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A Risk-Sensitive Inventory model with Random Demand and Capacity

Filiz Sayin, Fikri Karaesmen, Süleyman Özekici
Dept. of Industrial Engineering, Koç University, Istanbul, Turkey, fsayin@ku.edu.tr, fkaraesmen@ku.edu.tr, sozekici@ku.edu.tr

We consider single-period inventory problems with random demand in addition to uncertain supply due to supplier capacity problems. Focusing on risk-sensitive decision makers, we propose and analyze expected utility maximization formulations and investigate the structure of optimal ordering quantities under this formulation.

Key words: Inventory Management; Newsvendor Problem; Uncertain Supply; Risk-Sensitivity

1. Introduction

This paper investigates inventory problems that are subject to both demand and supply uncertainty from the perspective of risk-averse decision makers. It is clear that demand uncertainty has a significant effect how inventories are managed and a large portion of inventory literature focuses on modeling and analyzing this effect. On the other hand, there are cases where supply uncertainty is also a significant factor. In any case, most random inventory models consider the maximization of expected profits (or minimization of expected costs) as the objective. This is consistent when the decision makers are risk-neutral but is questionable otherwise. While there is a stream of papers that investigate the impacts of risk-aversion, to our knowledge, these papers mostly focus on the standard case with only demand uncertainty. In this paper, we propose and analyze a utility-maximization approach for an inventory problem with demand and capacity uncertainty.

It is well-known that supply randomness is a significant issue as documented in several examples. Chopra and Sodhi (2004) and Serel (2008) give examples of some real cases of supply uncertainty. These include a fire at a supplier's plant that disrupted the supply of radio-frequency chips to Ericsson in 2001 (Norrman and Jansson (2004)) and supplier problems resulting in manufacturing disruptions at Land Rover (Juttner (2005)). Recently, it was reported that the tsunami in Japan in 2011 led to major disruptions in global supply chains due to component shortages.

In an inventory model with random supply, the quantity ordered by the manager is not received in full with certainty. The literature on such inventory models is relatively mature. The earliest model seems to be the one by Karlin (1958) who assumes that the only decision available is whether to order, and that if an order is placed, a random quantity is delivered. Karlin also shows the optimality of a critical level order policy under certain conditions. Two excellent review papers Yano and Lee (1995) and Grosfeld-Nir and Gerchak (2004) cover most of the original work on this problem. A recent paper by Arifoğlu and Özekici (2010) present an overview of the literature and the recent developments in this area.

The objective of this paper is to present results for risk-averse inventory managers using a utility based objective function. There is a rich literature on risk-sensitive inventory models. Some of the early papers include Lau (1980) and Eeckhoudt et al. (1995) consider the newsvendor problem and Bouakiz and Sobel (1992) examine multi-period problems. Other relevant papers include Agrawal and Seshadri (2000), Schweitzer and Cachon (2000), Chen et al. (2007), Ahmed et al. (2007). Özler et al. (2009) present a review of some of the recent literature investigating alternative objective functions.

This paper is based on Okyay et al. (2013) who consider a newsvendor problem with random and dependent supply and demand and Sayin et al. (2013) who consider a utility-based formulation. We present relevant results from these two papers for the newsvendor problem focusing on a particular on a particular model of random supply that is referred to as random capacity. Let y be the amount ordered and $Q(y)$ be the amount received under the assumption that the supplier has some random replenishment capacity K, then

$$Q(y) = \min\{K, y\}. \tag{1}$$

When an order is placed for y units, the suppliers will ship y if the total amount K of on hand inventory that they poses is greater than y. Or else, they will send all the inventory they poses, which is K. Erdem and Özekici (2002) consider a periodically reviewed single-item inventory model in a random environment with random capacity and show that a base-stock policy is optimal.

In section 2, we present the notation and summarize the results for the standard newsvendor problem without supply uncertainty. Section 3 presents extensions of these results to the case with random capacity.

