Appendix A: Properties

A.1. Properties on the value function

i) and ii) Direct consequence of the definitions of $\mathtt{I}_{\boldsymbol{\alpha}}$, $\mathtt{D}_{\boldsymbol{\alpha}}$, $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\mathtt{S}^{ub}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

iii) We sum the two inequalities $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}v(\mathbf{x}+\boldsymbol{\gamma})\geq 0$ and $\Delta_{\boldsymbol{\gamma}}\Delta_{\boldsymbol{\beta}}v\geq 0$ to get $\Delta_{\boldsymbol{\alpha}+\boldsymbol{\gamma}}\Delta_{\boldsymbol{\beta}}v(\mathbf{x})\geq 0$.

A.2. Properties on the system state space

i) to v) Trivial

vi) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b})$ is equivalent to "for all $\mathbf{x}$ such that $\{\mathbf{x},\mathbf{x}+\mathbf{a}_1,\ldots,\mathbf{x}+\mathbf{a}_l\}\subset\mathcal{X}$, $\mathbf{x}+\mathbf{b}\in\mathcal{X}$". In this assertion we replace $\mathbf{x}$ by $\mathbf{x}+\mathbf{a}_l$ to obtain "for all $\mathbf{x}$ such that $\{\mathbf{x}-\mathbf{a}_l,\mathbf{x}+\mathbf{a}_1-\mathbf{a}_l,\ldots,\mathbf{x}\}\subset\mathcal{X}$, $\mathbf{x}+\mathbf{b}\in\mathcal{X}$'. So $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b})=\mathtt{R}_{-\mathbf{a}_l,\mathbf{a}_1-\mathbf{a}_l,\ldots,\mathbf{a}_{l-1}-\mathbf{a}_l,\mathbf{0}}(\mathbf{b}-\mathbf{a}_l)$.

Appendix B: Translation operator

With $\mathbf{y}=\mathbf{x}+\mathbf{b}$ and $\forall\mathbf{x},\mathbf{x}+\mathbf{b}\in\mathcal{X}$,

$$\mathcal{T}v(\mathbf{x})=\begin{cases}v(\mathbf{y}+\mathbf{a})+c_a & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ v(\mathbf{y})+c_r & \text{otherwise.}\end{cases}\tag{8}$$

B.1. Propagation of P and N (Cells 1 and 3)

We suppose that v is P (i.e. $v\geq 0$), then we want find conditions to have $\mathcal{T}$ which propagates P (i.e. $\mathcal{T}v\geq 0$). Given equation (8), we need to consider two cases:

- if $\mathbf{y}+\mathbf{a}\in\mathcal{X}$, then $\mathcal{T}v\geq 0$ if $c_a\geq 0$
- if $\mathbf{y}+\mathbf{a}\notin\mathcal{X}$, then $\mathcal{T}v\geq 0$ if $c_r\geq 0$. However this case is unreachable if $\mathcal{X}$ is $\mathtt{R}_{-\mathbf{b}}(\mathbf{a})$.

So $\mathcal{T}v\geq 0$ if $|c_a\geq 0|\wedge(\mathtt{R}_{-\mathbf{b}}(\mathbf{a})\vee|c_r\geq 0|)$. In the same way, $\mathcal{T}v\leq 0$ if $|c_a\leq 0|\wedge(\mathtt{R}_{-\mathbf{b}}(\mathbf{a})\vee|c_r\leq 0|)$.

B.2. Propagation of $\mathtt{I}_{\boldsymbol{\epsilon}}$ (Cell 5)

$$\Delta_{\boldsymbol{\epsilon}}\mathcal{T}v(\mathbf{x})=\begin{cases}\Delta_{\boldsymbol{\epsilon}}v(\mathbf{y}+\mathbf{a})+\epsilon_{c_a} & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\boldsymbol{\epsilon}}v(\mathbf{y})+\epsilon_{c_r} & \text{otherwise}\end{cases}$$

So $\mathcal{T}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if $|\epsilon_{c_a}\geq 0|\wedge(|\epsilon_{c_r}\geq 0|\vee\mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$

B.3. Propagation of $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ (Cells 7 and 9)

We make the assumption that $\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}v$ is positive (resp. negative), then we want find conditions to have $\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}\mathcal{T}$ positive (resp. negative).

$$\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}\mathcal{T}v(\mathbf{x})=\begin{cases}\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}v(\mathbf{y}+\mathbf{a}) & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}v(\mathbf{y}) & \text{otherwise}\end{cases}$$

So $\mathcal{T}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ or $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ without condition.

B.4. Propagation of $\mathtt{S}_{\mathbf{d},\epsilon}$ (Cell 11)

We make the assumption that v is $\mathtt{S}_{\mathbf{d},\epsilon}$ (i.e. $\Delta_\epsilon\Delta_{\mathbf{d}} v \geq 0$), then we want find conditions to have $\mathcal{T}$ which propagates $\mathtt{S}_{\mathbf{d},\epsilon}$ (i.e. $\Delta_\epsilon\Delta_{\mathbf{d}}\mathcal{T} v \geq 0$).

$$\Delta_\epsilon\Delta_{\mathbf{d}}\mathcal{T}v(\mathbf{x}) = \Delta_\epsilon\Delta_{\mathbf{d}} \begin{cases} v(\mathbf{y}+\mathbf{a}) + c_a & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X} \\ v(\mathbf{y}) + c_r & \text{otherwise} \end{cases}$$

The four possible cases are described in the following table

	$\mathbf{y}+\mathbf{a}\in\mathcal{X}$	$\mathbf{y}+\mathbf{a}\notin\mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d}\in\mathcal{X}$	Case 1	Case 3
$\mathbf{y}+\mathbf{a}+\mathbf{d}\notin\mathcal{X}$	Case 2	Case 4

- Case 1 $= 0$
- Case 2 $= \Delta_\epsilon[v(\mathbf{y}+\mathbf{d}) + c_r - v(\mathbf{y}+\mathbf{a}) - c_a]$

$= \Delta_\epsilon\Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}+\mathbf{a}) + \epsilon_{c_r} - \epsilon_{c_a}$

— Positive if $\mathtt{S}_{\mathbf{d}-\mathbf{a},\epsilon} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0|$

— Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})$

- Case 3 $= \Delta_\epsilon[v(\mathbf{y}+\mathbf{d}+\mathbf{a}) + c_a - v(\mathbf{y}) - c_r]$

$= \Delta_\epsilon\Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) - \epsilon_{c_r} + \epsilon_{c_a}$

— Positive if $\mathtt{S}_{\mathbf{d}+\mathbf{a},\epsilon} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0|$

— Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})$

- Case 4 $= 0$

So $\mathcal{T}$ propagates $\mathtt{S}_{\mathbf{d},\epsilon}$ if

$$(\mathtt{S}_{\mathbf{d}-\mathbf{a},\epsilon} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})) \wedge (\mathtt{S}_{\mathbf{d}+\mathbf{a},\epsilon} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0| \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}))$$

B.5. $\mathtt{PM}(\mathcal{T})$ and $\mathtt{NM}(\mathcal{T})$ (Cells 13 and 15)

$$\mathcal{T}v(\mathbf{x}) - v(\mathbf{x}) = \begin{cases} \Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + c_a & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X} & \text{Case 1} \\ \Delta_{\mathbf{b}} v(\mathbf{x}) + c_r & \text{otherwise} & \text{Case 2} \end{cases}$$

So v is $\mathtt{PM}(\mathcal{T})$ if $|\Delta_{\mathbf{a}+\mathbf{b}} v \geq -c_a| \wedge (|\Delta_{\mathbf{b}} v \geq -c_r| + \mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$ and v is $\mathtt{NM}(\mathcal{T})$ if $[|\Delta_{\mathbf{a}+\mathbf{b}} v \leq -c_a| \wedge (|\Delta_{\mathbf{b}} v \leq -c_r| + \mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$

B.6. $\mathtt{IM}_\epsilon(\mathcal{T})$ (Cell 17)

$$\Delta_\epsilon\Omega_{\mathcal{T}} v(\mathbf{x}) = \begin{cases} \Delta_\epsilon\Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + \epsilon_{c_a} & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X} \\ \Delta_\epsilon\Delta_{\mathbf{b}} v(\mathbf{x}) + \epsilon_{c_r} & \text{otherwise} \end{cases}$$

So, v is $\mathtt{IM}_\epsilon(\mathcal{T})$ if $\mathtt{S}_{\epsilon,\mathbf{a}+\mathbf{b}} \wedge |\epsilon_{c_a} \geq 0| \wedge (\mathtt{S}_{\epsilon,\mathbf{b}} \wedge |\epsilon_{c_r} \geq 0| \vee \mathtt{R}(\mathbf{a}+\mathbf{b}))$

	$\mathbf{y}+\mathbf{a}\in\mathcal{X}$	$\mathbf{y}+\mathbf{a}\notin\mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d}\in\mathcal{X}$	Case 1	Case 3
$\mathbf{y}+\mathbf{a}+\mathbf{d}\notin\mathcal{X}$	Case 2	Case 4

B.7. $\mathtt{IM}_{\mathbf{d}}(\mathcal{T})$ (Cell 19)

$$\Delta_{\mathbf{d}}\Omega_{\mathcal{T}}v(\mathbf{x}) = \Delta_{\mathbf{d}}\begin{cases}\Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x})+c_a & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\mathbf{b}}v(\mathbf{x})+c_r & \text{otherwise}\end{cases}$$

The four possible cases are described in the following table

- Case 1 $=\Delta_{\mathbf{d}}\Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x})$
  - —Positive if $\mathtt{S}_{\mathbf{d},\mathbf{a}+\mathbf{b}}$
- Case 2 $=\Delta_{\mathbf{b}}v(\mathbf{x}+\mathbf{d})+c_r-\Delta_{\mathbf{b}+\mathbf{a}}v(\mathbf{x})-c_a = \begin{cases}\Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x})-\Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{b})+c_r-c_a\\ \Delta_{\mathbf{d}-\mathbf{a}}\Delta_{\mathbf{b}}v(\mathbf{x}+\mathbf{a})-\Delta_{\mathbf{a}}v(\mathbf{x})+c_r-c_a\end{cases}$
  - —Positive if $|\Delta_{\mathbf{a}}v\le c_r-c_a|\wedge(\mathtt{S}_{\mathbf{b},\mathbf{d}}\vee\mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}})$
  - —Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}))$
- Case 3 $=\Delta_{\mathbf{b}+\mathbf{a}}v(\mathbf{x}+\mathbf{d})+c_a-\Delta_{\mathbf{b}}v(\mathbf{x})-c_r = \begin{cases}\Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x})+\Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{b}+\mathbf{d})-c_r+c_a\\ \Delta_{\mathbf{d}+\mathbf{a}}\Delta_{\mathbf{b}}v(\mathbf{x})+\Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{d})-c_r+c_a\end{cases}$
  - —Positive if $|\Delta_{\mathbf{a}}v\ge c_r-c_a|\wedge(\mathtt{S}_{\mathbf{b},\mathbf{d}}\vee\mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}})$
  - —Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}))$
- Case 4 $=\Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x})$
  - —Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$
  - —Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d})\vee\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b})$

So, v is $\mathtt{IM}_{\mathbf{d}}(\mathcal{T})$ if

$$\begin{gathered}\mathtt{S}_{\mathbf{d},\mathbf{a}+\mathbf{b}}\wedge(\mathtt{S}_{\mathbf{d},\mathbf{b}}\vee\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d})\vee\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}))\\ \wedge(|\Delta_{\mathbf{a}}v\le c_r-c_a|\wedge[\mathtt{S}_{\mathbf{d},\mathbf{b}}\vee\mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}}]\vee\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}))\\ \wedge(|\Delta_{\mathbf{a}}v\ge c_r-c_a|\wedge[\mathtt{S}_{\mathbf{d},\mathbf{b}}\vee\mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}}]\vee\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}))\end{gathered}$$

Appendix C: Choice operator

$$\mathcal{C}v(\mathbf{x}) = \begin{cases}\min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\} & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ v(\mathbf{y})+c_r, & \text{otherwise}\end{cases} \tag{9}$$

with $\mathbf{y}=\mathbf{x}+\mathbf{b}$ and $\forall\mathbf{x}$, $\mathbf{x}+\mathbf{b}\in\mathcal{X}$. In this section we may use $c_d=c_a-c_b$.

