Auctions under Shareability and Externality

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Abstract

Auctions for standard single indivisible goods are well understood. We consider auctioning goods with properties of shareability and externality where a bidder's valuation consists of the strict value (i.e., sole ownership), sharing value and the externality in case of not possessing any units. We first study a static auction in which a set of equilibria are examined and expected revenues are compared. We also propose a two-stage auction model in which we disentangle the standard and resale auctions. Both first- and second-price auctions in stage one are investigated and the unique perfect Bayesian equilibrium is characterized. More importantly, the revenue equivalence holds in the two-stage auction for the initial seller. Finally, we compare the static auction with the two-stage auction and show that under certain conditions the initial seller obtains a higher expected revenue in the one-shot auction.

1 Introduction

There are many contexts where introducing sharing as a solution can benefit the players. For example, a supercomputer can have excess capacity in addition to meeting the internal computing demand. The service provider can offer the extra capacity to the "public." Amazon.com has recently opened the extra capacity on its EC2 cloud for public consumption as a spot market. As a second example, a procurement contract can be split between suppliers and the buyer can take the advantage of the competition between the suppliers. A third example is when the externality of not possessing the good is considerable and exclusive ownership is also expensive. In such a case, sharing can be a feasible alternative.

We study auctions with two features: shareability and externality. Shareability is the property of a product or service that can be consumed by more than one user at the same time. Externality is the effect that is imposed by the presence of other players in the auction. Consider a manager of a firm that owns a valuable intellectual property such as a patent or trade secret. The intellectual property is digitally reproducible, and the firm can simply copy the specifications at zero cost. The firm is incapable of utilizing the intellectual property or its business model is based on selling intellectual property (ARM is an excellent example), so it seeks to sell it to one or more firms interested in licensing the technology. Exclusive ownership of the intellectual property is valued higher than being shared with other firms. A single winner of the sale can not only lower its cost (e.g., production), but also grab a significant market share. As a result, an exclusive winner can impose negative externality on other firms. The seller needs to know: (1) what is the optimal number of firms that it should license this technology to, and (2) what is the profit maximizing price to charge the firms for the license.

We model this example of the intellectual property as an auction with three players. The owner of the intellectual property has no interest to commercialize it and wants to sell it to two firms. The possible outcomes can be licensing either to an individual firm or to both firms depending upon the firms' valuation profiles. Each bidder incurs externality if it does not acquire the property. We call goods with the sharing property as shareable goods and auctions for a shareable good as shareable auctions. We study the shareable auction from different perspectives. The first approach is a static auction, which is comprehensive in the sense that we consider both shareability and externality within one auction. There are potentially multiple equilibria such as sharing or sole-winning in the static auction. We investigate all equilibrium bidding strategies of the shareable auction and demonstrate the relationships among the strategies. An alternative approach is studying the two properties in sequence. We decompose the shareable auction into two stages. The first stage represents a standard single-item auction under externality where at most one bidder can win the stage-1 auction and the winner of stage-1 auction is able to resell (i.e., share) the good to other bidders in the second stage. Note that stage 2 offers a sharing opportunity. The winner of the stage-1 auction does not completely pass the good to other bidders (provided that the stage-2 auction happens). Instead, he shares the shareable good in stage 2.

The main result of this work is the analysis of the two auctions proposed for selling shareable goods. The first auction is a one-shot auction where we consider both externality and shareability at once in the auction. Subsequently, the equilibrium outcomes can be sharing, sole-winning and hybrid, i.e., a mixture of the two. We exhibit the bidding strategy of each equilibrium and further compare them. The bidding strategy in each equilibrium is represented by a tuple of a strict bidding price and a sharing price. The sharing and hybrid equilibria coincide under certain conditions. Further, the sole-winning equilibrium in the one-shot auction is a special case of the hybrid equilibrium. The second auction is a two-stage dynamic auction where we exhibit externality in stage 1 and capture shareability in stage 2. The equilibrium in the two-stage auction is a unique perfect Bayesian equilibrium of two coordinates. The first coordinate represents the bidding strategy in stage 1 and the second coordinate is a take-it-or-leave-it offer in stage 2. The 2-stage auction is easier to implement because each stage captures one property and the equilibrium outcome in each stage is easier to predict. We compare and contrast the two auctions. Under certain conditions we show that the one-shot auction can generate a higher expected revenue.

The major contribution of this research is to simultaneously study shareability and externality in the auction context. Previous studies focus on exclusively either the sharing property or externality. To the best of our knowledge, this is the first work studying these two properties under the same auction. We unify shareability and externality together within a shareable auction. One of the characteristics of a shareable auction is the existence of multiple equilibria, unlike the unique equilibrium in an auction of standard single-indivisible goods. Another contribution of this paper is the disentanglement of shareability and externality in stages. The decomposition of the shareable auction into two successive stages secures the uniqueness of the Bayesian perfect equilibrium in both the first- and second-price auctions. This setting can even bring a higher expected revenue to the initial seller under certain conditions. The final contribution is a comparison of the performance of the decomposition to the one-shot shareable auction.

An auction for a shareable product is similar to patent licensing. The problem of licensing an innovation to firms who are competitors in a downstream market has been well studied. Kamien (1992) provides an excellent survey of patent licensing. Katz & Shapiro (1986) discuss a licensing game in which the bidders are identical and their signals are publicly observable. In our problem, the bidder's signal of willingness to pay for a license is private. Schmitz (2002) analyzes a revenue-maximizing auction for a sale of multiple licenses when each bidder's signal is private. However, he assumes no externality of allocation, i.e., a firm who gets the license will not affect the other firms without the license. Our problem adds the allocation externality into the auction. Many economic phenomena with embedded auctions involve competitions followed by an auction. Both Jehiel *et al.* (1996) and Jehiel & Moldovanu (2000) discuss auctions with externality. However, neither consider the possibility of sharing. Namely, it is impossible to share the product in the auction. Contrary to a single indivisible product auction, we allow multiple bidders to share a product.

Our work is also related to the literature on procurement auctions. Procurement auctions became of great importance for sourcing due to the advances in the internet technology. Usually, the final decision is made between two candidate suppliers. A bidder can take either a sole-sourcing auction, i.e., winner-takes-all, or a dual-sourcing auction, i.e., split-award. In particular, the split-award procurement auction is widely used in practice and buyers have experienced a huge success through dual-sourcing in select occasions, for example, "the great engine war" in 20th century, Drewes (1987). Two interesting papers on which we base our model on are Anton & Yao (1992) and Anton *et al.* (2010). They study bundling decisions in a procurement auction. The bundling decision in procurement resembles the sharing property in a shareable auction. However, they do not capture the externality of allocation. The auction with both shareability and externality shares common features with the menu auction examined by Bernhemand & Whinston (1986). In the menu auction, bidders present offers in the form of a "menu" depending on different outcomes. This is also known as a "contingent" auction where a bidder's valuation depends on the realized outcome. However, in Bernhemand & Whinston (1986) all signals of valuations are publicly observable. In our models, the signals are all private.

In literature, auctions with resale opportunity have already been studied, e.g., Zheng (2002) and Haile (2003). They discuss multi-stage auctions where resales follow standard auctions. Our two-stage model differs from these settings in all stages. In particular, the stage 1 of our model consists of a standard auction with externality. Neither Zheng (2002) nor Haile (2003) discusses the externality feature. Stage 2 in both papers is a pure resale opportunity. In other words, the object in stage 2 in the two papers is completely transferred. Our stage-2 model examines a sharing opportunity, i.e, every player obtains a copy of the object if a transaction in stage 2 is realized. This is different from both of the papers since the object in stage 2 of our model is a reduced (sharing) format.

The paper is structured as follows. In Section 2, we first study a one-shot shareable auction and characterize different classes of equilibria. We further investigate the bidding equilibria and compare revenue performances. In Section 3 we introduce a two-stage auction model for shareable goods and compare it with the one-shot static auction.

2 A One-stage Auction

In this section, we study the shareable auction that unifies externality and shareability. In the shareable auction a bidder's valuation depends on the outcome of the auction. At the same time, more than one outcome could happen in equilibrium in the shareable auction. Thus, a bid is multi-dimensional capturing his or her valuations contingent on the outcomes. We adopt the convention that the owner of the good is female and other buyers are male.

2.1 Model

We start with an auction problem of 3 players: one firm or the initial seller and two bidders where both sharing and externality exist. The firm, also called the auctioneer, is the owner of the good. We interchangeably use terms bidders and buyers in this auction. The two bidders compete for a shareable good. Namely, the final outcome could be either sharing the object between the two bidders or at most one bidder owning the object. For ease of exposition, we call such an auction with both shareability and externality as a shareable auction. We characterize bidder i's valuation as vector $\vec{t} = (t, c_1(t), c_2(t))$, where the first coordinate t represents the willingness of the bidder to pay if he exclusively wins the auction, the second coordinate $0 \leq c_1(t) \leq t$ is what he is willing to pay if he shares with his opponent, and the third coordinate $c_2(t) < 0$ captures the externality imposed on him given that he receives no allocation and his opponent wins. In this auction, the firm announces an auction rule according to which a bidder submits a bid $(p(t), p_s(t))$, where $p(t)$ is the winning price for the bidder and $p_s(t)$ is the sharing price between the two bidders. The firm collects the two bids $(p(t), p_s(t))$ and $(p(t_0), p_s(t_0))$ and then chooses $\max\{p(t), p(t_0), p_s(t) + p_s(t_0)\}$. We can view this shareable auction as a first-price auction for shareable goods. We assume the seller has a zero reserve value. Furthermore, we assume bidders' strict valuations follow an i.i.d. distribution, i.e., the first coordinate $t \sim F(t)$, where $t \in [\underline{t}, \bar{t}]$ and $F(t) = \int_{\underline{t}}^{t} f(x)dx$ for density f. We use $\bar{F}(t) = 1 - F(t)$. We assume an independent private-value model and symmetric risk-neutral bidders. Each buyer submits a bid by maximizing his expected utility. Specifically, the utility for a buyer with valuation $(t, c_1(t), c_2(t))$ is represented by

$$U((p(t), p_s(t)), (p(t_0), p_s(t_0)), \vec{t}) = \begin{cases} t - p(t) & \text{if } p(t) > \max\{p(t_0), p_s(t) + p_s(t_0)\}, \\ c_2(t) & \text{if } p(t_0) > \max\{p(t), p_s(t) + p_s(t_0)\}, \\ c_1(t) - p_s(t) & \text{if } p_s(t) + p_s(t_0) \geq \max\{p(t), p(t_0)\}. \end{cases}$$

Note that we assume that the auctioneer arbitrarily breaks a tie of sole-winning between the two buyers while favoring shared-winning more than sole-winning.