2. Standard Newsvendor Model: Risk-Neutral and Risk-Sensitive Cases

In this section, we consider the well-known single-item single-period inventory problem with random demand known as the newsvendor problem. The decision maker (newsvendor) decides how many items to order in the beginning of the period for sale during that period to maximize a profit objective. If the newsvendor is assumed to be risk-neutral, as in most of the papers, then the objective function is corresponds to the maximization of the expected profit (cash flow).

We denote by D the random demand with a known cumulative distribution function $G_D(x) = P\{D \leq x\}$ and a density function g_D.He buys items at a fixed unit purchase cost c and sells them at a unit sale price s. Unsold items can be salvaged at a unit salvage value v. Moreover, it is assumed that if demand exceeds the order quantity, the newsvendor can buy additional items from other source at a higher cost. As in Eeckhoudt et al. (1995), we model this by a shortage cost p which is taken to be negative by convention. To avoid trivial situations, $s > c > v \geq 0$ and $(c - s) < p \leq 0$. These parameters also satisfy $0 \leq s + p - c$ and $0 \leq s + p - v$. The cash flow for a given order quantity y under random demand D with the above assumptions is given by:

$$\begin{aligned} CF(D, y) &= -cy + s\min\{D, y\} + v\max\{y - D, 0\} - p\max\{D - y, 0\} \\ &= -(c - v)y + (s + p - v)\min\{D, y\} - pD. \end{aligned} \tag{2}$$

Let us first assume that the decision-maker is risk-neutral and therefore maximizes $E[CF(D, y)]$. It is well-known that the objective function is concave and the optimal order quantity, y^* is given by the solution of:

$$P(D \leq y^*) = \frac{s + p - c}{s + p - v}.$$

Let us denote by $\widehat{p}$ the right-hand side of 2, i.e. $\widehat{p} = (s + p - c)/(s + p - v)$. This is a quantity that will be frequently encountered.

Let us now assume that decision maker is risk-averse and his aim is to maximize the expected utility of the cash flow. To avoid trivial situations, we suppose that the utility function u is strictly increasing ($u' > 0$). Moreover, we assume that the utility function is concave with $u'' \leq 0$ modeling the risk-averse decision maker's respective. The aim of the newsvendor is maximizing the expected utility of cash flow by choosing an ordering quantity y, or

$$\max_{y \geq 0} H(y) = E[u(CF(D,y))] \tag{3}$$

The above setting has been considered by Eeckhoudt et al. (1995) who examine the newsvendor problem with a risk-averse decision maker when the aim is to maximize the expected utility of the cash flow. Let us review their main result to set up the stage for the more complicated case in the next section. Using a decomposition of the cashflow, Eeckhoudt et al. (1995) show that the optimal order quantity y^*satisfies

$$\frac{E\left[u'\left(CF\left(D,y^*\right)\right)1_{\{D \leq y^*\}}\right]}{E\left[u'\left(CF\left(D,y^*\right)\right)\right]} = \frac{s+p-c}{s+p-v} = \widehat{p} \tag{4}$$

Please note that if the decision maker is risk-neutral, the utility function is linear, that is $u(x) = a + bx$ and $u'(x) = b$. Then, the optimality condition in (4) reduces to

$$P\{D \leq y^*\} = \widehat{p}$$

which is the same condition as in the standard risk-neutral newsvendor problem.

Eeckhoudt et al. (1995) also establish that increased risk-aversion in a certain sense leads to a lower optimal order quantity. Next, we extend these results to a case with random capacity, in addition to random supply.

3. Newsvendor Model with Random Capacity

This section considers the newsvendor problems where the randomness results from both random demand and supplier's random capacity. In this problem, supplier may not fulfill all asked quantity because of limited capacity. Let us denote by $K \geq 0$ the random variable that represents the capacity of the supplier (maximum number of units) that can be shipped). Under such a capacity restriction, the amount received from ordering y units is $\min\{K, y\}$ where the random variable. We suppose that K has the distribution function $P\{K \leq z\} = G_K(z) > 0$ and density function g_K. Moreover, we assume that D and K may be dependent and then the conditional density function of demand given $K = z$ is $g_{D|z}$. The cash flow in (2) can be updated for the newsvendor problem with random capacity as

$$CF(D,K,y) = (s+p-v)\min\{D,K,y\} - (c-v)\min\{K,y\} - pD. \tag{5}$$

Let us first consider the risk-neutral case where the objective function is to maximize the expected cash flow. This case was analyzed by Okyay et al. who find the following optimality condition for the optimal order quantity y^*:

$$\frac{P\{D \leq y^*, K > y^*\}}{P\{K > y^*\}} = P\{D \leq y^* | K > y^*\} = \widehat{p}.$$