C.1. Propagation of $\mathtt{P}$ and $\mathtt{N}$ (Cells 2 and 4)

We suppose that v poisitve (resp. negative). From equation (9) the condition to have $\mathcal{C}v$ positive (resp. negative) is

$$|c_a\ge 0|\wedge|c_b\ge 0|\wedge(|c_r\ge 0|\vee\mathtt{R}_{-\mathbf{b}}(\mathbf{a})) \qquad (\text{resp. } |c_a\le 0|\wedge|c_b\le 0|\wedge(|c_r\le 0|\vee\mathtt{R}_{-\mathbf{b}}(\mathbf{a})))$$

C.2. Propagation of $\mathtt{I}_\epsilon$ (Cell 6)

$$\Delta_\epsilon\mathcal{C}v(\mathbf{x}) = \begin{cases}\Delta_\epsilon\min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\}\\ \qquad \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_\epsilon v(\mathbf{y})+c_r, \text{ otherwise}\end{cases}$$

The four cases of $\Delta_\epsilon\min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\}$ are described in the following table.

	$\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	$\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c_d{}'$	Case 1	Case 3
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c_d{}'$	Case 2	Case 4

- Case 1 $= \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}+\mathbf{a}) + \epsilon_{c_a}$
  —Positive if $|\epsilon_{c_a} \geq 0|$
- Case 2 $= v'(\mathbf{y}) + c_b{}' - v(\mathbf{y}+\mathbf{a}) - c_a \geq \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}) + \epsilon_{c_b}$
  —Positive if $\left|\epsilon_{c_b} \geq 0\right|$
- Case 3 $= v'(\mathbf{y}+\mathbf{a}) + c_a{}' - v(\mathbf{y}) - c_b \geq \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}+\mathbf{a}) + \epsilon_{c_a}$
  —Positive if $|\epsilon_{c_a} \geq 0|$
- Case 4 $Q = \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}) + \epsilon_{c_b}$
  —Positive if $\left|\epsilon_{c_b} \geq 0\right|$

Note that when $\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}$ (resp. $\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}$) the cases 2, 3, 4 (resp. 1, 2, 3) are Useless.

So $\mathcal{C}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if

$$\begin{pmatrix} |\epsilon_{c_a} \geq 0| \wedge \left|\epsilon_{c_b} \geq 0\right| \\ \vee \left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}\right| \wedge |\epsilon_{c_a} \geq 0| \\ \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}\right| \wedge \left|\epsilon_{c_b} \geq 0\right| \end{pmatrix} \wedge \begin{pmatrix} \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \\ \vee |\epsilon_{c_r} \geq 0| \end{pmatrix}$$

C.3. Propagation of $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ (Cells 8 and 10)

We make the assumption that $\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v$ is positive (resp. negative) then we want find conditions on v, and $\boldsymbol{\epsilon}$ to have $\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}\mathcal{C}$ positive (resp. negative).

$$\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}\mathcal{C}v(\mathbf{x}) = \begin{cases} \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\} \\ \qquad \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X} \\ \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}), \text{ otherwise} \end{cases}$$

We focus on $\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\}$. We use $v''(\mathbf{x})$ (resp. $c_b{}''$, $c_a{}''$) to denote $v(\mathbf{x}+2\boldsymbol{\epsilon})$ (resp. $c_b + 2\epsilon_{c_b}$, $c_a + 2\epsilon_{c_a}$).

$$\begin{aligned}\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\} = {} & \min\{v''(\mathbf{y}) + c_b{}'', v''(\mathbf{y}+\mathbf{a}) + c_a{}''\} \\ & -2\min\{v'(\mathbf{y}) + c_b{}', v'(\mathbf{y}+\mathbf{a}) + c_a{}'\} \\ & +\min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\}\end{aligned}$$

The 8 possible cases are given in the following table.

- Cases 1 and 8 are positive or negative without condition.
- Case 2 $= v''(\mathbf{y}+\mathbf{a}) + c_a{}'' - 2(v'(\mathbf{y}+\mathbf{a}) + c_a{}') + v(\mathbf{y}) + c_b = \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{a}} v(\mathbf{y}) - c_d$
  —Negative without condition
  —Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$
- Case 3 $= v''(\mathbf{y}+\mathbf{a}) + c_a{}'' - 2(v'(\mathbf{y}) + c_b{}') + v(\mathbf{y}+\mathbf{a}) + c_a = \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}+\mathbf{a}) + 2\Delta_{\mathbf{a}} v'(\mathbf{y}) + c_d + 2\epsilon_{c_d}$
  —Positive without condition

	$\Delta_{\mathbf{a}} v''(\mathbf{y}) \leq -c_d''$	$\Delta_{\mathbf{a}} v''(\mathbf{y}) \geq -c_d''$
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c_d'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	Case 1	Case 5
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c_d'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$	Case 2	Case 6
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c_d'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	Case 3	Case 7
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c_d'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$	Case 4	Case 8

— Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

- Case $4 = v''(\mathbf{y}+\mathbf{a}) + c_a'' - 2(v'(\mathbf{y}) + c_b') + v(\mathbf{y}) + c_b = \Delta_{\mathbf{a}} v''(\mathbf{y}) + \Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}) + c_d + 2\epsilon_{c_d}$
  - — Negative without condition
  - — Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$
- Case $5 = v''(\mathbf{y}) + c_b'' - 2(v'(\mathbf{y}+\mathbf{a}) + c_a') + v(\mathbf{y}+\mathbf{a}) + c_a = \Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{a}} v''(\mathbf{y}) - c_d - 2\epsilon_{c_d}$
  - — Negative without condition
  - — Useless if $\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$
- Case $6 = v''(\mathbf{y}) + c_b'' - 2(v'(\mathbf{y}+\mathbf{a}) + c_a') + v(\mathbf{y}) + c_b = \Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}) - 2\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) - c_d - 2\epsilon_{c_d}$
  - — Positive without condition
  - — Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$
- Case $7 = v''(\mathbf{y}) + c_b'' - 2(v'(\mathbf{y}) + c_b') + v(\mathbf{y}+\mathbf{a}) + c_a = \Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v(\mathbf{y}) + c_d$
  - — Negative without condition
  - — Useless if $\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

So $\mathcal{C}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ if

$$\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} = 0\right| \vee \left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}\right| \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}\right|$$

and propagate $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ if

$$\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}\right| \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}\right|$$

C.4. Propagation of $\mathtt{S}_{\mathbf{d},\epsilon}$ (Cell 12)

We make the assumption that v is $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ then we want find conditions on v, and $\boldsymbol{\epsilon}$ to have $\mathcal{C}$ which propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$.

$$\Delta_{\mathbf{d}} \Delta_{\boldsymbol{\epsilon}} \mathcal{C} v(\mathbf{x}) = \Delta_{\mathbf{d}} \begin{cases} \Delta_{\boldsymbol{\epsilon}} \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\} & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X} \\ \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}) + c_r, & \text{otherwise} \end{cases}$$

The 4 possible cases are given in the following table.

	$\mathbf{y}+\mathbf{a}\in\mathcal{X}$	$\mathbf{y}+\mathbf{a}\notin\mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d}\in\mathcal{X}$	Case A	Case C
$\mathbf{y}+\mathbf{a}+\mathbf{d}\notin\mathcal{X}$	Case B	Case D

C.4.1. Case A.

$$\begin{aligned}\text{Case A} &= \Delta_{\epsilon}\Delta_{\mathbf{d}}\min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\}\\ &= \min\{v'(\mathbf{y}+\mathbf{d})+c_b{}', v'(\mathbf{y}+\mathbf{a}+\mathbf{d})+c_a{}'\}-\min\{v'(\mathbf{y})+c_b{}', v'(\mathbf{y}+\mathbf{a})+c_a{}'\}\\ &\quad -\min\{v(\mathbf{y}+\mathbf{d})+c_b(, v(\mathbf{y}+\mathbf{a}+\mathbf{d})+c_a\}+\min\{v(\mathbf{y})+c_b{}', v(\mathbf{y}+\mathbf{a})+c_a\}\end{aligned}$$

The 16 possible cases of case A are described in Table 5

Table 5 Possible cases for Case A $=\Delta_{\epsilon}\Delta_{\mathbf{d}}\min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\}$.

Case A	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d})\le -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y})\le -c_d{}'$	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d})\le -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y})\ge -c_d{}'$	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d})\ge -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y})\le -c_d{}'$	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d})\ge -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y})\ge -c_d{}'$
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\le -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y})\le -c_d$	Case 1	Case 5	Case 9	Case 13
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\le -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y})\ge -c_d$	Case 2	Case 6	Case 10	Case 14
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\ge -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y})\le -c_d$	Case 3	Case 7	Case 11	Case 15
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\ge -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y})\ge -c_d$	Case 4	Case 8	Case 12	Case 16

- Case 1 $=\Delta_{\epsilon}\Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a})\ge 0$
- Case 2 $=\Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a})-\Delta_{\mathbf{d+a}}v(\mathbf{y})-c_d=\Delta_{\mathbf{d+a}}v'(\mathbf{y})-\Delta_{\mathbf{d+a}}v(\mathbf{y})-c_d-\Delta_{\mathbf{a}}v'(\mathbf{y})$
  — Positive if $\left|\epsilon_{c_d}\ge 0\right|\wedge \mathtt{S}_{\mathbf{d+a},\epsilon}$
  — Useless if $\mathtt{S}_{\mathbf{a},\mathbf{d}}\vee\mathtt{S}_{\mathbf{a},\epsilon}\wedge\left|\epsilon_{c_d}\ge 0\right|$
- Case 3 $=-\Delta_{\mathbf{d}-\mathbf{a}}v(\mathbf{y}+\mathbf{a})+\Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a})+c_d=-\Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a})+\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})+\Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a})+c_d\ge 0$
- Case 4 $=\Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a})-\Delta_{\mathbf{d}}v(\mathbf{y})\ge \begin{cases}\Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a})-\Delta_{\mathbf{d}}v(\mathbf{y})\\ \Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a})-\Delta_{\mathbf{d}}v(\mathbf{y})+\underbrace{\Delta_{\mathbf{a}}v'(\mathbf{y})-\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})}_{\ge 0 \text{ if } \epsilon_{c_d}\ge 0}\\ \quad =\Delta_{\mathbf{d+a}}v'(\mathbf{y})-\Delta_{\mathbf{d+a}}v(\mathbf{y})\end{cases}$
  — Positive if $\mathtt{S}_{\mathbf{d},\mathbf{a}}\vee\mathtt{S}_{\mathbf{d+a},\epsilon}\wedge\left|\epsilon_{c_d}\ge 0\right|$
  — Useless if $\mathtt{S}_{\mathbf{a},\epsilon}\wedge\left|\epsilon_{c_d}\ge 0\right|$
- Case 5 $=\Delta_{\mathbf{d+a}}v'(\mathbf{y})-\Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a})+c_d{}'=\Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a})-\Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a})+c_d{}'+\Delta_{\mathbf{a}}v'(\mathbf{y})\ge 0$
- Case 6 $=\Delta_{\mathbf{d+a}}v'(\mathbf{y})-\Delta_{\mathbf{d+a}}v(\mathbf{y})+c_d{}'-c_d$
  — Positive if $\mathtt{S}_{\mathbf{d+a},\epsilon}\wedge\left|\epsilon_{c_d}\ge 0\right|$
  — Useless if $\mathtt{S}_{\mathbf{d},\mathbf{a}}$
- Case 7 $=\Delta_{\mathbf{d+a}}v'(\mathbf{y})-\Delta_{\mathbf{d}-\mathbf{a}}v(\mathbf{y}+\mathbf{a})+c_d+c_d{}'$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$