We make the following assumptions about valuations and bids.

Assumption 1. Both $c_1(t)$ and $-c_2(t)$ are deterministic and increasing functions in t.

Assumption 2. The sole-winning bid $p(t)$ is continuous and monotone increasing in t over the range $[\underline{t}, \bar{t}]$.

Assumption 3. The function $t - c_1(t)$ is increasing in t over the range $[\underline{t}, \bar{t}]$.

Assumption 4. The inequality $\underline{t} + c_2(\underline{t}) \leq 2c_1(\underline{t})$ holds.

The first assumption implies that the sharing valuation increases in the strict valuation. The reverse direction is imposed to c_2 since it represents negative externality. Assumption 2 is standard in the sense that the bidders are "rational" and they are willing to bid high only if their valuations are high. Assumption 3 and 4 are technical assumptions. They are used to ensure that the sharing outcome is efficient for some bidders in an equilibrium. Notice that from these assumptions, it can be derived that both $c_1(t) - c_2(t)$ and $t - c_2(t)$ are increasing in t.

We want to find all Nash equilibria for this auction game. The game is a special first-price auction in which by inspection there are three classes of equilibria: sole-winning, sharing and the hybrid type. In the rest of this section, we show the bidding strategies for each equilibrium and compare the equilibrium revenues.

2.2 Sharing Equilibrium

In this section, we analyze the sharing equilibrium and the hybrid type is examined in Section 2.3. There is a possibility that sharing between two buyers is an equilibrium if neither strict valuation outweighs very much the other. A sharing equilibrium is a tuple $(p(t), p_s(t))$ such that $p_s(t) = p_s$ for all t and sharing is the auction equilibrium outcome, i.e., $2p_s \geq \max\{p(t), p(t_0)\}$. A sharing equilibrium $(p(t), p_s(t))$ for any t satisfies

$$(p(t), p_s(t)) \in \arg\max \int_{\underline{t}}^{\bar{t}} U((p(t), p_s(t)), (p(t_0), p_s(t_0)), \vec{t}) dF(t_0)$$

where $(p(t), p_s(t)) : [\underline{t}, \bar{t}] \longrightarrow \mathbb{R}_+^2$. By investigating the payoff function, the sharing prices $p_s(t)$ and $p_s(t_0)$ at the sharing equilibrium must be equal in order that the buyers have no incentive to deviate from an equilibrium strategy. We have $p_s(t) = p_s(t_0)$ for any t and t_0 at the sharing equilibrium. In other words, if $p_s(t) > p_s(t_0)$, then bidder i can lower his bidding price for sharing and share the object together with the other bidder. The same reasoning applies to the other bidder. However, there is no guarantee that a sharing equilibrium exists. We next characterize a condition under which such an equilibrium exists. First of all, we derive a necessary condition for its existence. At the equilibrium the sharing outcome must be at least as good as losing, i.e., $c_1(t) - p_s \geq c_2(t)$. Since $p_s(t) = p_s(t_0) = p_s$ we obtain

$$p_s \leq c_1(\underline{t}) - c_2(\underline{t}).$$

In addition, the sharing outcome at an equilibrium is also no worse than sole-winning, i.e., $c_1(t) - p_s \geq t - p(t)$. Suppose buyer j uses the strategy (p, p_s). Buyer i deviates to $(2p_s + \epsilon, p_s)$ for some $\epsilon > 0$ in order to veto the sharing equilibrium. We must have $c_1(t) - p_s \geq t - 2p_s - \epsilon$

for every t and all $\epsilon > 0$ to necessitate the sharing equilibrium. Thus we derive that

$$p_s \geq \bar{t} - c_1(\bar{t}).$$

We conclude that the necessary condition is

$$\bar{t} - c_1(\bar{t}) \leq p_s \leq c_1(\underline{t}) - c_2(\underline{t}).$$

This condition shows that the sharing valuation $c_1(t)$ for any given t must be high enough for sharing at equilibrium to be possible.

For example, let $t \sim U[a, b]$ and $c_1(t) = 0.9t$, $c_2(t) = -0.1t$. As long as $a < b \leq 10a$, the necessary condition holds. We next show a sufficient condition under which sharing is an equilibrium.

Theorem 1. *Let $B(\vec{t}, p_s) = c_1(t) - c_2(t) + p_s - \frac{c_1(t) - c_2(t) - p_s}{F(t)}$. If $p(t) \geq B(\vec{t}, p_s)$, then $(p(t), p_s)$ is a sharing equilibrium with $p_s \in [\bar{t} - c_1(\bar{t}), c_1(\underline{t}) - c_2(\underline{t})]$ and $p(\bar{t}) = 2p_s$.*

Proof. The detailed proof is deferred to Appendix. □

The sharing equilibrium is fairly general in the sense that the strict bidding price $p(t)$ is not uniquely determined. In other words, any $p(t) \geq B(\vec{t}, p_s)$ together with a fixed p_s comprise a sharing equilibrium, Figure 1. It is straightforward to verify that for a fixed p_s the function $B(\vec{t}, p_s)$ is continuous and increasing in t because

$$\begin{aligned} B(\vec{t}, p_s) &= c_1(t) - c_2(t) + p_s - \frac{c_1(t) - c_2(t) - p_s}{F(t)} \\ &= c_1(t) - c_2(t) - p_s - \frac{c_1(t) - c_2(t) - p_s}{F(t)} + 2p_s \\ &= [c_1(t) - c_2(t) - p_s]\left[1 - \frac{1}{F(t)}\right] + 2p_s. \end{aligned}$$

Note that $B(\vec{t}, p_s) = 2p_s$ when evaluating at $\bar{t}$. It can be seen that the tuple of $p(t) = 2[\bar{t} - c_1(\bar{t})]$ and $p_s = \bar{t} - c_1(\bar{t})$ is an instance of the sharing equilibrium class stated in

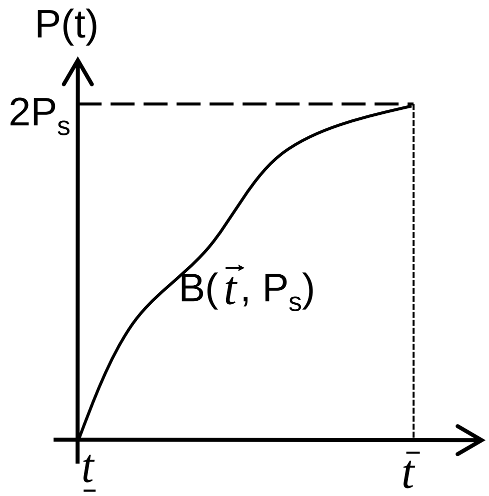

Figure 1: Sharing Equilibrium

Theorem 1.

By examining the sufficient condition of the sharing equilibrium, it is easy to see that the bidding price p is off the equilibrium path, and thus is only a supporting price. In other words, in the equilibrium the bidding price p does not directly contribute to the auctioneer's revenue while the sharing price p_s determines the equilibrium revenue. However, the bidding price p is important in the sense that it provides a lower bound for the sharing price. Furthermore, the two buyers coordinate at sharing price p_s in the equilibrium. This potentially creates a problem of collusion between the bidders since they could secretively exchange information and reach the lowest sharing price. The sharing equilibrium is endogenous. Namely, the bidding strategy depends solely on the strict valuation t and its distribution F. The sharing equilibrium is a range, not necessarily unique.

2.3 Hybrid Equilibrium

In Section 2.2, the sharing equilibrium of the sharable auction was investigated. Now, we turn to another class of equilibria of the auction, namely, a hybrid equilibrium where the equilibrium outcome could be either sharing or sole-winning, depending on the realization of bidders' valuations. To characterize the hybrid equilibrium, we need to make further assumptions.

Assumption 5. Function $c_2(t)$ is differentiable over the range $[\underline{t}, \bar{t}]$.

Assumption 6. There exists a $\tau \in [\underline{t}, \bar{t}]$ such that $\tau + c_2(\underline{t}) = c_1(\tau) + c_1(\underline{t})$.

With Assumption 6 about the existence of τ, we ensure that the range of the bidder's valuation is large enough that sharing is efficient at least for some bidders, as seen in Figure 2. Assumption 6 further shows that the sharing equilibrium except for $\tau = \bar{t}$ can not be sustained because the range $Q \equiv [\bar{t} - c_1(\bar{t}), c_1(\underline{t}) - c_2(\underline{t})]$ is now empty. We perceive that

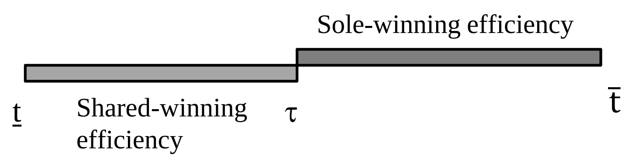

Figure 2: Overall efficiency

the structure of the hybrid equilibrium is parameterized by γ, above which the equilibrium outcome is sole-winning and below which it is a shared-winning equilibrium outcome, Figure 3. We call a bidder of a high type if his valuation is above γ and of a low type if it is below γ.