Let us now focus on the risk-sensitive case where the objective function is to maximize $E[u(CF(D,K,y))]$. It turns out that the first order condition for the optimal order quantity is given by (see Sayin et al. (2013)):

$$\frac{E\left[u'\left(CF(D,K,y^*)\right)1_{\{D \leq y^*, K > y^*\}}\right]}{E\left[u'\left(CF(D,K,y^*)\right)1_{\{K > y^*\}}\right]} = \widehat{p} \tag{6}$$

Equation (6) expresses the first-order optimality condition. However, unlike in the standard newsvendor problem, the objective function is not necessarily concave. It is found that the existence and uniqueness of y^* satisfying (6) depends on the structure of

$$h\left(y\right)=\frac{E\left[u'\left(CF(D,K,y)\right)1_{\{D\leq y,K>y\}}\right]}{E\left[u'\left(CF(D,K,y)\right)1_{\{K>y\}}\right]}. \tag{7}$$

It is shown in Sayin et al. (2013) that if $h\left(y\right)$ is strictly increasing in y and if $h\left(0\right)\leq\widehat{p}\leq h\left(\infty\right)$, then there exists a unique $0\leq y^*\leq\infty$ that satisfies the optimality condition in (6). In addition, we have $y^*=0$ if $h\left(0\right)>\widehat{p}$. Similarly, we can conclude that if

$$P\left\{D=\infty|K=\infty\right\}>1-\widehat{p} \tag{8}$$

then $y^*=\infty$.

In the special case, of risk-neutral newsvendor is risk-neutral with utility function $u(x)=a+bx$ (6) reduces to (3). Moreover, suppose that there is no capacity restriction, $K=\infty$, the optimality condition (6) reduces to (4).

3.1. The Effect of Risk-Aversion

This subsection explores the effect of risk-aversion on the optimal order quantity. In order to analyze this effect, we apply an approach by Pratt (1964) and replace the original utility function by its concave transformation. Suppose that we have two different decision makers with the same cash flow but different utility functions, the first one with utility function $u\left(x\right)$ and a more risk-averse one with utility function $k\left(u(x)\right)$, where $k'>0$ and $k''<0$.

The objective function of the more risk-averse newsvendor is:

$$\max_{y\geq 0}\widetilde{H}(y)=E[k\left(u(CF(D,K,y))\right)]$$

The first derivative of objective function is

$$\begin{aligned}\widetilde{g}\left(y\right)&=-(c-v)\int_y^\infty\left(\int_0^y k'\left(u\left(CF_-(x,y)\right)\right)u'\left(CF_-(x,y)\right)g_{D|z}\left(x\right)dx\right)g_K\left(z\right)dz\\&\quad+(s+p-c)\int_y^\infty\left(\int_y^\infty k'\left(u\left(CF_+(x,y)\right)\right)u'\left(CF_+(x,y)\right)g_{D|z}\left(x\right)dx\right)g_K\left(z\right)dz\\&=E\left[k'\left(u\left(\boldsymbol{CF}\right)\right)u'\left(\boldsymbol{CF}\right)1_{\{K>y\}}\right]\left(-(s+p-v)\widetilde{h}\left(y\right)+(s+p-c)\right)\end{aligned} \tag{9}$$

where

$$\widetilde{h}\left(y\right)=\frac{E\left[k'\left(u\left(\boldsymbol{CF}\right)\right)u'\left(\boldsymbol{CF}\right)1_{\{D\leq y,K>y\}}\right]}{E\left[k'\left(u\left(\boldsymbol{CF}\right)\right)u'\left(\boldsymbol{CF}\right)1_{\{K>y\}}\right]}. \tag{10}$$