- Case 8 $= -\Delta_{\mathbf{d}} v(\mathbf{y}) + \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) + c_d{}' = -\Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) + \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) + c_d{}'$

—Positive if $\left|\epsilon_{c_d} \geq 0\right| \wedge \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}$

—Useless if $\mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$

- Case 9 $= \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) - c_d{}' = \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{a}} v(\mathbf{y}) - c_d{}'$

—Positive if $\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

- Case 10 $= \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) + c_d{}' + c_d$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$

- Case 11 $= -\Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}+\mathbf{a}) + \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) + c_d - c_d{}'$

—Positive if $\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$

- Case 12 $= -\Delta_{\mathbf{d}} v(\mathbf{y}) + \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) - c_d{}'$

$= -\Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v'(\mathbf{y}) + \Delta_{\mathbf{d}} v'(\mathbf{y}) - c_d \geq 0$

- Case 13 $= \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) \geq \begin{cases} \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) \\ \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) + \underbrace{\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) - \Delta_{\mathbf{a}} v(\mathbf{y})}_{\geq 0 \text{ if } \epsilon_{c_d} \leq 0} \\ \quad = \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) \end{cases}$

—Positive if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

—Useless if $\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

- Case 14 $= \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) + c_d = \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) - c_d \geq 0$

- Case 15 $= \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}+\mathbf{a}) + c_d = \Delta_{\mathbf{d}-\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v'(\mathbf{y}) + c_d$

—Positive if $\left|\epsilon_{c_d} \leq 0\right| \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

- Case 16 $= -\Delta_{\mathbf{d}} v(\mathbf{y}) + \Delta_{\mathbf{d}} v'(\mathbf{y}) \geq 0$

Note that if $\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}$ or $\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}$ there is no condition because only cases 1 and 16 can be reach.

So Case A is positive if

$$
\begin{array}{lr}
\left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon_{c_d}^+\right| \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon_{c_d}^-\right| \vee & \\
(\left|\epsilon_{c_d} \geq 0\right| \wedge \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|) & \text{(Case 2)} \\
\wedge(\mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|) & \text{(Case 4)} \\
\wedge(\mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \vee \mathtt{S}_{\mathbf{d},\mathbf{a}}) & \text{(Case 6)} \\
\wedge(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|) & \text{(Case 7)} \\
\wedge(\left|\epsilon_{c_d} \geq 0\right| \wedge \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|) & \text{(Case 8)} \\
\wedge(\left|\epsilon_{c_d} \leq 0\right| \wedge \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|) & \text{(Case 9)} \\
\wedge(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|) & \text{(Case 10)} \\
\wedge(\mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}) & \text{(Case 11)} \\
\wedge(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|) & \text{(Case 13)} \\
\wedge(\left|\epsilon_{c_d} \leq 0\right| \wedge \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|) & \text{(Case 15)}
\end{array}
$$

With simplifications this condition reduces to

$$
\begin{array}{c}
\left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon_{c_d}^+\right| \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon_{c_d}^-\right| \\
\vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \\
\vee \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}) \wedge \left|\epsilon_{c_d} = 0\right|
\end{array}
$$

C.4.2. Case B.

$$
\begin{aligned}
\text{Case B} &= \Delta_{\boldsymbol{\epsilon}}[\mathcal{C}v(\mathbf{x}+\mathbf{d}) - \mathcal{C}v(\mathbf{x})] \\
&= v'(\mathbf{y}+\mathbf{d}) - v(\mathbf{y}+\mathbf{d}) + \epsilon_{c_r} - \min\{v'(\mathbf{y}) + c_b{}', v'(\mathbf{y}+\mathbf{a}) + c_a{}'\} + \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\}
\end{aligned}
$$

Case B	$\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c_d{}'$,	$\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c_d{}'$
$\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	Case 1	Case 3
$\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$	Case 2	Case 4

- Case 1 $= \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d-a}} v(\mathbf{y}+\mathbf{a}) - \epsilon_{c_a} + \epsilon_{c_r}$
  - —Positive if $\mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0|$
- Case 2 $= \Delta_{\mathbf{d-a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y}) - c_a{}' + c_b + \epsilon_{c_r} = \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v'(\mathbf{y}) - c_a{}' + c_b + \epsilon_{c_r}$
  - —Positive if $\left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right|$
  - —Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$
- Case 3 $= \Delta_{\mathbf{d}} v'(\mathbf{y}) - c_b{}' - \Delta_{\mathbf{d-a}} v(\mathbf{y}+\mathbf{a}) + c_a + \epsilon_{c_r} = \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d-a}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v'(\mathbf{y}) + c_a - c_b{}' + \epsilon_{c_r}$
  - —Positive if $\mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0|$
  - —Useless if $\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$
- Case 4 $= \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}} v(\mathbf{x}) - \epsilon_{c_b} + \epsilon_{c_r}$
  - —Positive if $\left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right|$

Note that when $\Delta_{\mathbf{a}} v \leq -c_d - \epsilon_{c_d}^+$ (resp. $\Delta_{\mathbf{a}} v \geq -c_d + \epsilon_{c_d}^-$) the cases 2, 3, 4 (resp. 1, 2, 3) are Useless. So case B is

- Positive if

$$
\begin{array}{c}
\mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \wedge \left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right| \\
\vee \left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon_{c_d}^+\right| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d-a}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \\
\vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon_{c_d}^-\right| \wedge \left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right|
\end{array}
$$

- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a+b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})$

C.4.3. Case C.

$$\begin{aligned}\text{Case C} &= \Delta_{\boldsymbol{\epsilon}}[\mathcal{C}v(\mathbf{x}+\mathbf{d})-\mathcal{C}v(\mathbf{x})]\\ &= \Delta_{\boldsymbol{\epsilon}}[\mathcal{C}v(\mathbf{x}+\mathbf{d})-v(\mathbf{y})]-\epsilon_{c_r}\\ &= \min\{v'(\mathbf{y}+\mathbf{d})+c_b{}',v'(\mathbf{y}+\mathbf{d}+\mathbf{a})+c_a{}'\}\\ &\quad -\min\{v(\mathbf{y}+\mathbf{d})+c_b,v(\mathbf{y}+\mathbf{d}+\mathbf{a})+c_a\}\\ &\quad -v'(\mathbf{y})+v(\mathbf{y})-\epsilon_{c_r}\end{aligned}$$

Case C	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d})\le -c'$,	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d})\ge -c'$
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\le -c$	Case 1	Case 3
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\ge -c$	Case 2	Case 4

- Case 1 $=\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}+\mathbf{a}}v(\mathbf{y})+\epsilon_{c_a}-\epsilon_{c_r}$

  —Positive if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}}\wedge|\epsilon_{c_a}-\epsilon_{c_r}\ge 0|$
- Case 2 $=\Delta_{\mathbf{d}+\mathbf{a}}v'(\mathbf{y})-\Delta_{\mathbf{d}}v(\mathbf{y})+c_a{}'-c_b-\epsilon_{c_r}=\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}+\mathbf{a}}v(\mathbf{y})+\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})+c_a{}'-c_b-\epsilon_{c_r}$

  —Positive if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}}\wedge|\epsilon_{c_a}-\epsilon_{c_r}\ge 0|$

  —Useless if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{a}}\wedge\left|\epsilon_{c_d}\ge 0\right|$
- Case 3 $\Delta_{\mathbf{d}}v'(\mathbf{y})-\Delta_{\mathbf{d}+\mathbf{a}}v(\mathbf{y})-c_a+c_b{}'-\epsilon_{c_r}=\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}v(\mathbf{y})-\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})-c_a+c_b{}'-\epsilon_{c_r}$

  —Positive if $\left|\epsilon_{c_b}-\epsilon_{c_r}\ge 0\right|$

  —Useless if $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\mathbf{a}}\wedge\left|\epsilon_{c_d}\le 0\right|$
- Case 4 $=\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}v(\mathbf{y})+\epsilon_{c_b}-\epsilon_{c_r}$

  —Positive if $\left|\epsilon_{c_b}-\epsilon_{c_r}\ge 0\right|$

Note that when $\Delta_{\mathbf{a}}v\le -c_d-\epsilon^+_{c_d}$ (resp.$\Delta_{\mathbf{a}}v\ge -c_d+\epsilon^-_{c_d}$) the cases 2, 3, and 4 (resp. 1, 2, and 3) are Useless. So case C is

- Positive if

$$\begin{gathered}\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}}\wedge|\epsilon_{c_a}-\epsilon_{c_r}\ge 0|\wedge\left|\epsilon_{c_b}-\epsilon_{c_r}\ge 0\right|\\ \vee\left|\Delta_{\mathbf{a}}v\le -c_d-\epsilon^+_{c_d}\right|\wedge\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}}\wedge|\epsilon_{c_a}-\epsilon_{c_r}\ge 0|\\ \vee\left|\Delta_{\mathbf{a}}v\ge -c_d+\epsilon^-_{c_d}\right|\wedge\left|\epsilon_{c_b}-\epsilon_{c_r}\ge 0\right|\end{gathered}$$

- Useless if $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})$

C.4.4. Case D.

$$\text{Case D}=\Delta_{\boldsymbol{\epsilon}}[\mathcal{C}v(\mathbf{x}+\mathbf{d})-\mathcal{C}v(\mathbf{x})]=\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}v(\mathbf{x})\ge 0$$

C.4.5. Conclusion. The operator $\mathcal{C}$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ if,

$$\begin{pmatrix}\left|\Delta_{\mathbf{a}}v\le -c_d-\epsilon^+_{c_d}\right|\vee\left|\Delta_{\mathbf{a}}v\ge -c_d+\epsilon^-_{c_d}\right|\\ \vee\mathtt{S}_{\mathbf{d},\mathbf{a}}\wedge\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}\wedge\left|\epsilon_{c_d}\le 0\right|\vee\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}\wedge\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}\wedge\left|\epsilon_{c_d}\ge 0\right|\\ \vee\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}}\wedge\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}})\wedge\left|\epsilon_{c_d}=0\right|\end{pmatrix}$$
$$\wedge\begin{pmatrix}\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}}\wedge|\epsilon_{c_r}-\epsilon_{c_a}\ge 0|\wedge\left|\epsilon_{c_r}-\epsilon_{c_b}\ge 0\right|\\ \vee\left|\Delta_{\mathbf{a}}v\le -c_d-\epsilon^+_{c_d}\right|\wedge\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}-\mathbf{a}}\wedge|\epsilon_{c_r}-\epsilon_{c_a}\ge 0|\\ \vee\left|\Delta_{\mathbf{a}}v\ge -c_d+\epsilon^-_{c_d}\right|\wedge\left|\epsilon_{c_r}-\epsilon_{c_b}\ge 0\right|\\ \vee\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})\end{pmatrix}\wedge\begin{pmatrix}\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}}\wedge|\epsilon_{c_a}-\epsilon_{c_r}\ge 0|\wedge\left|\epsilon_{c_b}-\epsilon_{c_r}\ge 0\right|\\ \vee\left|\Delta_{\mathbf{a}}v\le -c_d-\epsilon^+_{c_d}\right|\wedge\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}}\wedge|\epsilon_{c_a}-\epsilon_{c_r}\ge 0|\\ \vee\left|\Delta_{\mathbf{a}}v\ge -c_d+\epsilon^-_{c_d}\right|\wedge\left|\epsilon_{c_b}-\epsilon_{c_r}\ge 0\right|\\ \vee\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})\end{pmatrix}$$