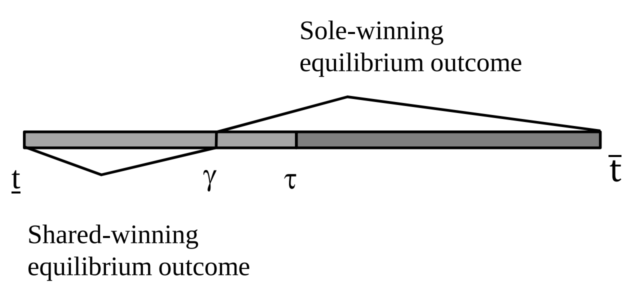

Figure 3: Equilibrium outcomes

Suppose the seller has to eventually transfer the object. Intuitively, when at least one bidder is a high type, i.e., a realization of the valuation higher than parameter γ, the equilibrium outcome is sole-winning. On the other hand, if both bidders are low types, the equilibrium is shared-winning. Let us define a tuple $(p(t), p_s)$ parameterized by γ as follows.

$$p(t) = \begin{cases} 2[\gamma - c_1(\gamma)] & \text{if } t \leq \gamma, \\ t - c_2(t) - \frac{\int_\gamma^t [1-c_2'(x)]F(x)dx}{F(t)} + \frac{F(\gamma)}{F(t)}[\gamma - 2c_1(\gamma) + c_2(\gamma)] & \text{if } t > \gamma, \end{cases}$$

and $p_s = \gamma - c_1(\gamma)$. As can be seen from Assumption 6, the existence of τ is determined by the valuation $\vec{t}$ and distribution F. The hybrid equilibrium is however parameterized by γ, i.e., each $\gamma \in [\underline{t}, \tau]$ corresponds to an equilibrium tuple. We characterize the hybrid equilibrium in the following theorem.

Theorem 2. *For any $\gamma \in [\underline{t}, \tau]$, $(p(t), p_s)$ is a symmetric Bayesian Nash equilibrium.*

Proof. See Appendix for a detailed proof. □

In the hybrid equilibrium bidding strategy, the first portion of $p(t)$ at $t \leq \gamma$ is off the equilibrium path because when $t \leq \gamma$ the equilibrium outcome can not be sole-winning for the bidder with value t. The second portion of $p(t)$ at $t > \gamma$ is the defining portion and contributes directly to the equilibrium revenue. The defining portion incorporates the effect of both externality and shareability on the equilibrium bidding. The first three terms of $p(t)$ at $t > \gamma$ together represent the externality effect, compared with the standard bidding strategy $t - \frac{\int_0^t F(x)dx}{F(t)}$ while the term $\frac{F(\gamma)}{F(t)}[\gamma - 2c_1(\gamma) + c_2(\gamma)]$ captures the shareability effect.

Note that when γ approaches $\bar{t}$, it implies that $\tau = \gamma$. We thus have $\bar{t} + c_2(\underline{t}) = c_1(\bar{t}) + c_1(\underline{t})$. The hybrid equilibrium corresponds to a particular sharing equilibrium studied in Section 2.2. In fact, the sharing equilibrium coincides with the hybrid equilibrium only when $\gamma = \bar{t}$ because the necessary condition of the sharing equilibrium, i.e., $\bar{t} - c_1(\bar{t}) < c_1(\underline{t}) - c_2(\underline{t})$, implies non-existence of τ. Unlike the bidding price $p(t)$ in the sharing equilibrium, the component $p(t)$ in the hybrid equilibrium is uniquely determined because of the existence of the sole-winning outcome in the equilibrium. The incorporation of both sharing and sole-winning in the equilibrium eliminates the indefiniteness of the bidding price $p(t)$.

We show the structure of the hybrid equilibrium with the following example. Suppose $t \sim U[0.1, 1.1]$ and $c_1(t) = 0.9t$, $c_2(t) = -0.1t$. It follows that $\tau = 1$ and we obtain the hybrid equilibrium

$$p(t) = \begin{cases} 0.2\gamma & \text{if } t \leq \gamma, \\ \frac{0.55t^2 - 0.35\gamma^2 - 0.02\gamma}{t - 0.1} & \text{if } t > \gamma, \end{cases}$$

and $p_s = 0.1\gamma$ for any $\gamma \in [0.1, 1]$. For every $\gamma \in [0.1, 1]$, the hybrid biding strategy is uniquely determined and the bidding component $p(t)$ is increasing in t.

2.4 Sole-winning Equilibrium

In this section, we consider the sole-winning equilibrium in the shareable auction. We further develop the connection of the shareable auction to standard auctions. We show that the sole-winning equilibrium in the shareable auction corresponds to the equilibrium in a standard auction with externality where the item is awarded to the bidder with the highest valuation and he pays a positive amount. The standard auction in our case is a single-object auction with externality but without sharing. That is, in the equilibrium at most one buyer is awarded the object. In Section 2.3, when parameter γ approaches $\underline{t}$, the equilibrium outcome becomes sole-winning regardless and the equilibrium bidding price for every $t > \underline{t}$ is

$$p(t) = t - c_2(t) - \frac{\int_{\underline{t}}^{t} [1 - c_2'(x)] F(x) dx}{F(t)}.$$

We characterize a sole-winning equilibrium in the following corollary to Theorem 2.

Corollary 1. *In the shareable auction, $(p(t), p_s)$ is a sole-winning equilibrium where for every $t > \underline{t}$ we have $p(t) = t - c_2(t) - \frac{\int_{\underline{t}}^{t} [1 - c_2'(x)] F(x) dx}{F(t)}$ and $p_s = \underline{t} - c_1(\underline{t}) = \frac{1}{2} p(\underline{t})$.*

The sole-winning equilibrium in the shareable auction consists of two parts: function $p(t)$ as shown in the corollary and p_s. Value p_s is a supporting price in the equilibrium and provides a lower bound to $p(t)$. Since the component p_s of the biding strategy in the shareable auction is off the sole-winning equilibrium path, we could set p_s aside and focus on a standard auction where a one-dimensional bid is allowed for each buyer. It can be shown that in the equilibrium the object has to be transferred. Jehiel *et al.* (1999) show that one of the optimal auctions according to the announced rule is the second-price auction with equilibrium bidding strategy $b(t) = t - c_2(t)$. It is a weakly dominant strategy. Function $p(t)$ corresponds to the equilibrium bidding strategy in a first-price auction. Thus we can analyze the component $p(t)$ in the setting of a first-price auction. We next derive the equilibrium bidding strategy $p(t)$ using arguments from the standard single-object auction.

It is well known that incentive compatibility of a mechanism with a single-parameter is equivalent to monotonicity of the allocation function $q(\mathbf{t})$, which is the derivative of bidder's expected surplus U over valuation $\mathbf{t}$. The two-dimensional version (e.g., the sole-winning equilibrium in the shareable auction) is that $q(\mathbf{t}) \in \partial U(\mathbf{t})$ is monotone and conservative, see Proposition 2 in Jehiel *et al.* (1999). By conservativeness of $q(\mathbf{t})$ we mean that $q(\mathbf{t})$ is the gradient of function $U(\mathbf{t})$.

Applying the fundamental theorem of calculus, we have

$$\begin{aligned} U(\mathbf{t}) &= U(\underline{\mathbf{t}}) + \int_{\underline{\mathbf{t}}}^{\mathbf{t}} q(\mathbf{t}) \cdot d\mathbf{t} \\ &= c_2(\underline{t}) + \int_{\underline{t}}^{t} F(x)dx + \int_{\underline{t}}^{t} \bar{F}(x)dc_2(x), \end{aligned}$$

where $\underline{\mathbf{t}} = (\underline{t}, c_1(\underline{t}), c_2(\underline{t}))$ and $U(\underline{\mathbf{t}}) = c_2(\underline{t})$. We further obtain that

$$U(\mathbf{t}) = c_2(t) + \int_{\underline{t}}^{t} [1 - c_2'(x)]F(x)dx.$$

By the first-price auction, the bidder's expected utility can be represented as

$$[t - p(t)]F(t) + c_2(t)\bar{F}(t) = c_2(t) + \int_{\underline{t}}^{t} [1 - c_2'(x)]F(x)dx.$$

We then obtain the equilibrium bidding strategy

$$p(t) = t - c_2(t) - \frac{\int_{\underline{t}}^{t} [1 - c_2'(x)] F(x) dx}{F(t)}.$$

Note that

$$\begin{aligned} p'(t) &= 1 - c_2'(t) - \frac{[1 - c_2'(t)] F^2(t) - f(t) \int_{\underline{t}}^{t} [1 - c_2'(t)] F(x) dx}{F^2(t)} \\ &= \frac{f(t)}{F^2(t)} \int_{\underline{t}}^{t} [1 - c_2'(t)] F(x) dx \\ &> 0. \end{aligned}$$

This shows that the equilibrium bidding strategy is continuous and monotone increasing.

We now demonstrate the structure of the sole-winning equilibrium by two examples. Suppose first that $\underline{t} = 0$, $c_1(\cdot) = c_1 \cdot t$ and $c_2(\cdot) = c_2 \cdot t$. Then the bidding strategy reduces to

$$\begin{aligned} p(t) &= t - c_2(t) - \frac{\int_{\underline{t}}^{t} [1 - c_2'(x)] F(x) dx}{F(t)} \\ &= (1 - c_2) \left[t - \frac{\int_0^t F(x) dx}{F(t)} \right], \\ p_s &= 0. \end{aligned}$$

This corresponds to the standard single-object auction subject to externality.

The second example is the one used in the hybrid equilibrium. We have $t \sim U[0.1, 1.1]$ and $c_1(\cdot) = 0.9t$, $c_2(\cdot) = -0.1t$. As shown in Figure 4, the sole-winning strategy in this example for every $t > \underline{t}$ is

$$p(t) = 0.55(t + 0.1),$$

$$p_s = \frac{1}{2} p(\underline{t}) = 0.01.$$

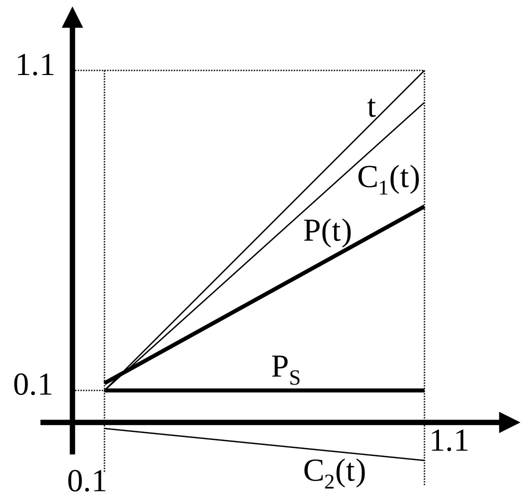

Figure 4: Sole-winning equilibrium

2.5 Equilibrium Revenues

In this section, we compare the equilibrium revenues. We start by analyzing the expected revenue of the hybrid equilibrium. The hybrid equilibrium in the shareable auction is parameterized by parameter γ at which a bidder is indifferent between sharing and sole-winning outcomes. When the two bidders are both low types, i.e., $t < \gamma$ and $t_0 < \gamma$, the equilibrium outcome is sharing and the equilibrium revenue is $2[\gamma - c_1(\gamma)]$. When at least one bidder is a high type, the equilibrium outcome is sole-winning and the equilibrium revenue is

$$2\int_{\gamma}^{\bar{t}}\{t - c_2(t) - \frac{\int_{\gamma}^{t}[1 - c_2'(x)]F(x)dx}{F(t)} + \frac{F(\gamma)}{F(t)}[\gamma - 2c_1(\gamma) + c_2(\gamma)]\}f(t)F(t)dt.$$

