^{\infty}\left(\frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^k}{k!}F^{(k)}(y)\right)\right] \\
&= -\mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)'\left(\sum_{k=0}^{\infty} - \frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^k}{k!}F^{(k)}(y) + \sum_{k=1}^{\infty}\frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^{k-1}}{(k-1)!}F^{(k)}(y)\right)\right] \\
&= -\mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)'\left(\sum_{k=0}^{\infty} - \frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^k}{k!}F^{(k)}(y) + \sum_{k=0}^{\infty}\frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^{k}}{k!}F^{(k+1)}(y)\right)\right] \\
&= -\mathbb{E}\left[(b+P_T)\left(M_T^\alpha\right)'\left(P\left\{D_{N_T^\alpha}+1 < y \mid \mathcal{P}\right\} - P\left\{D_{N_T^\alpha} < y \mid \mathcal{P}\right\}\right)\right]
\end{aligned}$$