It turns out that the comparison of the order quantities boils down to a comparison of $\widetilde{h}\left(y\right)$ with $h(y)$. In particular, if $\widetilde{h}\left(y\right)$ and $h(y)$ are both increasing in y (a necessary condition for a unique optimal solution), and if $\widetilde{h}\left(y\right)$ ¿ $h(y)$, then the optimal order quantity of the more risk-averse newsvendor is lower as in Eeckhoudt et al. Of course, the case with random capacity requires checking the above condition case-by-case for given utility functions and demand parameters.

y	$E[CF]$	$E[U(CF)]$ $(\gamma=500)$	$E[U(CF)]$ $(\gamma=1000)$	$E[U(CF)]$ $(\gamma=10000)$	σ_{CF}
90	473.214	0.8197	0.5978	0.0921	365.88
100	483.845	0.8198	0.5994	0.0930	386.23
110	492.433	0.8192	0.6005	0.0937	405.81
120	499.290	0.8182	0.6008	0.0943	422.60
130	504.681	0.8171	0.6004	0.0948	439.91
140	508.837	0.8159	0.6001	0.0950	455.07
150	511.954	0.8146	0.5994	0.0952	468.13
160	514.197	0.8132	0.5990	0.0954	480.92
170	515.712	0.8118	0.5978	0.0955	492.15
180	516.618	0.8104	0.5970	0.0955	502.39
190	517.021	0.8089	0.5961	0.0955	510.99
200	517.009	0.8077	0.5949	0.0954	520.82
210	516.657	0.8066	0.5936	0.0954	528.12

Table 1 **The Expected Cash Flow, Expected Utilities and Standard Deviation of the Cash Flow for Different Order Quantities**

3.2. Numerical Example

We present an example to numerically observe some of the above properties. To this end, we take demand D to be exponentially distributed with mean 100 and the capacity K to be exponentially distributed with mean 200. We assume that D and K are independent. In this case, for the risk-neutral optimality condition (3) reduces to:

$$P(D \leq y^*) = \frac{s+p-c}{s+p-v}.$$

Moreover for exponentially distributed demand, y^* can be obtained explicitly as:

$$y^* = -E[D] \ln \left(\frac{c-v}{s+p-v} \right)$$

As for the risk sensitive case, we take a utility function given by:

$$U(x) = 1 - e^{-(x+500)/\gamma}$$

where γ denotes the risk tolerance parameter of the decision maker. A lower γ designates a more risk averse decision maker.

The other parameters are given as follows: $s = 10$, $c = 2$, $v = 1$ and $p = 4$. The risk neutral optimal order quantity is equal to: 194.59. We vary the order quantity y between 90 and 210 and numerically compute the corresponding expected cash flow and the utilities for three different values of the risk-tolerance parameter γ.

The results are reported in the table below.

While the risk-neutral optimal order quantity is approximately 195, a decision-maker with a γ of 500 has an optimal order quantity that is close to 100 and a decision-maker whose γ is 1000 has an optimal order quantity that is around 120. Finally, a less risk-averse decision maker with higher risk tolerance of $\gamma = 10000$ has an optimal order quantity that is around 180. In this example, increased risk aversion leads to lower order quantities and depending on the risk tolerance parameter, the change in the order quantity may be significant.

The last column of Table 1 reports the standard deviation of the cash flow which is frequently taken as an approximate measure of risk. As expected, the standard deviation is increasing in the order quantity. These values give an indication of the risk-reward trade-off. For instance, the risk-averse decision maker with a with a γ of 1000 chooses to order 120 units instead of 190 for the risk-neutral counterpart. This results in savings of 17.3% in the standard deviation of the cash flow for a corresponding loss of only 3.4% in expected cash flow. This explains why some decision-makers may prefer to avoid a purely risk-neutral consideration.

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