Hence, the expected revenue of the hybrid equilibrium $E[R_h]$ is expressed as

$$\begin{aligned}\frac{E[R_h]}{2} &= [\gamma - c_1(\gamma)]F^2(\gamma) + \int_{\gamma}^{\bar{t}}\{t - c_2(t) - \frac{\int_{\gamma}^{t}[1 - c_2'(x)]F(x)dx}{F(t)} \\ &\quad + \frac{F(\gamma)}{F(t)}[\gamma - 2c_1(\gamma) + c_2(\gamma)]\}f(t)F(t)dt.\end{aligned}$$

We further have

$$\begin{aligned}
\frac{E[R_h]}{2} &= [\gamma - c_1(\gamma)]F^2(\gamma) + \int_\gamma^{\bar{t}} [t - c_2(t)]f(t)F(t)dt - \int_\gamma^{\bar{t}} \int_\gamma^{t} [1 - c_2'(x)]F(x)dxdF(t) \\
&\quad + [\gamma - 2c_1(\gamma) + c_2(\gamma)]F(\gamma)\bar{F}(\gamma) \\
&= [\gamma - c_1(\gamma)]F^2(\gamma) + \int_\gamma^{\bar{t}} [t - c_2(t)]f(t)F(t)dt - \left\{ F(t) \int_\gamma^{t} [1 - c_2'(x)]F(x)dx \right\} \Bigg|_\gamma^{\bar{t}} \\
&\quad + \int_\gamma^{\bar{t}} [1 - c_2'(t)]F^2(t)dt + [\gamma - 2c_1(\gamma) + c_2(\gamma)]F(\gamma)\bar{F}(\gamma) \\
&= [\gamma - c_1(\gamma)]F^2(\gamma) + [\gamma - 2c_1(\gamma) + c_2(\gamma)]F(\gamma)\bar{F}(\gamma) \\
&\quad + \int_\gamma^{\bar{t}} \{[t - c_2(t)]f(t)F(t) - [1 - c_2'(t)]F(t) + [1 - c_2'(t)]F^2(t)\}dt.
\end{aligned}$$

In particular, when γ approaches $\underline{t}$, the expected revenue associated with the sole-winning equilibrium or the standard single-object auction $E[R_b]$ is

$$\begin{aligned}
\frac{E[R_b]}{2} &= \int_{\underline{t}}^{\bar{t}} p(t)f(t)F(t)dt \\
&= \int_{\underline{t}}^{\bar{t}} [t - c_2(t)]f(t)F(t)dt - \int_{\underline{t}}^{\bar{t}} \int_{\underline{t}}^{t} [1 - c_2'(x)]F(x)dxdF(t) \\
&= \int_{\underline{t}}^{\bar{t}} [t - c_2(t)]f(t)F(t)dt - \left\{ F(t) \int_{\underline{t}}^{t} [1 - c_2'(x)]F(x)dx \right\} \Bigg|_{\underline{t}}^{\bar{t}} \\
&\quad + \int_{\underline{t}}^{\bar{t}} [1 - c_2'(t)]F^2(t)dt \\
&= \int_{\underline{t}}^{\bar{t}} \{[t - c_2(t)]f(t)F(t) - [1 - c_2'(t)]F(t)\bar{F}(t)\}dt.
\end{aligned}$$

We would like to find conditions when the hybrid and sharing equilibria are better than the sole-winning equilibrium in terms of the expected revenue. Recall that at the sharing equilibrium the expected revenue is $E[R_s] = 2p_s$, where p_s is within $Q = [\bar{t} - c_1(\bar{t}), c_1(\underline{t}) - c_2(\underline{t})]$. Since the sharing equilibrium is a range, we take the lower bound of the range and compare the revenue with the sole-winning equilibrium. In other words, the expected revenue of the sharing equilibrium is assumed to be $E[R_s] = 2p_s = 2[\bar{t} - c_1(\bar{t})]$. By Theorem 1 and the derivation shown above, we have the following corollary.

Corollary 2. *If $\bar{t} - c_1(\bar{t}) \leq c_1(\underline{t}) - c_2(\underline{t})$, i.e., Q is non-empty, and*

$$\int_{\underline{t}}^{\bar{t}} [1 - \frac{F(t)}{2}][1 - c_2'(t)]F(t)dt > \frac{\bar{t} - c_2(\bar{t})}{2} - \bar{t} + c_1(\bar{t}),$$

then the sharing equilibrium has a higher revenue than the sole-winning equilibrium.

Furthermore, by Theorem 2 and the derivation shown before, we also obtain the next corollary.

Corollary 3. *If $\bar{t} - c_1(\bar{t}) > c_1(\underline{t}) - c_2(\underline{t}) > \underline{t} - c_1(\underline{t})$, i.e., there exists a $\tau \in [\underline{t}, \bar{t}]$ as shown in Assumption 6, and for any $\gamma \in [\underline{t}, \tau]$ it holds*

$$\int_{\underline{t}}^{\gamma} [1 - \frac{F(t)}{2}][1 - c_2'(t)]F(t)dt > \frac{[\gamma - c_2(\gamma)]F^2(\gamma)}{2} + [c_1(\gamma) - c_2(\gamma)]F(\gamma)\bar{F}(\gamma) - [\gamma - c_1(\gamma)]F(\gamma),$$

then the expected revenue of the hybrid equilibrium parameterized by γ is larger than the expected revenue of the sole-winning equilibrium.

Note that the sharing and hybrid equilibria coincide only at $\gamma = \bar{t}$. In other words, we observe either a sharing equilibrium or a hybrid equilibrium at an auction instance, but not both unless $\gamma = \bar{t}$. The sole-winning equilibrium is a special case of the hybrid equilibrium in the shareable auction. On the other hand, we showed earlier in this section that the sole-winning equilibrium of the shareable auction is equivalent to a standard single-object auction with externality. Hence, the expected revenue of the sole-winning equilibrium can be regarded as the expected revenue of the standard auction with externality for a single object. If conditions in either Corollary 2 or 3 hold, the shareable auction yields a higher revenue than the single-object auction with externality regardless of the equilibrium outcomes.

We illustrate Corollary 2 and 3 by the following two examples. In the first example, we assume that $c_1(t) = c_1 \cdot t = 0.6t$, $c_2(t) = \frac{1}{2}c_2 \cdot t^2 = -0.1t^2$ and $F \sim U[0.6, 0.99]$. The conditions in Corollary 2 are fully satisfied. We show that the sharing equilibrium has a higher expected revenue than in the single-object auction with externality. The sharing equilibrium with $p_s = 0.396$ appears in the one-shot shareable auction. Note that in this example $Q = 0.396$ is a singleton. The expected equilibrium revenue $E[R_s]$ of the one-shot shareable auction is 0.792. On the other hand, the bidding strategy $b(t)$ in the single-object auction corresponding to the sole-winning equilibrium is

$$b(t) = \begin{cases} \frac{-0.167t^3+0.62t^2-0.098}{t-0.6} & \text{if } t > 0.6, \\ 0.48 & \text{if } t = 0.6. \end{cases}$$

The expected revenue $E[R_b]$ of the single-object auction is $E[b(\min\{t, t_0\})] = 0.299$, where both t and t_0 follow distribution $U[0.6, 0.99]$. We have shown that $E[R_s] > E[R_b]$.

In the second example $c_1(t)$ and $c_2(t)$ remain the same, but we use $F \sim U[0.6, 1]$. Since the conditions in Corollary 3 are satisfied, we show that the expected revenue of the hybrid equilibrium is higher than the expected revenue of the single-object auction with externality. It can be shown that $\tau = 0.99$ satisfies Assumption 6. There exists a hybrid equilibrium corresponding to $\gamma = \tau = 0.99$. The expected revenue $E[R_h]$ of the hybrid equilibrium under $\gamma = 0.99$ is 1.038. The single-object auction in this example remains the same. The expected revenue $E[R_b]$ is still 0.299. Thus we obtain that $E[R_h] > E[R_b]$.

3 A Two-stage Auction

In Section 2 we considered one seller and two bidders. The seller conducts a one-shot auction between the two bidders. The class of equilibria was investigated. In this section we examine this problem from a different perspective, namely, a dynamic auction game in which each stage of the auction is simpler than the one-shot shareable auction and collectively includes the same outcomes as the shareable auction, as in Figure 5. In stage 1, there can be at most one winner who becomes the seller in the second stage. It is not allowed for bidders to share the good in stage 1, but they incur externality. However, the winner has the opportunity to resell or share with the other bidder in stage 2. The second stage is a take-it-or-leave-it offer for the bidder who loses the stage 1 auction. A revenue comparison between a one- and two-stage auctions for shareable goods is provided.