We can simplify this results because if $|\Delta_{\mathbf{a}} \leq -c_d|$ the state $\mathbf{x}+\mathbf{a}+\mathbf{b}$ is always chosen in the minimization, so the operator is equivalent to $\mathcal{T}$ (plus the cost c_a), and if $|\Delta_{\mathbf{a}} \leq -c_d|$ the state $\mathbf{x}+\mathbf{a}+\mathbf{b}$ is never chosen in the minimization, so the operator is equivalent to $\mathcal{T}$ or $\mathcal{C}$ with $\mathbf{a}=\mathbf{0}$. So we can consider that $|\Delta_{\mathbf{a}} \leq -c_d| = |\Delta_{\mathbf{a}} \geq -c_d| = false$. Then the relation reduces to

$$\begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \\ \vee \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}) \wedge \left|\epsilon_{c_d} = 0\right| \end{pmatrix}$$
$$\wedge \begin{pmatrix} \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_r} - \epsilon_{c_a} \geq 0\right| \wedge \left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}} \wedge \left|\epsilon_{c_a} - \epsilon_{c_r} \geq 0\right| \wedge \left|\epsilon_{c_b} - \epsilon_{c_r} \geq 0\right| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$$

C.5. $\mathtt{PM}(\mathcal{T})$ and $\mathtt{NM}(\mathcal{T})$ (Cells 14 and 16)

$$\mathcal{C}v(\mathbf{x}) - v(\mathbf{x}) = \begin{cases} \min\{\Delta_{\mathbf{b}}v(\mathbf{x}) + c_b, \Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x}) + c_a\} & \text{if } \mathbf{x}+\mathbf{a}+\mathbf{b} \in \mathcal{X} \\ \Delta_{\mathbf{b}}v(\mathbf{x}) + c_r, & \text{otherwise} \end{cases}$$

So v is $\mathtt{PM}(\mathcal{C})$ if

$$(|\Delta_{\mathbf{b}}v \geq -c_b| \vee |\Delta_{\mathbf{a}}v \leq -c_d|) \wedge (|\Delta_{\mathbf{a}+\mathbf{b}}v \geq -c_a| \vee |\Delta_{\mathbf{a}}v \geq -c_d|) \wedge (|\Delta_{\mathbf{b}}v \geq -c_r| \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$$

and v is $\mathtt{NM}(\mathcal{C})$ if

$$\left(|\Delta_{\mathbf{b}}v \leq -c_b| \wedge \overline{|\Delta_{\mathbf{a}}v \leq -c_d|} \vee |\Delta_{\mathbf{a}+\mathbf{b}}v \leq -c_a| \wedge \overline{|\Delta_{\mathbf{a}}v \geq -c_d|}\right) \wedge (|\Delta_{\mathbf{b}}v \leq -c_r| \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$$

C.6. $\mathtt{IM}_{\boldsymbol{\epsilon}}(\mathcal{T})$ (Cell 18)

$$\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{C}}v(\mathbf{x}) = \begin{cases} \Delta_{\boldsymbol{\epsilon}} \min\{\Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x}) + c_a, \Delta_{\mathbf{b}}v(\mathbf{x}) + c_b\} & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X} \\ \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{b}}v(\mathbf{x}) + \epsilon_{c_r} & \text{otherwise} \end{cases}$$

The 4 possible cases for $\Delta_{\boldsymbol{\epsilon}} \min\{\Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x}) + c_a, \Delta_{\mathbf{b}}v(\mathbf{x}) + c_b\}$ are given in the following table.

	$\Delta_{\mathbf{a}}v'(\mathbf{y}) \leq -c_d$	$\Delta_{\mathbf{a}}v'(\mathbf{y}) \geq -c_d$
$\Delta_{\mathbf{a}}v(\mathbf{y}) \leq -c_d$	Case 1	Case 3
$\Delta_{\mathbf{a}}v(\mathbf{y}) \geq -c_d$	Case 2	Case 4

- Case $1 = \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x}) + \epsilon_{c_a}$
  — Positive if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}+\mathbf{a}} \wedge |\epsilon_{c_a} \geq 0|$
- Case $2 = \Delta_{\mathbf{a}+\mathbf{b}}v'(\mathbf{x}) + c_a{}' - \Delta_{\mathbf{b}}v(\mathbf{x}) + c_b$
  — Useless if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{a}}$
- Case $3 = \Delta_{\mathbf{b}}v'(\mathbf{x}) + c_b{}' - \Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x}) - c_a \geq \Delta_{\boldsymbol{\epsilon}}v(\mathbf{x}+\mathbf{b}) - \Delta_{\boldsymbol{\epsilon}}v(\mathbf{x}) + \epsilon_{c_b}$
  — Positive if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \left|\epsilon_{c_b} \geq 0\right|$
  — Useless if $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\mathbf{a}}$
- Case $4 = \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{b}}v(\mathbf{x}) + \epsilon_{c_b}$
  — Positive if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \left|\epsilon_{c_b} \geq 0\right|$

Note that when $\Delta_{\mathbf{a}} v \leq -c_d$ (resp.$\Delta_{\mathbf{a}} v \geq -c_d$) the cases 2, 3, and 4 (resp. 1, 2, and 3) are Useless. So $\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{C}} v$ is positive if

$$\begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{a}} \wedge \left|\epsilon_{c_a} \geq 0\right| \wedge \left|\epsilon_{c_b} \geq 0\right| \\ \vee \left|\Delta_{\mathbf{a}} v \leq -c_d\right| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}+\mathbf{a}} \wedge \left|\epsilon_{c_a} \geq 0\right| \\ \vee \left|\Delta_{\mathbf{a}} v \geq -c_d\right| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \left|\epsilon_{c_b} \geq 0\right| \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \left|\epsilon_{c_r} \geq 0\right| \\ \vee \mathtt{R}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$$

C.7. $\mathtt{IM}_{\mathbf{d}}(\mathcal{T})$ (Cell 20)

$$\Delta_{\mathbf{d}} \Omega_{\mathcal{C}} v(\mathbf{x}) = \Delta_{\mathbf{d}}(\mathcal{C} v(\mathbf{x}) - v(\mathbf{x}))$$

The 4 possible cases are given in the following table.

	$\mathbf{y}+\mathbf{a} \in \mathcal{X}$	$\mathbf{y}+\mathbf{a} \notin \mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d} \in \mathcal{X}$	Case A	Case C
$\mathbf{y}+\mathbf{a}+\mathbf{d} \notin \mathcal{X}$	Case B	Case D

C.7.1. Case A.

$$\begin{aligned} \Delta_{\mathbf{d}} \Omega_{\mathcal{C}} v(\mathbf{x}) = \min\{v(\mathbf{y}+\mathbf{d}) + c_b, v(\mathbf{y}+\mathbf{d}+\mathbf{a}) + c_a\} - v(\mathbf{x}+\mathbf{d}) \\ - \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\} + v(\mathbf{x}) \end{aligned}$$

The 4 possible cases are given in the following table.

	$\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	$\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \leq -c_d$	Case 1	Case 3
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \geq -c_d$	Case 2	Case 4

- Case 1 $= \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{x}) = \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{x})$
  - Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}}$
- Case 2 $= \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{x}) - \Delta_{\mathbf{a}} v(\mathbf{y}) - c_d \geq \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}) - \Delta_{\mathbf{d}} v(\mathbf{x})$
  - Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$
  - Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$
- Case 3 $= \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{x}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) + c_d \leq \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}) - \Delta_{\mathbf{d}} v(\mathbf{x})$
  - Useless if $\mathtt{S}_{\mathbf{d},\mathbf{a}}$
- Case 4 $= \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}) - \Delta_{\mathbf{d}} v(\mathbf{x})$
  - Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$

Note that when $\Delta_{\mathbf{a}} v \leq -c_d$ (resp.$\Delta_{\mathbf{a}} v \geq -c_d$) the cases 2, 3, and 4 (resp. 1, 2, and 3) are Useless. So Case A is

- Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \left|\Delta_{\mathbf{a}} v \leq -c_d\right| \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}} \vee \left|\Delta_{\mathbf{a}} v \geq -c_d\right| \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}}$

C.7.2. Case B. Case B$= v(\mathbf{y}+\mathbf{d}) + c_r - v(\mathbf{x}+\mathbf{d}) - \mathcal{C}v(\mathbf{x}) + v(\mathbf{x})$

- If $\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$ then Case B $= \begin{cases} \Delta_{\mathbf{d}}\Delta_{\mathbf{b}} v(\mathbf{x}) - \Delta_{\mathbf{a}} v(\mathbf{y}) + c_r - c_a \geq \Delta_{\mathbf{b}}\Delta_{\mathbf{d}} v(\mathbf{x}) + c_r + c_b \\ \Delta_{\mathbf{d}-\mathbf{a}}\Delta_{\mathbf{b}} v(\mathbf{x}+\mathbf{a}) - \Delta_{\mathbf{a}} v(\mathbf{x}) + c_r - c_a \end{cases}$

  —Positive if $\left(\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r + c_b \geq 0| \vee \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}} \wedge |\Delta_{\mathbf{a}} v \leq c_r - c_a|\right)$

  —Useless if $\Delta_{\mathbf{a}} v \geq -c_d$

- If $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$ then Case B $= \Delta_{\mathbf{b}}\Delta_{\mathbf{d}} v(\mathbf{x}) + c_r - c_b$

  —Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r - c_b \geq 0|$

  —Useless if $\Delta_{\mathbf{a}} v \leq -c_d$

So Case B is

- Positive if

$$\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r + c_b \geq 0| \vee \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}} \wedge |\Delta_{\mathbf{a}} v \leq c_r - c_a| \vee |\Delta_{\mathbf{a}} v \geq -c_d| \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r - c_b \geq 0| \vee |\Delta_{\mathbf{a}} v \leq -c_d|)$$

- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})$

C.7.3. Case C. Case C $= \mathcal{C}v(\mathbf{x}+\mathbf{d}) - v(\mathbf{x}+\mathbf{d}) - v(\mathbf{y}) - c_r + v(\mathbf{x})$

- If $\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \leq -c_d$ then Case C $= \begin{cases} \Delta_{\mathbf{d}}\Delta_{\mathbf{b}} v(\mathbf{x}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) - c_r + c_a \\ \Delta_{\mathbf{d}+\mathbf{a}}\Delta_{\mathbf{b}} v(\mathbf{x}) + \Delta_{\mathbf{a}} v(\mathbf{x}+\mathbf{d}) - c_r + c_a \end{cases}$

  —Positive if $|\Delta_{\mathbf{a}} v \geq c_r - c_a| \wedge (\mathtt{S}_{\mathbf{b},\mathbf{d}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}})$

  —Useless if $\Delta_{\mathbf{a}} v \geq -c_d$

- If $\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \geq -c_d$ then Case C $= \Delta_{\mathbf{b}}\Delta_{\mathbf{d}} v(\mathbf{x}) - c_r + c_b$

