A Note on the Influence of Variability in Make-to-Stock Queues

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1. Introduction

This note is intended as a complement to Jemai and Karaesmen [1] which investigates the influence of inter-arrival time variability in a GI/M/1 Make-to-Stock queue. The primary result therein is an ordering of optimal base stock levels and costs. In particular, it is shown that when two GI/M/1 make-to-stock queues with identical parameters except for the demand inter-arrival time distributions are compared, the system with the less variable (in terms of the convex stochastic order) inter-arrival time distribution has lower optimal base-stock levels and costs. Below, we present some approximate extensions to this comparison under different assumptions.

2. Extensions to Other Systems

2.1 Processing Time Variability: An Extension to M/G/1 Make-to-Stock queues

In this section, we mention some parallel results on an M/G/1 make-to-stock queue where the demand arrivals occur according to a Poison process and the influence of variability is in processing times. Processing times now have a general (independent) distribution function with rate μ with the standard stability condition $\rho=\lambda/\mu$ <1. Unfortunately, in this case, the stationary queue length distribution cannot be expressed in terms of a single parameter (unlike in the GI/M/1 case) and few stochastic comparison results exist for the queue length process. On the other hand, if we focus on a particular regime where the ratio *b/h* is high, we can make use of accurate tail probability approximations based on asymptotic expansions to obtain approximate analytical results.

For instance, Tijms [2] proposes the following approximation which is empirically found to be extremely accurate. Let *f(x)* the density function of the processing time, *F(x)* its cumulative distribution function and *L(s)* its Laplace Transform. Under the assumption that

$\int_0^\infty e^{sx}(1-F(x))dx < \infty$ for some *s>0,* an asymptotic expansion yields the following distributional form for the *M/G/1* queue:

$$p_j \approx \sigma \times \tau^{-j} \qquad \text{for } j \text{ large}$$

where τ is the unique solution of the equation $\lambda\int_0^\infty e^{-\lambda(1-\tau)x}(1-F(x))dx = 1$ on the interval $(1,1+B/\lambda)$; $B = \sup\left[s \middle| \int_0^\infty e^{sx}(1-F(x))dx < \infty\right]$. Furthermore, letting $\eta=1/\tau$, σ can be approximated by: $\sigma = \rho \times (1-\eta)/\eta$. These approximations require that the service time distribution has no extremely long tail as is the case in most situations of practical interest.

Using then the fact that $F(\infty)=1$, the stationary distribution of the *M/G/1* queue is approximated by: $p_0 = 1-\rho$; $p_j = \rho \times (1-\eta) \times \eta^{j-1}$. Note that under this approximation, the steady-state probabilities of the *M/G/1* queue parameterized by η are the same as those for the *G/M/1* queue parametrized by r_0 when the utilization rates are identical.

In addition, τ is obtained from an identical computation to r_0, i.e., it is the unique root of the equation $L(\lambda(1-\tau)) = \tau$, in the interval $(1, 1+B/\lambda)$.

The optimal base stock level S^* and its associated cost $C(S^*)$ can then be expressed in terms of η. Recall that $F_N(\widetilde{S}^* - 1) = b/(h+b)$ and $S^* = \lfloor \widetilde{S}^* \rfloor$ thus:

$$S^* = \left\lfloor \frac{\log(1 - \frac{\alpha-(1-\rho)}{\rho})}{\log \eta} \right\rfloor,$$

where $\alpha = b/(b+h)$. The cost, for a given base-stock level is given by:

$$C(S) = h\left[S - \rho - \rho \times \eta \times \frac{1-\eta^{S-2}}{1-\eta} - \rho \times \eta^{S-1}\right] + b\left[\rho \times \frac{\eta^S}{1-\eta}\right].$$

In order to justify the use of the above approximation, note that for *b>>h, S** has to be large. This implies, in turn, that the computation of the optimal base stock level is a tail probability calculation in a regime where the tail asymptotics are highly accurate. Lemmas 1, 2, and 3, and Propositions 1 and 2 of Jemai and Karaesmen [1] then apply for M/G/1 make-to-stock queues. Increased variability in processing times then has exactly the same effects as increased variability in demand inter-arrival times.

2.2 The G/G/1 Make-to-Stock Queue in Heavy Traffic

When both demand inter-arrival and processing times are generally distributed, the stationary inventory level (or queue length) distribution cannot be expressed in closed-form. In addition, there no general stochastic comparison results that are based on second order properties of the underlying random variables. There is, however, one limiting regime in which explicit results can be obtained. Let us denote by cv_a and cv_s, the cv's of the arrival time and service distributions. In fact, through heavy traffic theory it is known that as the utilization rate (traffic load), ρ, approaches 1, the queue-length distribution in a G/G/1 queue weakly converges to a Reflected Brownian Motion with drift $\alpha=(\lambda-\mu)$ and variance $\sigma=\lambda \mathrm{cv}_a^2+\mu \mathrm{cv}_s^2$. The stationary distribution of this process is exponential with parameter $2\alpha/\sigma^2$. This leads to the following expressions for the optimal base stock level and the optimal cost of the Make-to-Stock Queue (see Wein [3] for a complete exposition):

$$S^* = \frac{\log\left(\frac{h}{h+b}\right)}{2\alpha/\sigma^2} \text{ and } C(S^*) = h\frac{\log\left(\frac{h}{h+b}\right)}{2\alpha/\sigma^2}$$

Variability effects are immediately visible from the above formulas. Both S^* and $C(S^*)$ are linearly increasing in both the demand and the processing time variance for fixed values of λ and μ.

References

[1] Z. Jemai and F. Karaesmen, « The Influence of Demand Variability on the Performance of a Make-to-Stock Queue», *Technical Report, Ecole Centrale Paris and Koç University*, 2003.

[2] H.C. Tijms, *Stochastic Models: An Algorithmic Approach*, John Wiley and Sons, New York, 1995.

[3] L.M. Wein, « Dynamic Scheduling of a Multiclass Make-to-Stock Queue », *Operations Research*, Vol. 40, 1992, pp.724-735.