In this section, we study a perfect Bayesian equilibrium $(b(t), p_2)$ in the two-stage auction, namely, a two-stage equilibrium where the first stage bidding strategy $b(t)$ is assumed to be increasing in t and the second stage offer p_2 is weakly undominated. We again assume that t follows the distribution $F(\cdot)$ with positive density $f(\cdot)$. We assume that potential buyer's private information in stage 2 is represented by $c_1(t_0) - c_2(t_0)$ that follows a conditional distribution of $G(\cdot|t)$ with density $g(\cdot|t)$. We further assume that the conditional distribution satisfies the monotone hazard rate property, i.e., $\frac{g(\cdot)}{1-G(\cdot)}$ is strictly increasing. The stage-1

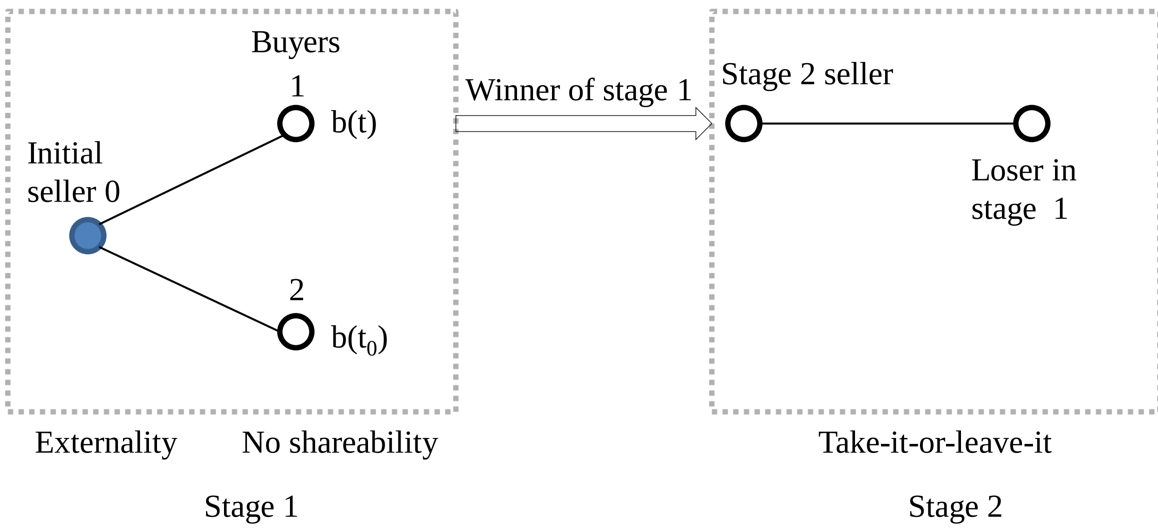

Figure 5: Two-stage auction model

model is a standard auction mechanism for a single-indivisible object subject to externality. The bidding strategy is $b_i : \mathbb{R}^3 \longrightarrow \mathbb{R}$, for $i = 1, 2$. The mechanism (ρ, p, x) in stage 1 is stated as follows. A profile $t = (\vec{t}, \vec{t_0}) \in T$ consists of $\vec{t} = (t, c_1(t), c_2(t)) \in \mathbb{R}^3$ and $\vec{t_0} = (t_0, c_1(t_0), c_2(t_0)) \in \mathbb{R}^3$. Function $p_i : T \longrightarrow \mathbb{R}, i = 1, 2$ specifies the probability that bidder i wins the first stage auction, and $x_i : T \longrightarrow \mathbb{R}$ determines the money transfer between the winner and initial seller. On the other hand, $\rho^i_j : T \longrightarrow \mathbb{R}$ defines the winning probability of bidder j if bidder i refuses to participate. In particular, $\rho^i_j = 1$, for every $i \neq j$ and $x_j = 0$; namely, the winner does not have to pay to the initial seller. In addition, $\rho^i_i = 0$, for all i.

Since the initial seller values the good zero, namely, the good has to be transferred to a bidder in stage 1, she has no incentive to participate in the stage-2 auction. We first investigate the second stage offer p_2 since it is easier and then study the first stage bidding strategy $b(t)$. In the stage-2 auction the seller, winner of the stage-1 auction, adopts a take-it-or-leave-it offer to the other buyer.

3.1 Stage-2 model

In stage 2, the current owner, say bidder 1, makes an optimal sharing offer to the other buyer at a price p_2. The offer is sharing since if accepted, both posses the item. More importantly, the current seller has learned from stage 1 about the buyer's private information. In other words, the stage-2 buyer's bid is no more than the stage-2 seller's bid, i.e., $t_0 < t$, because the bidding strategy b in stage 1 is increasing. Hence, the optimal offering price is determined by

$$p_2(t) = \max\{0, p\},$$

where p solves

$$\psi(p) = p - \frac{1 - G(p|t)}{g(p|t)} = t - c_1(t).$$

Note that the player's valuation in stage 2 is different from the valuation in stage 1 since in stage 2 the two players are faced with a sharing offer or resale opportunity. A player's valuation in stage 2 is the difference between the valuations due to the sharing outcome. The current owner 1 posts a price $p_2(t)$ at which buyer 2 could either accept or reject depending on his valuation in stage 2, i.e., $c_1(t_0) - c_2(t_0)$. Hence, the current owner's expected payoff in stage 2 is $p_2(t)[1 - G(p_2(t))]$.

3.2 Stage-1 model

As discussed, if a bidder refuses to participate in the stage-1 auction, then the other bidder automatically wins the stage-1 auction and pays nothing to the seller as a penalty or threat to the nonparticipant, i.e., $\rho_j^i = 1$, for every $i \neq j$ and $x_j = 0$. Hence, we can drop the ρ term and use a standard auction mechanism (p, x) for the stage-1 problem. In the standard auction mechanism, the bidder with the highest bid wins the auction. By the revelation principle, we can focus on a direct mechanism where the bidders truthfully reveal their private information. Without loss of generality, we set $\underline{t} = 0$. We propose two stage-1 models: a first- and second-price model.

3.2.1 A first-price sealed-bid auction model

In this model, the good is allocated to the buyer with the higher bid and he is charged for his own bid. A tie is randomly broken. Suppose $b(t)$ is a symmetric and increasing bidding strategy and bidder i with value t is the winner of the stage-1 auction. The private value of the winner of the stage-1 auction is captured by $t - c_1(t)$ in stage 2. Hence, the expected payoff of bidder i (with strict valuation t) when he reports $\tilde{t}$ in stage 1 is

$$\begin{aligned} R(t, \tilde{t}) &= \int_0^{\tilde{t}} \{t - b(\tilde{t}) + p_2(t)[1 - G(p_2(t))]\} dF(t_0) \\ &+ \int_{\tilde{t}}^{\bar{t}} \{c_2(t) + [c_1(t) - c_2(t) - p_2(t_0)]\} dF(t_0). \end{aligned}$$

In the expected payoff function, the first term corresponds to the case when buyer i wins the stage-1 auction and becomes the seller in stage 2, and the second term captures the case when buyer i loses in stage 1 and turns to be the potential buyer in stage 2.

We differentiate the payoff function with respect to $\tilde{t}$ and set it to be 0 at t, i.e.,

$$\begin{aligned} \left.\frac{\partial}{\partial \tilde{t}} R(t, \tilde{t})\right|_{\tilde{t}=t} &= \{t - b(t) + p_2(t)[1 - G(p_2(t))]\} f(t) - b'(t)F(t) \\ &\quad -[c_1(t) - p_2(t)]f(t) \\ &= 0. \end{aligned}$$

Let us denote $b_0(t) = t - c_1(t) + 2p_2(t) - p_2(t)G(p_2(t))$. Function $b_0(t)$ is the comprehensive valuation since it captures the valuation in the two stages. The term $t - c_1(t) + p_2(t)$ is the net valuation across the two stages and the term $p_2(t)[1 - G(p_2(t))]$ captures the expected payoff that buyer i can obtain as a seller in stage 2. It implies that $b'(t) + \frac{f(t)}{F(t)} b(t) = \frac{f(t)}{F(t)} b_0(t)$. We solve this differential equation and obtain

$$\begin{aligned} b(t) &= \frac{\int_0^t \frac{f(t)}{F(t)} b_0(t) e^{\int \frac{f(t)}{F(t)} dt} dt}{F(t)} \\ &= \frac{\int_0^t b_0(x) dF(x)}{F(t)} \\ &= b_0(t) - \frac{\int_0^t b_0'(x) F(x) dx}{F(t)}. \end{aligned}$$

Comparing with the standard symmetric bidding strategy $b(t) = t - \frac{\int_0^t F(x)dx}{F(t)}$ in a first-price auction, we observe that they differ in the comprehensive valuation term $b_0(t)$. It is straightforward to verify that $b(t) \leq b_0(t)$. The derivative of the bidding function reads

$$\begin{aligned} b'(t) &= -\frac{f(t)}{F(t)} b(t) + \frac{f(t)}{F(t)} b_0(t) \\ &= \frac{f(t)}{F^2(t)} \int_0^t b_0'(x) F(x) dx. \end{aligned}$$

We show the equilibrium of the first-price sealed-bid auction in the following theorem.

Theorem 3. *Let us suppose the first-price sealed-bid auction is applied in stage 1. If* $t - c_1(t)$ *is an increasing and differentiable function and* $g(p_2(t)) \leq \frac{1}{t - c_1(t)}$ *for every* t*, then* $(b(t), p_2(t))$ *with* $b(t) = b_0(t) - \frac{\int_0^t b_0'(x)F(x)dx}{F(t)}$ *and* $p_2(t) \geq 0$ *solving* $p_2(t) - \frac{1 - G(p_2(t))}{g(p_2(t))} = t - c_1(t)$ *characterizes a unique perfect Bayesian equilibrium in the two-stage auction.*

Proof. We need to show that: (1) the bidding strategy $b(t)$ in stage 1 is increasing in t, and (2) the first-order condition characterizes an optimal action for the bidder. We first show the increasing bidding strategy in stage 1. Recall that $p_2(t) \geq 0$ solves

$$p_2(t) - \frac{1 - G(p_2(t))}{g(p_2(t))} = t - c_1(t).$$

We differentiate on both sides and obtain

$$p_2'(t)\left\{1+\frac{g(p_2(t))g(p_2(t))+[1-G(p_2(t)]g'(p_2(t))}{g(p_2(t))g(p_2(t))}\right\}=1-c_1'(t).$$

Since the hazard rate function of distribution $G(\cdot)$ is strictly increasing, we have $-\frac{g'(p_2(t))}{g(p_2(t))} \leq h(p_2(t))$. By using the assumption of increasing $t-c_1(t)$, it can be shown that $p_2'(t) \geq 0$. In addition, $g(p_2(t)) \leq \frac{1}{t-c_1(t)}$ for every t implies that $2-G(p_2(t))-p_2(t)g(p_2(t)) \geq 0$. We thus obtain $b_0'(t) \geq 0$ for all t. It is straightforward to verify that if $b_0'(t) \geq 0$, then $b'(t) \geq 0$ for every t.