  —Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_b - c_r \geq 0|$

  —Useless if $\Delta_{\mathbf{a}} v \leq -c_d$

So Case C is

- Positive if $(|\Delta_{\mathbf{a}} v \geq c_r - c_a| \wedge (\mathtt{S}_{\mathbf{b},\mathbf{d}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}}) \vee |\Delta_{\mathbf{a}} v \geq -c_d|) \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_b - c_r \geq 0| \vee |\Delta_{\mathbf{a}} v \leq -c_d|)$
- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})$

C.7.4. Case D. Case D $= \Delta_{\mathbf{d}}\Delta_{\mathbf{b}} v(\mathbf{x})$

- Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$
- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b})) \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d}))$

C.7.5. Conclusion. $\Delta_{\mathbf{d}}\Omega_{\mathcal{C}} v \geq 0$ if,

$$\begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge \mathtt{S}_{\mathbf{d},\mathbf{a}} \\ \vee |\Delta_{\mathbf{a}} v \leq -c_d| \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}} \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d| \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}} \end{pmatrix} \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}))$$
$$\wedge \begin{pmatrix} \begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r + c_b \geq 0| \\ \vee \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}} \wedge |\Delta_{\mathbf{a}} v \leq c_r - c_a| \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d| \end{pmatrix} \\ \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r - c_b \geq 0| \vee |\Delta_{\mathbf{a}} v \leq -c_d|) \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \begin{pmatrix} \mathtt{S}_{\mathbf{b},\mathbf{d}} \wedge |\Delta_{\mathbf{a}} v \geq c_r - c_a| \\ \vee \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}} \wedge |\Delta_{\mathbf{a}} v \geq c_r - c_a|) \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d| \end{pmatrix} \\ \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_b - c_r \geq 0| \vee |\Delta_{\mathbf{a}} v \leq -c_d|) \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$$

With $|\Delta_{\mathbf{a}} v \leq -c_d| = |\Delta_{\mathbf{a}} v \geq -c_d| = false$ this expression reduces to

$$(|c_r \geq 0| \wedge |c_b = 0| \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})) \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})$$

Appendix D: Admission control

$$
\begin{aligned}
\boldsymbol{\mathcal{M}}v &= \mathcal{H} + \mu\mathcal{O}_0 v + \textstyle\sum_{i=1}^{n} \lambda_i \mathcal{O}_i v + p_0 v,\\
\mathcal{H}(\mathbf{x}) &= hx,\\
\mathcal{O}_0 v(\mathbf{x}) &= \mathcal{T}v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = -\mathbf{e}_1, \mathbf{b} = \mathbf{0},\\ c_a = c_r = 0, \end{cases}\\
\mathcal{O}_i v(\mathbf{x}) &= \mathcal{C}v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = \mathbf{e}_1, \mathbf{b} = \mathbf{0},\\ c_b = c_i, c_a = c_r = 0. \end{cases}
\end{aligned}
$$

The state space is $\mathcal{S}_1 = \mathbb{Z}^+$.

From Stidham (1985) we know that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$ and $\mathtt{I}_{\mathbf{e}_1}$.

D.1. Proof of Theorem 1

D.1.1. Monotonicity. We look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$. From Proposition 2 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if the following condition is satisfied, knowing that v is $\mathtt{I}_{\boldsymbol{\epsilon}}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$, and $\mathtt{I}_{\mathbf{e}_1}$.

$$
\begin{aligned}
&|\Delta_{\boldsymbol{\epsilon}}(hx) \geq 0|\\
&\bigwedge \left[\begin{array}{c} |\mathcal{O}_0 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_\mu < 0| \wedge |\Omega_{\mathcal{O}_0} v \leq 0| \\ \vee |\epsilon_\mu > 0| \wedge |\Omega_{\mathcal{O}_0} v \geq 0| \\ \vee |\epsilon_\mu = 0| \end{pmatrix} \end{array} \right] \bigwedge_{i=1}^{l} \left[\begin{array}{c} |\mathcal{O}_i \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_{\lambda_i} < 0| \wedge |\Omega_{\mathcal{O}_i} v \leq 0| \\ \vee |\epsilon_{\lambda_i} > 0| \wedge |\Omega_{\mathcal{O}_i} v \geq 0| \\ \vee |\epsilon_{\lambda_i} = 0| \end{pmatrix} \end{array} \right] \bigwedge \begin{pmatrix} |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{P}| \\ \vee |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{N}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}.
\end{aligned}
\tag{10}
$$

From Table 4 we obtain the following relations.

- $|\Delta_{\boldsymbol{\epsilon}}(hx) \geq 0| = |\epsilon_h \geq 0|$
- $|\mathcal{O}_0 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| = true$ (see cell 5).
- $|\Omega_{\mathcal{O}_0} v \leq 0| = |\Delta_{-\mathbf{e}_1} v \leq 0| = true$ (see cell 15).
- $|\Omega_{\mathcal{O}_0} v \geq 0| = |\Delta_{-\mathbf{e}_1} v \geq 0| = false$ (see cell 13).
- $|\mathcal{O}_i \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| = |\epsilon_{c_i} \geq 0|$ because $\mathtt{R}(\mathbf{e}_1) = true$ (see cell 6).
- $|\Omega_{\mathcal{O}_i} v \leq 0| = |\Delta_{\mathbf{e}_1} v \leq 0| = false$ (see cell 16).
- $|\Omega_{\mathcal{O}_i} v \geq 0| = |\Delta_{\mathbf{e}_1} v \geq 0| = true$ (see cell 14).
- $|v \text{ is } \mathtt{P}| = true$ because costs are positive (see cells 1 and 2).
- $|v \text{ is } \mathtt{N}| = false$ because costs are not negative (see cells 3 and 4).

So equation (10) can be reduced to

$$
|\epsilon_h \geq 0| \wedge |\epsilon_\mu \leq 0| \wedge |\epsilon_\eta \leq 0| \bigwedge_{i=1}^{l} (|\epsilon_{c_i} \geq 0| \wedge |\epsilon_{\lambda_i} \geq 0|) \tag{11}
$$

Conclusion, the optimal value function is increasing in the arrival rates λ_i, the rejection costs c_i, the holding cost h and decreasing in the service rate μ and the discount rate η.

D.1.2. Convexity/Concavity. First we look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$. However $|\mathcal{O}_i \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}| = false$, so $\boldsymbol{\mathcal{M}}$ does not propagate $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ (see Proposition 3 and cell 10 in Table 4).

Now we look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ if the following condition is satisfied, knowing that v is $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$, and $\mathtt{I}_{\mathbf{e}_1}$.

$$|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}(hx) \le 0| \bigwedge \left[\begin{array}{c} |\mathcal{O}_0 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_\mu > 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_0} v \le 0| \\ \vee |\epsilon_\mu < 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_0} v \ge 0| \\ \vee |\epsilon_\mu = 0| \end{pmatrix} \end{array}\right] \bigwedge_{i=1}^{l} \left[\begin{array}{c} |\mathcal{O}_i \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_{\lambda_i} > 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i} v \le 0| \\ \vee |\epsilon_{\lambda_i} < 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i} v \ge 0| \\ \vee |\epsilon_{\lambda_i} = 0| \end{pmatrix} \end{array}\right] \bigwedge \begin{pmatrix} |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \vee |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{I}_{-\boldsymbol{\epsilon}}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}. \tag{12}$$

From Table 4 we obtain the following relations.

- $|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}(hx) \le 0| = true$.
- $|\mathcal{O}_0 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| = true$ (see cell 9).
- $|\Delta_{-\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_0} v \ge 0| = \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ (see cell 17).
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_0} v \ge 0| = \mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_1}$ (see cell 17).
- $|\mathcal{O}_i \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| = \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}} \wedge |\epsilon_{c_i} \le 0| \vee \mathtt{S}^{ub}_{\mathbf{e}_1,\boldsymbol{\epsilon}} \wedge |\epsilon_{c_i} \ge 0|$ (see cell 10).
- $|\Delta_{-\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i} v \ge 0| = \mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_1} \wedge |\epsilon_{c_i} \le 0|$ (see cell 18).
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i} v \ge 0| = \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1} \wedge |\epsilon_{c_i} \ge 0|$ (see cell 18).
- $|v \text{ is } \mathtt{I}_{\boldsymbol{\epsilon}}|$ if (see equation 11) $|\epsilon_h \ge 0| \wedge |\epsilon_\mu \le 0| \wedge |\epsilon_\eta \le 0| \bigwedge_{i=1}^{l} |\epsilon_{c_i} \ge 0| \wedge |\epsilon_{\lambda_i} \ge 0|$.
- $|v \text{ is } \mathtt{I}_{-\boldsymbol{\epsilon}}|$ if (see equation 11) $|\epsilon_h \le 0| \wedge |\epsilon_\mu \ge 0| \wedge |\epsilon_\eta \ge 0| \bigwedge_{i=1}^{l} |\epsilon_{c_i} \le 0| \wedge |\epsilon_{\lambda_i} \le 0|$.

So equation (12) reduces to

$$\begin{pmatrix} |\epsilon_\mu > 0| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1} \\ \vee |\epsilon_\mu < 0| \wedge \mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_1} \\ \vee |\epsilon_\mu = 0| \end{pmatrix} \bigwedge_{i=1}^{l} \left[\begin{array}{c} \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}} \wedge |\epsilon_{c_i} \le 0| \vee \mathtt{S}^{ub}_{\mathbf{e}_1,\boldsymbol{\epsilon}} \wedge |\epsilon_{c_i} \ge 0| \\ \bigwedge \begin{pmatrix} |\epsilon_{\lambda_i} > 0| \wedge \mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_1} \wedge |\epsilon_{c_i} \le 0| \\ \vee |\epsilon_{\lambda_i} < 0| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1} \wedge |\epsilon_{c_i} \ge 0| \\ \vee |\epsilon_{\lambda_i} = 0| \end{pmatrix} \end{array}\right] \bigwedge |\epsilon_\eta = 0| \ . \tag{13}$$

In the following section (see equation 15) we will see that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ if

$$|\epsilon_h \ge 0| \wedge |\epsilon_{c_i} \ge 0| \wedge |\epsilon_{\lambda_i} \ge 0| \wedge |\epsilon_\mu \le 0| \wedge |\epsilon_\eta \le 0|,$$

so equation (13) reduces to

$$|\epsilon_{c_i} = 0| \wedge |\epsilon_{\lambda_i} = 0| \wedge |\epsilon_\mu = 0| \wedge |\epsilon_\eta = 0| \ .$$

Conclusion, the optimal value function is concave in the holding cost h.

D.1.3. Monotonicity of the optimal policy. We look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ if the following condition is satisfied, knowing that v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$, and $\mathtt{I}_{\mathbf{e}_1}$.

$$|\Delta_{\mathbf{e}_1}\Delta_{\boldsymbol{\epsilon}}(hx)\geq 0| \wedge \left[\begin{array}{c} |\mathcal{O}_0 \text{ propagates } \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}| \\ \wedge\left(\begin{array}{c} |\epsilon_\mu<0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_0}v\leq 0| \\ \vee|\epsilon_\mu>0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_0}v\geq 0| \\ \vee|\epsilon_\mu=0|\end{array}\right)\end{array}\right] \bigwedge_{i=1}^{l}\left[\begin{array}{c} |\mathcal{O}_i \text{ propagates } \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}| \\ \wedge\left(\begin{array}{c} |\epsilon_{\lambda_i}<0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i}v\leq 0| \\ \vee|\epsilon_{\lambda_i}>0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i}v\geq 0| \\ \vee|\epsilon_{\lambda_i}=0|\end{array}\right)\end{array}\right] \wedge \left(\begin{array}{c} |\epsilon_\eta<0|\wedge|v \text{ is } \mathtt{I}_{\mathbf{e}_1}| \\ \vee|\epsilon_\eta>0|\wedge|v \text{ is } \mathtt{I}_{-\mathbf{e}_1}| \\ \vee|\epsilon_\eta=0|\end{array}\right). \tag{14}$$

From Table 4 we obtain the following relations.