Secondly, p_2 is a monopoly price since the optimal strategy for the stage-2 seller is just making a take-it-or-leave-it offer regardless. Thus, p_2 is weakly undominated in stage 2. It remains to show that the bidding strategy $b(t)$ is an equilibrium strategy. It suffices to show that $\frac{\partial}{\partial t}R(t,\tilde{t})$ increases in t, which is known as the single-crossing condition. The single-crossing property ensures that the bidders in stage-1 auction behave truthfully. In other words, it ensures that $b(t)$ is the equilibrium bidding strategy in stage 1. We have

$$\begin{aligned}\frac{\partial}{\partial \tilde{t}}R(t,\tilde{t}) &= \{t-b(\tilde{t})+p_2(t)[1-G(p_2(t)]\}f(\tilde{t}) \\ &\quad -b'(\tilde{t})F(\tilde{t})-[c_1(t)-p_2(\tilde{t})]f(\tilde{t}),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial \tilde{t}}R(t,\tilde{t}) &= f(\tilde{t})\{1-c_1'(t)+p_2'(t)[1-G(p_2(t))-p_2(t)g(p_2(t))]\} \\ &\geq f(\tilde{t})[1-c_1'(t)-p_2'(t)] \\ &\geq 0.\end{aligned}$$

The first inequality follows from $2-G(p_2(t))-p_2(t)g(p_2(t)) \geq 0$ and the second inequality follows by the definition of $p_2(t)$ and the assumption of the monotone hazard rate function of $h(\cdot)$, i.e., $t-c_1(t)-p_2(t) = -\frac{1}{h(p_2(t))}$ is increasing in t since $p_2(t)$ is increasing. We have shown that the first-order condition $\frac{\partial}{\partial \tilde{t}} R(t,\tilde{t})\Big|_{\tilde{t}=t} = 0$ is sufficient. This completes the proof. □

3.2.2 A second-price sealed-bid auction model

In this model the good is allocated to the buyer with higher bid and the winner is charged for the other buyer's bid. We expect the bidding strategy to be more aggressive than in the first-price auction. We express the expected payoff function as

$$\begin{aligned} R(t,\tilde{t}) &= \int_0^{\tilde{t}} \{t - b(t_0) + p_2(t)[1 - G(p_2(t))]\} dF(t_0) \\ &\quad + \int_{\tilde{t}}^{\bar{t}} \{c_2(t) + [c_1(t) - c_2(t) - p_2(t_0)]\} dF(t_0). \end{aligned}$$

The payoff function is different from before only in the payment term $b(t_0)$ in the first integral. Following similar arguments, the bidding strategy in the second-price auction is stated in the following corollary to Theorem 3.

Corollary 4. *Suppose the second-price sealed-bid auction is held in stage 1. Then* $(b(t), p_2(t))$ *with* $b(t) = t - c_1(t) + 2p_2(t) - p_2(t)G(p_2(t))$ *and* $p_2(t)$ *the same as in Theorem 3 characterizes a unique perfect Bayesian equilibrium in the two-stage auction.*

It is easy to see that the bid $b(t)$ in the second-price auction is higher than in the first-price auction. This is consistent with the intuition about the first- and second-price auctions.

3.3 Revenue Equivalence in Stage 1

It is apparent that in the two-stage auction game, the initial seller can gain profits only in the first stage by selling to one of the buyers. Further, the initial seller expects that the buyers will bid aggressively after taking the resale opportunity into account. We can compute the equilibrium revenue for the initial seller by simply looking into the bids. In the model of the first-price sealed-bid auction in stage 1, the expected revenue for the initial seller is

$$
\begin{aligned}
\mathbf{E}[b(t_{max})] &= \mathbf{E}\left[\frac{\int_0^t b_0(x)dF(x)}{F(t)}\right] \\
&= \int_0^{\bar{t}} 2f(y)\left[\int_0^y b_0(x)dF(x)\right]dy \\
&= \left[2F(y)\int_0^y b_0(x)dF(x)\right]\Big|_0^{\bar{t}} - \int_0^{\bar{t}} 2F(y)f(y)b_0(y)dy \\
&= 2\int_0^{\bar{t}} b_0(x)dF(x) - \int_0^{\bar{t}} 2F(y)f(y)b_0(y)dy \\
&= \int_0^{\bar{t}} 2\bar{F}(y)f(y)b_0(y)dy.
\end{aligned}
$$

Note that in $\mathbf{E}[b(t_{max})]$ the quantity t_{max} is the maximal valuation and $b(t_{max})$ is the maximal bid between the two bidders in the stage-1 auction. On the other hand, the expected revenue for the initial seller in the model of the second-price sealed-bid auction in stage 1 is

$$
\begin{aligned}
\mathbf{E}[b(t_{min})] &= \mathbf{E}[b_0(t_{min})] \\
&= \int_0^{\bar{t}} 2\bar{F}(y)f(y)b_0(y)dy.
\end{aligned}
$$

Likewise, in $\mathbf{E}[b(t_{min})]$ the quantity t_{min} is the minimal valuation and $b(t_{min})$ is the minimal bid between the two bidders in the stage-1 auction. We next show that the initial seller gets the same expected revenue between the first- and second-price sealed-bid auction in stage 1 given the owner in stage 2 adopts an optimal leave-it-or-take-it offer.

3.4 Revenue Comparison to the One-stage Auction

In the previous section we showed that the initial seller has the same expected revenue by using either a first- or second-price auction in stage 1. It is apparent that each of the two stages in the dynamic auction model is simpler than the one-shot auction in Section 2. In terms of implementation, the initial seller only needs to conduct a simple single-object auction in stage 1 while awarding the future resale right to the winner. An interesting question is whether the two-stage auction is better than the one-shot auction for the initial seller in terms of the expected revenue. In general it depends on the valuation profile $(t, c_1(t), c_2(t))$ and distribution F. In this section, we compare the expected revenues for the initial seller between the two auction models under certain conditions.

We assume that $c_1(t) = c_1 \cdot t$, $c_2(t) = c_2 \cdot t$, where $-1 \leq c_2 \leq 0 \leq c_1 \leq 1$ and $F \sim U[0,1]$. As shown in Section 2.4, these assumptions imply that $\gamma = \underline{t}$, namely, the sole-winning equilibrium in the one-shot auction model. It can be derived from Section 2.5 that the expected revenue in the static auction model is

$$\begin{aligned}
E[R_1] &= \int_0^1 2[t - c_2(t)]f(t)F(t)dt - \int_0^1 2[1 - c_2'(x)]F(x)\bar{F}(x)dx \\
&= 2(1-c_2)\int_0^1 (2t^2 - t)dt \\
&= \frac{1-c_2}{3}.
\end{aligned}$$

Under the same assumptions, the conditional distribution G in the two-stage auction model is expressed as

$$G(p_2|t) = \begin{cases} \frac{p_2}{(c_1-c_2)t} & \text{if } p_2 \leq (c_1-c_2)t, \\ 1 & \text{if } (c_1-c_2)t < p_2 < c_1 - c_2. \end{cases}$$

The optimal price $p_2(t)$ in stage 2 solves

$$p_2 - \frac{1 - \frac{p_2}{(c_1-c_2)t}}{\frac{1}{(c_1-c_2)t}} = (1-c_1)t.$$

We obtain that $p_2 = \frac{(1-c_2)t}{2}$. There are two cases for the two-stage auction model. The first case is $1 + c_2 \leq 2c_1$. In this case, $G(p_2|t) = \frac{p_2}{(c_1-c_2)t}$. Hence, the expected revenue for the initial seller in the two-stage model is

$$\begin{aligned}
E[R_2] &= 2\int_0^1 f(t)\bar{F}(t)b_0(t)dt \\
&= 2\int_0^1 (1-t)\left[(1-c_1)t + (1-c_2)t - \frac{(1-c_2)^2 t}{4(c_1-c_2)}\right] dt \\
&= \frac{2-c_1-c_2}{3} - \frac{(1-c_2)^2}{12(c_1-c_2)}.
\end{aligned}$$

It is easy to check that $4(1-c_1)(c_1-c_2) \leq (1-c_2)^2$. Thus, we conclude $E[R_1] \geq E[R_2]$ in case 1. The second case is $1+c_2 > 2c_1$. In this case, $G(p_2|t) = 1$, and we have

$$\begin{aligned}
E[R_2] &= 2\int_0^1 f(t)\bar{F}(t)b_0(t)dt \\
&= 2\int_0^1 (1-t)\left[(1-c_1)t + \frac{(1-c_2)t}{2}\right] dt \\
&= \frac{3-2c_1-c_2}{6}.
\end{aligned}$$

It is straightforward to verify that $E[R_1] < E[R_2]$ in case 2 since $1+c_2 > 2c_1$.

As shown in this section, the initial seller can implement a shareable auction in a simple format, e.g., a sequence of standard auctions, and achieve a higher expected revenue when assuming distribution $F \sim U[0,1]$ and a linear valuation function with $1+c_2 > 2c_1$. It is worth pointing out that the one-shot auction may have a higher expected revenue than the two-stage auction if $1+c_2 \leq 2c_1$ as shown in case 1 above. The result of a possible higher revenue of the one-shot auction is consistent with the intuition that shareable goods can be better harnessed through sharing within a shareable auction than through a secondary market, e.g., a resale opportunity. Furthermore, incorporation of externality in a shareable auction mitigates the effect of shareability. Subsequently, we have shown that the revenue inequalities between the one-shot and two-stage auctions hold only under certain conditions.

4 Summary and Concluding Remarks

Externality has been studied more extensively than shareability in literature. Yet we believe that shareability is also an important dimension. Auctions for shareable goods have received limited attention. Previous studies either do not take the sharing property of the goods into account or fail to combine sharing with externality. We investigate both shareability and externality simultaneously in the shareable auction.

In Section 2, we showed a one-shot auction in which $c_1(t)$ and $c_2(t)$ are assumed to be deterministic functions in strict valuation t. We classify three sets of equilibria, namely, the sharing, hybrid and sole-winning equilibrium. In particular, sufficient and necessary conditions of the equilibrium bidding strategy in a sharing equilibrium are analyzed. It is shown that the sharing equilibrium is endogenous, depending upon the valuation. Furthermore, it is not unique. This issue can be softened under the complete information setting as shown in Bernhemand & Whinston (1986). Then, we investigate the hybrid equilibrium that could lead to both sharing and sole-winning outcomes, depending upon the observation of valuations. We observed that the hybrid equilibrium is so general that the sole-winning equilibrium is a special case. In addition, the bidding strategy of the sole-winning equilibrium is a modified version of the bidding strategy in the standard first-price auction. We compare the expected revenues across different equilibria in the one-shot auction and rank the revenues depending on parameter γ.

In Section 3, we propose a two-stage auction in which we disentangle the standard and resale auction. In other words, the stage-1 problem resembles a standard single-object auction with externality. It is anticipated that a bidder perceiving a future resale (sharing) opportunity would bid accordingly. The stage-2 auction can be reduced to a take-it-or-leave-it offer. We study both the first- and second-price auction in stage 1 and characterize the unique perfect Bayesian equilibrium in either case. More importantly, the revenue equivalence between the two auctions holds in our two-stage auctions for the initial seller. We also compare the equilibrium revenues in the two-stage auction with the one-shot auction.

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Appendix

Proof of Theorem 1

Proof. We argue that the sharing equilibrium bidding strategy has form $(p(t), p_s)$ for bidder 1 with strict valuation t. The identical argument also applies to bidder 2 with strict valuation t_0. Let $(\hat{p}, \hat{p_s})$ and (p, p_s) represent buyer 1 and 2's bidding strategies. We analyze the deviation from the sharing equilibrium by distinguishing three different cases.

- Case 1 (Sharing deviation). In this case, we suppose $\hat{p_s} < p_s$ and $\hat{p} < \hat{p_s} + p_s$. Buyer 1 can not win the sole award and thus receives the sharing award. The resulting utility is $c_1(t) - \hat{p_s}$. Note that any $\hat{p_s}$ is dominated if $\hat{p_s} \geq p_s$. If $\hat{p} < \hat{p_s} + p_s < p(\underline{t})$, his opponent wins solely and his resulting utility is $c_2(t)$, which is negative and less than the sharing utility, i.e., $c_1(t) - p_s \geq c_2(t)$. Hence, we obtain $p(\underline{t}) - p_s \leq \hat{p_s} < p_s$. There exists a $\tilde{t} \in [\underline{t}, \bar{t}]$ such that $p(\tilde{t}) - p_s = \hat{p_s}(t)$. Consequently, the expected utility is

$$
\begin{aligned}
&[c_1(t) - \hat{p_s}(t)] \cdot P(\hat{p_s}(t) + p_s > p(t_0)) + c_2(t)[1 - P(\hat{p_s}(t) + p_s > p(t_0))] \\
&= [c_1(t) - c_2(t) - \hat{p_s}(t)] \cdot P(\hat{p} > p(t_0)) + c_2(t) \\
&= [c_1(t) - c_2(t) - p(\tilde{t}) + p_s] \cdot F(\tilde{t}) + c_2(t).
\end{aligned}
$$

  Note that the first equality follows by substituting $\hat{p_s}(t)$ and the second equality is immediate because of the i.i.d. assumption of distributions and increasing bidding function $p(t)$ in t. It remains to show in the equilibrium that $[c_1(t) - c_2(t) - p(t) + p_s] \cdot F(t) + c_2(t) \leq [c_1(t) - p_s]$, where $[c_1(t) - c_2(t) - p(t) + p_s] \cdot F(t) + c_2(t)$ is the expected utility $[c_1(t) - c_2(t) - p(\tilde{t}) + p_s] \cdot F(\tilde{t}) + c_2(t)$ evaluated at $\tilde{t} = t$. This follows from the assumption in Theorem 1 of $\hat{p} \geq B(\vec{t}, P_s)$. In conclusion, we showed in this case that a deviation from the sharing equilibrium by submitting a low sharing price is not more profitable in terms of the expected utility.

- Case 2 (sole-winning deviation). Suppose $\hat{p} > \hat{p_s} + p_s$. Buyer 1 has incentive to win the sole-award. If $\hat{p} < p(\underline{t})$, buyer 1 earns $c_2(t)$. If $\hat{p} > p(\bar{t})$, the bid $\hat{p}$ is dominated by $p(\bar{t})$ that ensures the sole-award. Hence, we have $p(\underline{t}) \leq \hat{p} \leq p(\bar{t})$. There exists a $t^* \in [\underline{t}, \bar{t}]$ such that $\hat{p} = p(t^*)$. The expected utility is $(t-\hat{p})P(\hat{p} > p(t_0)) + c_2(t)[1-P(\hat{p} > p(t_0))]$. We further have

$$\begin{aligned}&(t-\hat{p})P(\hat{p} > p(t_0)) + c_2(t)[1-P(\hat{p} > p(t_0))]\\&= [t - c_2(t) - p(t^*)]F(t^*) + c_2(t).\end{aligned}$$

  We want to show in the equilibrium that $[t - c_2(t) - p(t)]F(t) + c_2(t) \leq [c_1(t) - p_s]$, where $[t - c_2(t) - p(t)]F(t) + c_2(t)$ is the expected utility $[t - c_2(t) - p(t^*)]F(t^*) + c_2(t)$ evaluated at $t^* = t$. Since $\bar{t} - c_1(\bar{t}) \leq p_s \leq c_1(\underline{t}) - c_2(\underline{t})$, we have

$$\begin{aligned}p(t) &\geq c_1(t) - c_2(t) + p_s - \frac{c_1(t) - c_2(t) - p_s}{F(t)}\\&\geq t - c_2(t) - \frac{c_1(t) - c_2(t) - p_s}{F(t)},\end{aligned}$$

  which implies that $[t - c_2(t) - p(t)]F(t) + c_2(t) \leq [c_1(t) - p_s]$. In conclusion, the sole-winning deviation is also not more profitable.

- Case 3. Let now $\hat{p} = \hat{p_s} + p_s$. Firstly, if $\hat{p} = \hat{p_s} + p_s < p$, buyer 1's utility is $c_2(t)$. Secondly, if $\hat{p} = \hat{p_s} + p_s \geq p$, the buyer gets either the sole- or shared-award with equal probability. Notice that when $t = \bar{t}$, the equilibrium bid has the form $\hat{p} = \hat{p_s} + p_s$. By the necessary condition $t - c_1(t) \leq p_s$, we conclude that the shared-award is weakly preferred.

Since the three cases shown above cover all the possibilities for alternative bids for buyer 1 with strict valuation t, the proposed pair of $(p(t), p_s)$ is a sharing equilibrium. □

Proof of Theorem 2:

Proof. Let us denote by $\vec{t} = (t, c_1(t), c_2(t))$ the bidder 1's type and by $\vec{t_0} = (t_0, c_1(t_0), c_2(t_0))$ the bidder 2's type. In addition, suppose bidder 1 uses strategy $(\hat{p}(t), \hat{p}_s)$ while bidder 2 follows the equilibrium bidding strategy, namely $(p(t_0), p_s)$. We argue the structure of the equilibrium biding strategy by focusing on bidder 1. The identical argument also applies to bidder 2. The ex-post utility function for bidder 1 is

$$U[(\hat{p}, \hat{p}_s), (p(t_0), p_s), \vec{t}] = \begin{cases} t - \hat{p} & \text{if } \hat{p} > \max\{p(t_0), \hat{p}_s + p_s\}, \\ c_2(t) & \text{if } p(t_0) > \max\{\hat{p}, \hat{p}_s + p_s\}, \\ c_1(t) - \hat{p}_s & \text{if } \hat{p}_s + p_s \geq \max\{\hat{p}, p(t_0)\}. \end{cases}$$

From the indifference of outcomes at γ we have

$$[\gamma - p(\gamma)]F(\gamma) + c_2(\gamma)\bar{F}(\gamma) = [c_1(\gamma) - p_s]F(\gamma) + c_2(\gamma)\bar{F}(\gamma).$$

It implies that $\gamma - p(\gamma) = c_1(\gamma) - p_s$. The continuity of $p(t)$ implies that $p(\gamma) = 2p_s$. Hence, we have that $p_s = \gamma - c_1(\gamma)$.

Notice that as argued before in the sharing equilibrium, both bidders agree at the constant equilibrium sharing price p_s. For a high type bidder, i.e., $t \geq \gamma$, the expected bidder surplus is

$$EU(p, p_s, \vec{t}) = [t - p(t)]F(t) + c_2(t)\bar{F}(t).$$

For a low type bidder with $t < \gamma$, the expected bidder surplus is

$$EU(p, p_s, \vec{t}) = [c_1(t) - p_s]F(\gamma) + c_2(t)\bar{F}(\gamma).$$

We next show the proposed tuple $(p(t), p_s)$ is indeed a hybrid equilibrium by examining all possible deviations. There are two possible deviations from equilibrium bidding.

The first is a sharing deviation, i.e., $\hat{p}_s + p_s > \hat{p}$. Bidder 1 can not win the sole-winning outcome, and hence wins based on shared-winning when $\hat{p}_s + p_s > p(t_0)$. The range of $\hat{p}_s$ is $[p(\underline{t}) - p_s, p(\bar{t}) - p_s]$, i.e., $p(\underline{t}) \leq \hat{p}_s + p_s \leq p(\bar{t})$. Equilibrium bidding has the property that for all $t \leq \gamma$ we have $p(t) = 2[\gamma - c_1(\gamma)]$. In particular, we obtain $p(\underline{t}) = 2[\gamma - c_1(\gamma)]$, which implies

$$\hat{p}_s \in [p_s, p(\bar{t}) - p_s].$$

For each $\hat{p}_s$, there exists a $t^* \in [\gamma, \bar{t}]$ such that $\hat{p}_s \in [p_s, p(\bar{t}) - p_s]$ and $\hat{p}_s = p(t^*) - p_s$. The expected utility is

$$EU(p, p_s, \vec{t}) = [c_1(t) - p(t^*) + \gamma - c_1(\gamma)]F(t^*) + c_2(t)\bar{F}(t^*).$$

The second deviation is the sole-winning deviation, i.e., $\hat{p}_s + p_s \leq \hat{p}$. In this case, bidder 1 can not win the shared-winning outcome. However, he wins the sole-winning outcome if $\hat{p} > p(t_0)$. The range of $\hat{p}$ is $[2p_s, p(\bar{t})]$, i.e., for each $\hat{p} \in [2p_s, p(\bar{t})]$, there exists a $t^{**} \in [\gamma, \bar{t}]$ such that $\hat{p} = p(t^{**})$. Bidder 1's expected surplus is

$$EU(p, p_s, \vec{t}) = [t - p(t^{**}]F(t^{**}) + c_2(t)\bar{F}(t^{**}).$$

In summary, for bidder 1 with strict valuation $t \in [\underline{t}, \bar{t}]$ and bidding prices $(\hat{p}, \hat{p}_s)$, we have a unique pair (t^*, t^{**}) such that

$$\begin{cases} \hat{p}_s = p(t^*) - p_s & t^* \in [\gamma, \bar{t}], \\ \hat{p} = p(t^{**}) & t^{**} \in [\gamma, \bar{t}]. \end{cases}$$

Thus, the expected utility can be represented as

$$EU(t^*, t^{**}, \vec{t}) = \begin{cases} [c_1(t) - p(t^*) + \gamma - c_1(\gamma)]F(t^*) + c_2(t)\bar{F}(t^*) & \text{if } \hat{p}_s + p_s > \hat{p}, \\ [t - p(t^{**})]F(t^{**}) + c_2(t)\bar{F}(t^{**}) & \text{if } \hat{p}_s + p_s \leq \hat{p}. \end{cases}$$

We next provide details of the two deviations.