- $|\Delta_{\mathbf{e}_1}\Delta_{\boldsymbol{\epsilon}}(hx)\geq 0| = |\epsilon_h\geq 0|$
- $|\mathcal{O}_0$ propagates $\mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}| = true$ (see cell 11).
- $|\Delta_{-\mathbf{e}_1}\Omega_{\mathcal{O}_0}v\leq 0| = \mathtt{S}_{-\mathbf{e}_1,-\mathbf{e}_1}\wedge|\Delta_{-\mathbf{e}_1}v\leq 0| = true$ (see cell 19).
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_0}v\geq 0| = \mathtt{S}_{\mathbf{e}_1,-\mathbf{e}_1}\wedge\cdots = false$ (see cell 19).
- $|\mathcal{O}_i$ propagates $\mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}| = \mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}\wedge|\epsilon_{c_i}\geq 0|$ (see cell 12).
- $|\Delta_{-\mathbf{e}_1}\Omega_{\mathcal{O}_i}v\geq 0| = \mathtt{S}_{-\mathbf{e}_1,\mathbf{e}_1} = false$ (see cell 20).
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i}v\geq 0| = \mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1} = true$ (see cell 20).
- $|v$ is $\mathtt{I}_{\mathbf{e}_1}| = true$ (Stidham 1985).
- $|v$ is $\mathtt{I}_{-\mathbf{e}_1}| = false$ (Stidham 1985).

So equation (14) can be reduced, and $\boldsymbol{\mathcal{M}}$ propages $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ if

$$|\epsilon_h\geq 0|\wedge|\epsilon_{c_i}\geq 0|\wedge|\epsilon_{\lambda_i}\geq 0|\wedge|\epsilon_\mu\leq 0|\wedge|\epsilon_\eta\leq 0|. \tag{15}$$

Given that the optimal thresholds t_i decrease if

$$|\boldsymbol{\mathcal{M}} \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}}|\wedge|\epsilon_{c_i}\leq 0|,$$

the optimal thresholds t_i are decreasing in the arrival rate λ_i, the holding cost h, and increasing in the service rate μ and the discount rate η.

D.2. Proof of Theorem 2

D.2.1. Effect of λ and μ : Piecewise convexity. Let $[\mu_l,\mu_u]$ (resp. $[\lambda_l,\lambda_u]$) be a set such that for all $\mu\in[\mu_l,\mu_u]$ (resp. $\lambda_i\in[\lambda_l,\lambda_u]$) the optimal thresholds S_i^* do not change. For all $\mu\in[\mu_l,\mu_u]$ (resp. $\lambda_i\in[\lambda_l,\lambda_u]$) the MDP formulation can be rewritten.

Let ϵ_μ (resp. ϵ_{λ_i}) be positive such that $\mu+\epsilon_\mu\in[\mu_l,\mu_u]$ (resp. $\lambda_i+\epsilon_{\lambda_i}\in[\lambda_l,\lambda_u]$).

- For all state space $\mathcal{X}$ and for all direction $\mathbf{a}$, $\mathcal{T}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ without conditions.
- $\mathtt{IM}_{\boldsymbol{\epsilon}}(\mathcal{O}_0)$ is positive if v is $\mathtt{S}_{\boldsymbol{\epsilon},-\mathbf{e}}$ which is true because ϵ_μ is positive. (resp. $\mathtt{IM}_{\boldsymbol{\epsilon}}(\mathcal{O}_{i>0})$ is positive if v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}}$ which is true because ϵ_{λ_i} is positive.)

So $v^\star(\mathbf{x})$ is convex in $\mu\in[\mu_l,\mu_u]$ resp. $\lambda_i\in[\lambda_l,\lambda_u]$) if the optimal thresholds S_i^* do not change on the set $[\mu_l,\mu_u]$ (resp. $[\lambda_l,\lambda_u]$).

D.2.2. Effect of h and c_i : concavity and piecewise linearity. With $\epsilon_h \geq 0$ and $\epsilon_{c_i} \leq 0$, v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}}$ and operators $\mathcal{C}$ (with $\mathbf{a}=\mathbf{e}$) and $\mathcal{T}$ (with $\mathbf{a}=-\mathbf{e}$) propagate $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$. So v is concave in ϵ_h and ϵ_c.

We consider a set of parameters $[h_l,h_u]$ (resp. $[c_l,c_u]$) such that the optimal thresholds S_i^* do not change on this set. As previously the MDP formulation can be rewritten on this set with translation operator only.

With $\epsilon_h \geq 0$ (resp. $\epsilon_{c_i} \geq 0$) such that $h+\epsilon_h \in [h_l,h_u]$ (resp. $c_i+\epsilon_{c_i} \in [c_l,c_u]$), then $\mathcal{T}$ propagates $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ without conditions $\forall \mathcal{X}$ and $\forall \mathbf{a}$.

Given that v $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ imply that v is linear in $\boldsymbol{\epsilon}$, the optimal value function $v^\star(\mathbf{x})$ is linear in $h \in [h_l,h_u]$ (resp. $c_i \in [c_l,c_u]$) if the optimal thresholds S_i^* do not change on the set $[h_l,h_u]$ (resp. $[c_l,c_u]$).

Appendix E: Tandem queue, proof of Theorem 3

The optimality equations for the tandem queue problem are

$$\begin{aligned}
\boldsymbol{\mathcal{M}}v &= \mathcal{H} + \mu_1\mathcal{O}_1 v + \mu_2\mathcal{O}_2 v + \lambda\mathcal{O}_3 v + p_0 v,\\
\mathcal{H}(\mathbf{x}) &= h_1x_1 + h_2\max\{x_2,0\} + b\max\{-x_2,0\},\\
\mathcal{O}_1 v(\mathbf{x}) &= \mathcal{C}v(\mathbf{x}) \text{ with } \begin{cases}\mathbf{a}=\mathbf{e}_1, \mathbf{b}=\mathbf{0},\\ c_a=c_r=0,\end{cases}\\
\mathcal{O}_2 v(\mathbf{x}) &= \mathcal{C}v(\mathbf{x}) \text{ with } \begin{cases}\mathbf{a}=\mathbf{e}_2-\mathbf{e}_1, \mathbf{b}=\mathbf{0},\\ c_a=c_b=c_r=0,\end{cases}\\
\mathcal{O}_3 v(\mathbf{x}) &= \mathcal{T}v(\mathbf{x}) \text{ with } \begin{cases}\mathbf{a}=-\mathbf{e}_2, \mathbf{b}=\mathbf{0},\\ c_a=c_b=c_r=0.\end{cases}
\end{aligned}$$

From Veatch and Wein (1992) we know that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

E.1. Monotonicity

We look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$. From Proposition 2 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if the following condition is satisfied, knowing that v is $\mathtt{I}_{\boldsymbol{\epsilon}}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

$$\begin{aligned}
&\left|\Delta_{\boldsymbol{\epsilon}}(h_1x_1 + h_2x_2^+ + b(-x_2)^+) \geq 0\right|,\\
&\wedge \left[\begin{matrix}|\mathcal{O}_1 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}|\\ \wedge\begin{pmatrix}|\epsilon_{\mu_1}<0| \wedge |\Omega_{\mathcal{O}_1}v \leq 0|\\ \vee|\epsilon_{\mu_1}>0| \wedge |\Omega_{\mathcal{O}_1}v \geq 0|\\ \vee|\epsilon_{\mu_1}=0|\end{pmatrix}\end{matrix}\right] \wedge \left[\begin{matrix}|\mathcal{O}_2 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}|\\ \wedge\begin{pmatrix}|\epsilon_{\mu_2}<0| \wedge |\Omega_{\mathcal{O}_2}v \leq 0|\\ \vee|\epsilon_{\mu_2}>0| \wedge |\Omega_{\mathcal{O}_2}v \geq 0|\\ \vee|\epsilon_{\mu_2}=0|\end{pmatrix}\end{matrix}\right]\\
&\wedge \left[\begin{matrix}|\mathcal{O}_3 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}|\\ \wedge\begin{pmatrix}|\epsilon_{\lambda}<0| \wedge |\Omega_{\mathcal{O}_3}v \leq 0|\\ \vee|\epsilon_{\lambda}>0| \wedge |\Omega_{\mathcal{O}_3}v \geq 0|\\ \vee|\epsilon_{\lambda}=0|\end{pmatrix}\end{matrix}\right] \wedge \begin{pmatrix}|\epsilon_\eta<0| \wedge |v \text{ is } \mathtt{P}|\\ \vee|\epsilon_\eta>0| \wedge |v \text{ is } \mathtt{N}|\\ \vee|\epsilon_\eta=0|\end{pmatrix}.
\end{aligned} \tag{16}$$

From Table 4 we obtain the following relations.

- $\left|\Delta_{\boldsymbol{\epsilon}}(h_1x_1+h_2x_2^+ + b(-x_2)^+)\geq 0\right| = |\epsilon_{h_1}\geq 0|\wedge|\epsilon_{h_2}\geq 0|\wedge|\epsilon_b\geq 0|$,
- $|\mathcal{O}_1$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}| = true$ (see cell 6).
- $|\Omega_{\mathcal{O}_1}v\leq 0| = true$ (see cell 16).
- $|\Omega_{\mathcal{O}_1}v\geq 0| = |\Delta_{\mathbf{e}_1}v\geq 0| = false$ (see cell 14).
- $|\mathcal{O}_2$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}| = true$ (see cell 6).
- $|\Omega_{\mathcal{O}_2}v\leq 0| = true$ (see cell 16).
- $|\Omega_{\mathcal{O}_2}v\geq 0| = |\Delta_{\mathbf{e}_2-\mathbf{e}_1}v\geq 0|$ false when $h_1\leq h_2$ (see cell 14).
- $|\mathcal{O}_3$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}| = true$ (see cell 5).
- $|\Omega_{\mathcal{O}_3}v\leq 0| = |\Delta_{-\mathbf{e}_2}v\leq 0| = false$ (see cell 15).
- $|\Omega_{\mathcal{O}_3}v\geq 0| = |\Delta_{-\mathbf{e}_2}v\geq 0| = false$ (see cell 13).
- $|v$ is $\mathtt{P}| = true$ because all costs are positive.
- $|v$ is $\mathtt{N}| = false$ because all costs are positive.

So equation (16) can be reduced, and $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if

$$|\epsilon_{h_1}\geq 0|\wedge|\epsilon_{h_2}\geq 0|\wedge|\epsilon_b\geq 0|\wedge|\epsilon_{\mu_1}\leq 0|\wedge|\epsilon_{\mu_2}\leq 0|\wedge|\epsilon_\lambda = 0|\wedge|\epsilon_\eta\leq 0|. \tag{17}$$

Conclusion, the optimal value function is increasing in the costs h_i and b, and decreasing in the service rate μ_i and the discount rate η.

E.2. Convexity/concavity

First we look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$. However $|\mathcal{O}_1$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}| = false$, so $\boldsymbol{\mathcal{M}}$ does not propagate $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ (see Proposition 3 and cell 10 in Table 4).

Now we look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ if the following condition is satisfied, knowing that v is $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

$$\begin{aligned}
&\left|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}(h_1x_1+h_2x_2^+ + b(-x_2)^+)\leq 0\right| \\
&\bigwedge\left[\begin{array}{c}|\mathcal{O}_1 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge\begin{pmatrix}|\epsilon_{\mu_1}>0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_1}v\leq 0| \\ \vee|\epsilon_{\mu_1}<0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_1}v\geq 0| \\ \vee|\epsilon_{\mu_1}=0|\end{pmatrix}\end{array}\right]\bigwedge\left[\begin{array}{c}|\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge\begin{pmatrix}|\epsilon_{\mu_2}>0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_2}v\leq 0| \\ \vee|\epsilon_{\mu_2}<0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_2}v\geq 0| \\ \vee|\epsilon_{\mu_2}=0|\end{pmatrix}\end{array}\right] \\
&\bigwedge\left[\begin{array}{c}|\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge\begin{pmatrix}|\epsilon_{\lambda}>0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_3}v\leq 0| \\ \vee|\epsilon_{\lambda}<0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_3}v\geq 0| \\ \vee|\epsilon_{\lambda}=0|\end{pmatrix}\end{array}\right]\bigwedge\begin{pmatrix}|\epsilon_\eta>0|\wedge|v \text{ is } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \vee|\epsilon_\eta<0|\wedge|v \text{ is } \mathtt{I}_{-\boldsymbol{\epsilon}}| \\ \vee|\epsilon_\eta=0|\end{pmatrix}.
\end{aligned} \tag{18}$$

From Table 4 we obtain the following relations.

- $\left|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}(h_1x_1+h_2x_2^+ + b(-x_2)^+)\leq 0\right| = true$,
- $|\mathcal{O}_1$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| = \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}\vee\mathtt{S}^{ub}_{\mathbf{e}_1,\boldsymbol{\epsilon}}$ (see cell 10),

- $|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_1} v \le 0| = \mathtt{S}_{-\boldsymbol\epsilon,\mathbf{e}_1}$ (see cell 18),
- $|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_1} v \ge 0| = \mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1}$ (see cell 18),
- $|\mathcal{O}_2$ propagates $\mathtt{S}_{\boldsymbol\epsilon,-\boldsymbol\epsilon}| = \mathtt{S}_{\mathbf{e}_2-\mathbf{e}_1,\boldsymbol\epsilon} \vee \mathtt{S}^{ub}_{\mathbf{e}_2-\mathbf{e}_1,\boldsymbol\epsilon}$ (see cell 10),
- $|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_2} v \le 0| = \mathtt{S}_{-\boldsymbol\epsilon,\mathbf{e}_2-\mathbf{e}_1}$ (see cell 18),
- $|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_2} v \ge 0| = \mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_2-\mathbf{e}_1}$ (see cell 18),
- $|\mathcal{O}_3$ propagates $\mathtt{S}_{\boldsymbol\epsilon,-\boldsymbol\epsilon}| = \mathtt{S}_{-\mathbf{e}_2,\boldsymbol\epsilon} \vee \mathtt{S}^{ub}_{-\mathbf{e}_2,\boldsymbol\epsilon}$ (see cell 9),
- $|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_3} v \le 0| = \mathtt{S}_{-\boldsymbol\epsilon,-\mathbf{e}_2}$ (see cell 17),
- $|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_3} v \ge 0| = \mathtt{S}_{\boldsymbol\epsilon,-\mathbf{e}_2}$ (see cell 17),
- $|v$ is $\mathtt{I}_{\boldsymbol\epsilon}|$ (see equation 17). $|\epsilon_{h_1} \ge 0| \wedge |\epsilon_{h_2} \ge 0| \wedge |\epsilon_b \ge 0| \wedge |\epsilon_{\mu_1} \le 0| \wedge |\epsilon_{\mu_2} < 0| \wedge |\epsilon_\lambda = 0| \wedge |\epsilon_\eta \le 0|$.
- $|v$ is $\mathtt{I}_{-\boldsymbol\epsilon}|$ if (see equation 17) $|\epsilon_{h_1} \le 0| \wedge |\epsilon_{h_2} \le 0| \wedge |\epsilon_b \le 0| \wedge |\epsilon_{\mu_1} \ge 0| \wedge |\epsilon_{\mu_2} \ge 0| \wedge |\epsilon_\lambda = 0| \wedge |\epsilon_\eta \ge 0|$.

In the following section (see equation 15) we will see that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1}$, $\mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_2-\mathbf{e}_1}$, and $\mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_2}$ if

$$|\epsilon_{h_1} = 0| \wedge |\epsilon_{h_2} \ge 0| \wedge |\epsilon_b \ge 0| \wedge |\epsilon_{\mu_1} = 0| \wedge |\epsilon_{\mu_2} = 0| \wedge |\epsilon_\lambda \le 0|.$$

So $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol\epsilon,-\boldsymbol\epsilon}$ if

$$|\epsilon_{h_1} = 0| \wedge |\epsilon_{h_2} \ge 0| \wedge |\epsilon_b \ge 0| \wedge |\epsilon_{\mu_1} = 0| \wedge |\epsilon_{\mu_2} = 0| \wedge |\epsilon_\lambda = 0|.$$

Conclusion, the optimal value function is concave in the costs h_2 and b.

E.3. Monotonicity of the optimal policy

We look for the condition on v and $\boldsymbol\epsilon$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1}$ and $\mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_2-\mathbf{e}_1}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol\epsilon,\mathbf{d}}$ if the conditions (19) and (20) are satisfied, knowing that v is $\mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1}$, $\mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_2-\mathbf{e}_1}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

$$\begin{aligned}
&\left|\Delta_{\mathbf{e}_1}\Delta_{\boldsymbol\epsilon}(h_1x_1 + h_2x_2^+ + b(-x_2)^+) \le 0\right| \\
&\bigwedge \left[\begin{array}{c} |\mathcal{O}_1 \text{ propagates } \mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1}| \\ \bigwedge \left(\begin{array}{c} |\epsilon_{\mu_1} > 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \le 0| \\ \vee |\epsilon_{\mu_1} < 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \ge 0| \\ \vee |\epsilon_{\mu_1} = 0| \end{array}\right) \end{array}\right] \bigwedge \left[\begin{array}{c} |\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1}| \\ \bigwedge \left(\begin{array}{c} |\epsilon_{\mu_2} > 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \le 0| \\ \vee |\epsilon_{\mu_2} < 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \ge 0| \\ \vee |\epsilon_{\mu_2} = 0| \end{array}\right) \end{array}\right] \\
&\bigwedge \left[\begin{array}{c} |\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1}| \\ \bigwedge \left(\begin{array}{c} |\epsilon_\lambda > 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \le 0| \\ \vee |\epsilon_\lambda < 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \ge 0| \\ \vee |\epsilon_\lambda = 0| \end{array}\right) \end{array}\right] \bigwedge \left(\begin{array}{c} |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{I}_{\mathbf{e}_1}| \\ \vee |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{I}_{-\mathbf{e}_1}| \\ \vee |\epsilon_\eta = 0| \end{array}\right).
\end{aligned} \tag{19}$$

From Table 4 we obtain the following relations.

- $\left|\Delta_{\boldsymbol\epsilon}\Delta_{\mathbf{e}_1}(h_1x_1 + h_2x_2^+ + b(-x_2)^+) \le 0\right| = |\epsilon_{h_1} \ge 0| \wedge |\epsilon_{h_2} \ge 0| \wedge |\epsilon_b \ge 0|$,
- $|\mathcal{O}_1$ propagates $\mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \le 0| = false$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \ge 0| = true$,

- $|\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \leq 0| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \geq 0| = false$,
- $|\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \leq 0| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \geq 0| = false$,
- $|v \text{ is } \mathtt{I}_{\mathbf{e}_1}| = false$,
- $|v \text{ is } \mathtt{I}_{-\mathbf{e}_1}| = false$.

$$
\begin{aligned}
&\left|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Delta_{\boldsymbol{\epsilon}}(h_1x_1 + h_2x_2^+ + b(-x_2)^+) \leq 0\right| \\
&\bigwedge \left[\begin{array}{c} |\mathcal{O}_1 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| \\ \bigwedge \begin{pmatrix} |\epsilon_{\mu_1} > 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \leq 0| \\ \vee |\epsilon_{\mu_1} < 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \geq 0| \\ \vee |\epsilon_{\mu_1} = 0| \end{pmatrix} \end{array}\right] \bigwedge \left[\begin{array}{c} |\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| \\ \bigwedge \begin{pmatrix} |\epsilon_{\mu_2} > 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \leq 0| \\ \vee |\epsilon_{\mu_2} < 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \geq 0| \\ \vee |\epsilon_{\mu_2} = 0| \end{pmatrix} \end{array}\right] \\
&\bigwedge \left[\begin{array}{c} |\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| \\ \bigwedge \begin{pmatrix} |\epsilon_{\lambda} > 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \leq 0| \\ \vee |\epsilon_{\lambda} < 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \geq 0| \\ \vee |\epsilon_{\lambda} = 0| \end{pmatrix} \end{array}\right] \bigwedge \begin{pmatrix} |\epsilon_{\eta} > 0| \wedge |v \text{ is } \mathtt{I}_{\mathbf{e}_2-\mathbf{e}_1}| \\ \vee |\epsilon_{\eta} < 0| \wedge |v \text{ is } \mathtt{I}_{\mathbf{e}_1-\mathbf{e}_2}| \\ \vee |\epsilon_{\eta} = 0| \end{pmatrix}.
\end{aligned} \tag{20}
$$

From Table 4 we obtain the following relations.

- $\left|\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{e}_2-\mathbf{e}_1}(h_1x_1 + h_2x_2^+ + b(-x_2)^+) \leq 0\right| = |\epsilon_{h_1} \leq 0| \wedge |\epsilon_{h_2} \geq 0| \wedge |\epsilon_b \geq 0|$,
- $|\mathcal{O}_1 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \leq 0| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \geq 0| = false$,
- $|\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \leq 0| = false$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \geq 0| = true$,
- $|\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \leq 0| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \geq 0| = false$,
- $|v \text{ is } \mathtt{I}_{\mathbf{e}_2-\mathbf{e}_1}| = false$,
- $|v \text{ is } \mathtt{I}_{-\mathbf{e}_2-\mathbf{e}_1}| = false$.

So equations (19) and (20) reduce to

$$
|\epsilon_{h_1} = 0| \wedge |\epsilon_{h_2} \geq 0| \wedge |\epsilon_b \geq 0| \wedge |\epsilon_{\mu_1} = 0| \wedge |\epsilon_{\mu_2} = 0| \wedge |\epsilon_{\lambda} \leq 0|.
$$

Conclusion, the optimal switching curves $s_i(x_1)$ are increasing in the demand rate λ, the backlog costs b, and decreasing in the holding cost h_2.