Deviation 1: shared-winning deviation, i.e., $\hat{p}_s + p_s > \hat{p}$.

The expected utility in this deviation is $[c_1(t) - p(t^*) + \gamma - c_1(\gamma)]F(t^*) + c_2(t)\bar{F}(t^*)$. We want to show that the shared-winning deviation is not more profitable. In other words, we want to show that for each $t \in [\underline{t}, \bar{t}]$ we have

$$[c_1(t) - p(t^*) + \gamma - c_1(\gamma)]F(t^*) + c_2(t)\bar{F}(t^*) \leq [t - p(t)]F(t) + c_2(t)\bar{F}(t).$$

We substitute $p(t^*) = t^* - c_2(t^*) - \frac{\int_\gamma^{t^*}[1-c_2'(x)]F(x)dx}{F(t^*)} + \frac{F(\gamma)}{F(t^*)}[\gamma - 2c_1(\gamma) + c_2(\gamma)]$ in the expected utility function and get

$$\begin{aligned} EU(t^*, t^{**}, \vec{t}) \;\; = \;\; & c_2(t) + [c_1(t) - c_2(t) + \gamma - c_1(\gamma) - t^* + c_2(t^*)]F(t^*) - F(\gamma)[\gamma - 2c_1(\gamma) + c_2(\gamma)] \\ & + \int_\gamma^{t^*} [1 - c_2'(x)]F(x)dx. \end{aligned}$$

Hence, we obtain that

$$\frac{\partial}{\partial t^*} EU(t^*, t^{**}, \vec{t}) = [c_1(t) - c_2(t) + \gamma - c_1(\gamma) - t^* + c_2(t^*)]f(t^*).$$

The first order condition $\frac{\partial}{\partial t^*} EU(t^*, t^{**}, \vec{t}) = 0$ implies that

$$c_1(t) - c_2(t) + \gamma - c_1(\gamma) = t^* - c_2(t^*).$$

Thus, the optimal expected utility EU^* is achieved at an optimal t^* where

$$EU^*(t^*, t^{**}, \vec{t}) = c_2(t) - F(\gamma)[\gamma - 2c_1(\gamma) + c_2(\gamma)] + \int_\gamma^{t^*} [1 - c_2'(x)]F(x)dx.$$

If bidder 1 is a high type bidder, for each $t \in [\gamma, \bar{t}]$ we have

$$
\begin{aligned}
& t - c_1(t) > \gamma - c_1(\gamma) \\
& \Longrightarrow t - c_2(t) - [c_1(t) - c_2(t)] > \gamma - c_1(\gamma) \\
& \Longrightarrow t - c_2(t) > [c_1(t) - c_2(t)] + \gamma - c_1(\gamma) = t^* - c_2(t^*) \\
& \Longrightarrow t > t^*.
\end{aligned}
$$

Hence, we further have

$$\int_{\gamma}^{t^*} [1 - c_2'(x)]F(x)dx < \int_{\gamma}^{t} [1 - c_2'(x)]F(x)dx$$

from $\gamma < t^* < t$ and $1 - c_2'(x) > 0$. Eventually, we obtain

$$
\begin{aligned}
EU^* &= c_2(t) - F(\gamma)[\gamma - 2c_1(\gamma) + c_2(\gamma)] + \int_{\gamma}^{t^*} [1 - c_2'(x)]F(x)dx \\
&< c_2(t) - F(\gamma)[\gamma - 2c_1(\gamma) + c_2(\gamma)] + \int_{\gamma}^{t} [1 - c_2'(x)]F(x)dx \\
&= [t - p(t)]F(t) + c_2(t)\bar{F}(t).
\end{aligned}
$$

In conclusion, this shows that for all $t \in [\gamma, \bar{t}]$, the shared-winning deviation is not more profitable, comparing to the sole-winning equilibrium under (t^*, t^{**}).

It remains to show the result for the low type bidder 1. That is, we need to show that for each $t < \gamma$ we have

$$[c_1(t) - p(t^*) + \gamma - c_1(\gamma)]F(t^*) + c_2(t)\bar{F}(t^*) \leq [c_1(t) - p_s]F(\gamma) + c_2(t)\bar{F}(\gamma).$$

Let us substitute $p(t^*)$ in the expected utility function and define

$$\begin{aligned} LHS &= c_2(t) + [c_1(t) - c_2(t) + \gamma - c_1(\gamma) - t^* + c_2(t^*)]F(t^*) - F(\gamma)[\gamma - 2c_1(\gamma) + c_2(\gamma)] \\ &\quad + \int_{\gamma}^{t^*} [1 - c_2'(x)]F(x)dx \end{aligned}$$

and

$$\begin{aligned} RHS &= [c_1(t) - p_s]F(\gamma) + c_2(t)\bar{F}(\gamma) \\ &= c_2(t) + F(\gamma)[c_1(t) - c_2(t) - \gamma + c_1(\gamma)]. \end{aligned}$$

Now we observe that

$$\frac{\partial}{\partial t^*} LHS = [c_1(t) - c_2(t) + \gamma - c_1(\gamma) - t^* + c_2(t^*)]f(t^*) < 0,$$

because we assume $f(t^*) > 0$ and for every $t^* > \gamma$ and $t < \gamma$ we have

$$\begin{aligned} c_1(t) - c_2(t) + \gamma - c_1(\gamma) - t^* + c_2(t^*) &< c_1(t) - c_2(t) - [c_1(\gamma) - c_2(t^*)] \\ &< c_1(t) - c_2(t) - [c_1(\gamma) - c_2(\gamma)] \\ &< 0. \end{aligned}$$

It implies that the optimal t^* equals γ. Thus, for all $t < \gamma$, we have

$$\begin{aligned} LHS &< c_2(t) + [c_1(t) - c_2(t) + \gamma - c_1(\gamma) - \gamma + c_2(\gamma)]F(\gamma) - F(\gamma)[\gamma - 2c_1(\gamma) + c_2(\gamma)] \\ &= c_2(t) + [c_1(t) - c_2(t) - \gamma + c_1(\gamma)]F(\gamma) \\ &= RHS. \end{aligned}$$

We have shown that the shared-winning deviation for $t < \gamma$ is not more profitable.

Deviation 2: sole-winning deviation, i.e., $\hat{p}_s + p_s \leq \hat{p}$.

We need to show the results for bidder 1 in two situations: low and high type. If bidder 1 is a high type, i.e., $t \in [\gamma, \bar{t}]$, we want to show that for each $t \in [\gamma, \bar{t}]$ the optimal t^{**} is exactly t. For $t \in [\gamma, \bar{t}]$ we have

$$p(t^{**}) = t^{**} - c_2(t^{**}) - \frac{\int_{\gamma}^{t^{**}} [1 - c_2'(x)]F(x)dx}{F(t^{**})} + \frac{F(\gamma)}{F(t^{**})}[\gamma - 2c_1(\gamma) + c_2(\gamma)],$$

and

$$\begin{aligned} p'(t^{**}) =& 1 - c_2'(t^{**}) - \frac{[1 - c_2'(t^{**})]F^2(t^{**}) - f(t^{**}) \int_{\gamma}^{t^{**}} [1 - c_2'(x)]F(x)dx}{F^2(t^{**})} \\ & - \frac{f(t^{**})F(\gamma)}{F^2(t^{**})}[\gamma - 2c_1(\gamma) + c_2(\gamma)]. \end{aligned}$$

In addition, we have

$$\frac{\partial}{\partial t^{**}} EU(t^*, t^{**}, \vec{t}) = -p'(t^{**})F(t^{**}) + [t - p(t^{**}) - c_2(t)]f(t^{**}).$$

The first order condition requires that $\frac{\partial}{\partial t^{**}} EU(t^*, t^{**}, \vec{t}) = 0$. It implies that

$$[t - p(t^{**}) - c_2(t)]f(t^{**}) = p'(t^{**})F(t^{**}).$$

We substitute $p(t^{**})$ and $p'(t^{**})$ and obtain $t - c_2(t) = t^{**} - c_2(t^{**})$ since by assumption $f(t^{**}) > 0$. Thus, we show that optimal t^{**} is t. This shows that the sole-winning deviation for the high type bidder 1 is not more profitable.

If bidder 1 is a low type bidder, i.e., $t \in [\underline{t}, \gamma]$, we have

$$\begin{aligned} \frac{\partial}{\partial t^{**}} EU(t^*, t^{**}, \vec{t}) &= -p'(t^{**})F(t^{**}) + [t - p(t^{**}) - c_2(t)]f(t^{**}) \\ &= [t - c_2(t) - (t^{**} - c_2(t^{**}))]f(t^{**}) \\ &< 0, \end{aligned}$$

because $f(t^{**}) > 0$ and $t - c_2(t) \leq t^{**} - c_2(t^{**})$ by assumptions. Hence, the optimal t^{**} equals γ. We have shown in the situation of the sole-winning deviation that bidder 1 can not do better than adopting the proposed bidding strategy $(p(t), p_s)$.

In conclusion, we showed that $(p(t), p_s)$ is the hybrid equilibrium bidding strategy. $\square$