Appendix F: Detailed tables

	$T_{A(i)}$	$T_{D(i)}$	T_{PD} $\left(\sum_k \mathbf{a}_k = -\mathbf{e}_i - \mathbf{e}_j\right)$	$T_{T(i,j)}$
$\mathtt{P}$	*true*	*true*	*true*	*true*
$\mathtt{N}$	*true*	*true*	*true*	*true*
$\mathtt{I}_{\boldsymbol{\epsilon}}$	*true*	*true*	*true*	*true*
$\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$	*true*	*true*	*true*	*true*
$\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$	*true*	*true*	*true*	*true*
$\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}$	*true*	*true*	*true*	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$
$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$
$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*	*true*	*true*	*true*
$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}$	*true*	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$ (=*false* in most cases)	*true*
$\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}$	*true*	*true*	*true*	$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$
$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*	$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$
$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*	*true*	*true*	*true*
$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*	$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*
$\Omega_{\mathcal{O}} v \geq 0$	$\mathtt{I}_{\mathbf{e}_i}$	$\mathtt{D}_{\mathbf{e}_i}$	$\mathtt{D}_{\mathbf{e}_i}$	$\mathtt{I}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Omega_{\mathcal{O}} v \leq 0$	$\mathtt{D}_{\mathbf{e}_i}$	$\mathtt{I}_{\mathbf{e}_i}$	$\mathtt{I}_{\mathbf{e}_i}$	$\mathtt{D}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_i}$	$\mathtt{S}_{\boldsymbol{\epsilon},-\mathbf{e}_i}$	$\mathtt{S}_{\boldsymbol{\epsilon},-\mathbf{e}_i}$	$\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i+\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_j-\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	*false*	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_i-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_j,-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i}$	*false*	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_j-\mathbf{e}_i}$

Table 6 Detailed results for Arrival, Departure, Parallel Departure, and Tandem server operators

	$T_{CA(i)}$ and $T_{BA(i)}$	$T_{CD(i)}$
$\mathtt{P}$	$\lvert c \geq 0\rvert$	$\lvert c \geq 0\rvert$
$\mathtt{N}$	$\lvert c \leq 0\rvert$	$\lvert c \leq 0\rvert$
$\mathtt{I}_{\epsilon}$	$\lvert \epsilon_c \geq 0\rvert$	$\lvert \epsilon_c \geq 0\rvert$
$\mathtt{S}_{\epsilon,\epsilon}$	$\mathtt{S}_{\mathbf{e}_i,\epsilon}\wedge\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c=0\rvert$	$\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon}\wedge\mathtt{S}_{\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\epsilon,-\epsilon}$	$\mathtt{S}_{\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\geq0\rvert\vee\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq0\rvert$	$\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\geq0\rvert\vee\mathtt{S}_{\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq0\rvert$
$\mathtt{S}_{\mathbf{e}_i,\epsilon}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i}\wedge\lvert\epsilon_c\leq0\rvert\vee\mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i}\wedge\lvert\epsilon_c\geq0\rvert\vee\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i}\wedge\lvert\epsilon_c\geq0\rvert\vee\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{2\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{2\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert$	$\left(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,-\mathbf{e}_i}\ \vee\mathtt{S}_{2\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}_{\mathbf{e}_i,\epsilon}\vee\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert\right)$ $\wedge\mathtt{S}_{\epsilon,\mathbf{e}_j}\wedge\lvert\epsilon_c\geq0\rvert$
$\mathtt{S}_{\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_j+\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_j+\mathbf{e}_i,\epsilon}\wedge\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_j+\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_j,-\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\mathtt{S}_{\mathbf{e}_j+\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}_{\mathbf{e}_i,\epsilon}\vee\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge\mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert$	$\left(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,-\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\epsilon}\ \vee\mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}_{\mathbf{e}_i,\epsilon}\vee\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert\right)$ $\wedge\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge\lvert0\geq\epsilon_c\rvert$
$\mathtt{S}_{-\mathbf{e}_i,\epsilon}$	$\mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i}\wedge\lvert\epsilon_c\leq0\rvert\vee\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i}\wedge\lvert\epsilon_c\geq0\rvert\vee\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i}\wedge\lvert\epsilon_c\leq0\rvert\vee\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{-2\mathbf{e}_i-\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{-\mathbf{e}_j,\epsilon}\wedge\mathtt{S}_{-2\mathbf{e}_i-\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert$	$\left(\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}\ \vee\mathtt{S}_{-2\mathbf{e}_i-\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}^{ub}_{-\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{-\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert\right)$ $\wedge\mathtt{S}_{-\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\leq0\rvert$
$\mathtt{S}_{-\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{-\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_j+\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{-\mathbf{e}_j+\mathbf{e}_i,\epsilon}\wedge\mathtt{S}_{-\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_j+\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{-\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\mathtt{S}_{-\mathbf{e}_j+\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}_{\mathbf{e}_i,\epsilon}\vee\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon})$ $\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}\wedge\mathtt{S}_{2\mathbf{e}_i-\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{2\mathbf{e}_i-\mathbf{e}_j,\epsilon}\wedge\mathtt{S}_{-\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert$	$\left(\mathtt{S}^{ub}_{\mathbf{e}_i-\mathbf{e}_j,-\mathbf{e}_i}\wedge\lvert\epsilon_c\geq0\rvert\ \vee\mathtt{S}_{2\mathbf{e}_i-\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}^{ub}_{-\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{-\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert\right)$ $\wedge\mathtt{S}_{\epsilon,-\mathbf{e}_j}\wedge\lvert\epsilon_c\geq0\rvert$
$\Omega_{\mathcal{O}}v\geq0$	$\lvert\Delta_{\mathbf{e}_i}v\geq-c\rvert$	$\lvert\Delta_{-\mathbf{e}_i}v\geq-c\rvert$
$\Omega_{\mathcal{O}}v\leq0$	*true*	*true*
$\Delta_{\epsilon}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\epsilon,\mathbf{e}_i}\wedge\lvert\epsilon_c\geq0\rvert$	$\mathtt{S}_{\epsilon,-\mathbf{e}_i}\wedge\lvert\epsilon_c\geq0\rvert$
$\Delta_{\mathbf{e}_i}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i}\wedge\lvert\Delta_{-\mathbf{e}_i}v\geq-c\rvert$
$\Delta_{\mathbf{e}_i+\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,-\mathbf{e}_i}\wedge\lvert\Delta_{-\mathbf{e}_i}v\geq-c\rvert$
$\Delta_{\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_j-\mathbf{e}_i}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,-\mathbf{e}_i}\wedge\lvert\Delta_{-\mathbf{e}_i}v\leq-c\rvert$
$\Delta_{-\mathbf{e}_i}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_i-\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_j,-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i-\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,-\mathbf{e}_i}\wedge\lvert\Delta_{-\mathbf{e}_i}v\geq-c\rvert$

Table 7 Detailed results for Controlled Arrival, Batch Arrival, and Controlled Departure operators

	$T_{CT(i,j)}$	T_{CAF} $(\mathbf{a}=\mathbf{e}_i+\mathbf{e}_j)$
$\mathtt{P}$	$\lvert c\geq 0\rvert$	$\lvert c\geq 0\rvert$
$\mathtt{N}$	$\lvert c\leq 0\rvert$	$\lvert c\leq 0\rvert$
$\mathtt{I}_{\boldsymbol{\epsilon}}$	$\lvert \epsilon_c\geq 0\rvert$	$\lvert \epsilon_c\geq 0\rvert$
$\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\geq0\rvert\vee\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\geq0\rvert\vee\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$
$\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}$	$\mathtt{S}^{ub}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_j}\wedge\lvert\epsilon_c\geq0\rvert$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{2\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{2\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\left(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\vee\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert\right)$ $\wedge\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_j}\wedge\lvert\epsilon_c\geq0\rvert$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{2\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{2\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_i+2\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_i+2\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\epsilon_c\leq0\rvert\vee\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}$	$\left(\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}\vee\mathtt{S}_{-2\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert\right)$ $\wedge\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-2\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\mathtt{S}_{-2\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\left(\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}\vee\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert\right)$ $\wedge\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{-2\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\mathtt{S}_{-2\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-2\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\mathtt{S}_{-2\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}^{ub}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\epsilon_c\geq0\rvert\vee\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\leq0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\lvert\epsilon_c\geq0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}\wedge\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}})\wedge\lvert\epsilon_c=0\rvert$
$\Omega_{\mathcal{O}}v\geq0$	$\lvert\Delta_{\mathbf{e}_j-\mathbf{e}_i}v\geq-c\rvert$	$\lvert\Delta_{\mathbf{e}_i+\mathbf{e}_j}v\geq-c\rvert$
$\Omega_{\mathcal{O}}v\leq0$	$true$	$true$
$\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\epsilon_c\geq0\rvert$	$\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_i+\mathbf{e}_j}\wedge\lvert\epsilon_c\geq0\rvert$
$\Delta_{\mathbf{e}_i}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\Delta_{\mathbf{e}_j-\mathbf{e}_i}v\geq-c\rvert$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{\mathbf{e}_i+\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\Delta_{\mathbf{e}_j-\mathbf{e}_i}v\geq-c\rvert$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{\mathbf{e}_j-\mathbf{e}_i}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{-\mathbf{e}_i}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{-\mathbf{e}_i-\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{-\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{\mathbf{e}_i-\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq0$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\Delta_{\mathbf{e}_j-\mathbf{e}_i}v\geq-c\rvert$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}$

Table 8 Detailed results for Controlled Tandem and Controled Arrival as Fork operators

	$T_{R(i,j)}$
$\mathtt{P}$	$\left\|c^j \geq 0\right\| \wedge \left\|c^i \geq 0\right\|$
$\mathtt{N}$	$\left\|c^j \leq 0\right\| \wedge \left\|c^i \leq 0\right\|$
$\mathtt{I}_{\epsilon}$	$\left\|\epsilon_c^j \geq 0\right\| \wedge \left\|\epsilon_c^i \geq 0\right\|$
$\mathtt{S}_{\epsilon,\epsilon}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \wedge \mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} = \epsilon_{c^i}\|$
$\mathtt{S}_{\epsilon,-\epsilon}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} \geq \epsilon_{c^i}\| \vee \mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} \leq \epsilon_{c^i}\|$
$\mathtt{S}_{\mathbf{e}_i,\epsilon}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{2\mathbf{e}_i-\mathbf{e}_j,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{\mathbf{e}_j,\epsilon} \wedge \mathtt{S}_{2\mathbf{e}_i-\mathbf{e}_j,\epsilon} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{\mathbf{e}_j,\epsilon} \wedge \mathtt{S}_{\mathbf{e}_i,\epsilon} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{2\mathbf{e}_j-\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{2\mathbf{e}_j-\mathbf{e}_i,\epsilon} \wedge \mathtt{S}_{\mathbf{e}_i,\epsilon} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{-\mathbf{e}_i,\epsilon}$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\epsilon} \wedge \mathtt{S}_{-\mathbf{e}_j,\epsilon} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{-\mathbf{e}_i,\epsilon} \wedge \mathtt{S}_{-\mathbf{e}_j,\epsilon} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{-\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{-2\mathbf{e}_j+\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_i,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{-\mathbf{e}_i,\epsilon} \wedge \mathtt{S}_{-2\mathbf{e}_j+\mathbf{e}_i,\epsilon} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\epsilon} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \left\|\epsilon_{c^j - c^i} = 0\right\|$
$\Omega_{\mathcal{O}} v \geq 0$	$\left\|\Delta_{\mathbf{e}_i} v \geq -c^i\right\| \wedge \left\|\Delta_{\mathbf{e}_j} v \geq -c^j\right\|$
$\Omega_{\mathcal{O}} v \leq 0$	$\left\|\Delta_{\mathbf{e}_i} v \leq -c^i\right\| \vee \left\|\Delta_{\mathbf{e}_j} v \leq -c^j\right\|$
$\Delta_{\epsilon} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\epsilon,\mathbf{e}_i} \wedge \mathtt{S}_{\epsilon,\mathbf{e}_j-\mathbf{e}_i} \wedge \left\|\epsilon_c^j \geq 0\right\| \wedge \left\|\epsilon_c^i \geq 0\right\|$
$\Delta_{\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i+\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_j-\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_i-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$

Table 9 Detailed results for Routing operator