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Multi-product newsvendor problem with value-at-risk considerations

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ABSTRACT

We consider the single period stochastic inventory (newsvendor) problem with downside risk constraints. The aim in the classical newsvendor problem is maximizing the expected profit. This formulation does not take into account the risk of earning less than a desired target profit or losing more than an acceptable level due to the randomness of demand. We utilize Value at Risk (VaR) as the risk measure in a newsvendor framework and investigate the multi-product newsvendor problem under a VaR constraint. To this end, we first derive the exact distribution function for the two-product newsvendor problem and develop an approximation method for the profit distribution of the N-product case ($N>2$). A mathematical programming approach is used to determine the solution of the newsvendor problem with a VaR constraint. This approach allows us to handle a wide range of cases including the correlated demand case that yields new results and insights. The accuracy of the approximation method and the effects of the system parameters on the solution are investigated numerically.

1. Introduction

In the vast literature of inventory control, most of the models employ maximization of the expected profit as the main optimality criterion. In recent studies, the financial risk in management of inventory systems is considered from various perspectives. There are many risk measures that are used in risk management in stochastic inventory models such as the satisficing probability maximization, utility functions, Value at Risk (VaR) and conditional VaR (CVaR).

In this study, we focus on VaR as a measure of downside risk and incorporate this risk measure in the multi-product newsvendor problem. VaR measures the maximum value of the random function or the variable in a β confidence interval, see for example, Jorion (1997), Artzner et al. (2000), and Simons (1996). VaR is a measure of the maximum potential change in the value of a portfolio of financial instruments over a pre-set horizon. VaR answers the question of how much one can lose with x% probability over a given time horizon. If a portfolio is expressed as a 95% one-day VaR of \$100 million, this means that there is only a 5% chance that the portfolio will lose more than \$100 million over the next day.

Multi-product newsvendor problem can also be considered as a problem of determining the best product portfolio among all the possible alternatives. Similar to the case of investing in a financial portfolio, a retailer faces a substantial risk in its ordering decisions. In a multi-product portfolio, if the retailer ends up with a high number of unsold products at the end of the season, the financial losses can be devastating. From this perspective, the objective of the retailer is maximizing the expected return while it takes calculated risks, e.g., the retailer

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knows in advance that the probability of losing more than a pre-determined level is less than the desired probability.

In this paper, we present a mathematical programming methodology to solve the multi-product newsvendor problem with a VaR constraint. The VaR constraint is expressed explicitly by using the probability distribution of the total profit.

For the two-product case, we give a compact expression that yields the total profit distribution based on the demand distributions. Once the VaR constraint can be expressed explicitly, the resulting optimization problem can be solved by a mathematical programming approach. We illustrate our approach for the cases with independent and bivariate exponential distributions.

For the multi-product case, we present an approximate method that is based on approximating the distribution of the total profit with a normal distribution following the central limit theorem. Numerical comparisons with simulation shows that this approximation is quite accurate in representing downside risk constraints.

The organization of the remaining part of this paper is as follows. In Section 2, we review the pertinent literature. We introduce the model and summarize the results for a single-product newsvendor problem with a VaR constraint in Section 3. In Section 4, the two-product newsvendor problem with a VaR constraint is formulated as a mathematical program by deriving the probability distribution of the total profit. Results for two products with independent and correlated demands are also given in this section. Section 5 extends this approach to multi-product case by approximating the probability distribution of the total profit with a normal distribution. Finally, conclusions are given in Section 6.

2. Literature review

Incorporating risk in inventory management decisions received some attention in recent years. Sankarasubramanian and Kumarasamy (1983) consider a single-period stochastic inventory problem in which it is required to determine the product order quantity which maximizes the probability of realizing a predetermined level of profit. A condition for deciding the optimal order quantity is found and explicit expressions for the optimal order quantities in three special cases are given. Schweitzer and Cachon (2000) investigate the decision bias in the newsvendor problem with a known demand distribution.

The satisfaction probability is used as an objective function in a number of studies (Lau, 1980; Lau and Lau, 1988; Li et al., 1990, 1991; Parlar and Weng, 2003). Satisficing probability is defined as the probability of exceeding a prespecified fixed target profit level. The aim is to maximize the satisfaction probability function in terms of the product order quantity. Lau (1980) solves the satisfaction probability maximization problem for a single product under the assumption of zero salvage value.

Lau and Lau (1988) consider the maximization of the probability of achieving a target profit in a two-product newsvendor problem. Solution procedures are developed to find the optimal order quantities of each product that will maximize the probability of achieving the target profit value. Li et al. (1990, 1991) present an analytical solution procedure to maximize the probability of achieving a target profit in a two-product newsvendor problem for uniformly and exponentially distributed demands respectively. Some analytical results are presented for these restrictive cases.

Parlar and Weng (2003) investigate the satisfaction probability value maximization objective in the classical newsvendor problem. They also develop a model that integrates this objective with the standard expected profit maximization objective. In our setting, the satisfaction probability function is used as a constraint of the classical newsvendor problem and the aim is to solve this model for N product case.

An alternative approach for modeling risk preferences in inventory management, has been using utility functions. Lau (1980) maximizes the expected utility of the newsvendor problem. The utility function is defined in terms of the expected value of the random profit and its standard deviation. This corresponds to the well-known mean-standard deviation trade-off approach. Eeckhoudt et al. (1995) and Bouakiz and Sobel (1992) examine the risk aversion in the newsvendor problem with an exponential utility function and show that a base-stock policy is optimal when a dynamic version of the newsvendor model is optimized with respect to an exponential utility criterion. In a recent paper, Chen et al. (2007) incorporate risk aversion through utility maximization in multiperiod inventory models involving pricing strategies.

There is a huge interest in hedging operational risks using financial instruments. Anvari (1987) uses the well-known capital asset pricing model (CAPM) in finance to investigate a newsvendor problem. Gaur and Seshadri (2005) investigate the impact of financial hedging on operational decisions in the framework of the newsvendor problem. They develop optimal hedging transactions that minimize the variance of profit and increase the expected utility for a risk-averse decision maker.

Luciano et al. (2003) investigate VaR as a risk measure in the context of a single-product multi-period Economic Order Quantity type inventory model. They present an exact analysis to obtain the VaR and also establish useful bounds. In contrast, we investigate a single-period problem but focus on the interaction of multiple products that are related through the VaR constraint.

In Tapiero (2005), an asymmetric valuation between ex-ante expected costs above and below an appropriate target cost, provides an explanation for the VaR criterion when it is used as a tool for design. This approach gives some insights regarding the selection of the VaR probability that turns out to be the ratio of the asymmetric linear cost parameters in this case. In this setting, it is proposed to optimize the planned (targeted cost) that is defined as the sum of the expected newsvendor cost and risk specification quantile (defined to be "index of risk aversion") times the standard deviation of the newsvendor cost under the assumption of normally distributed newsvendor cost function. We do not consider the design problem here and assume that a desired profit level and a corresponding level is specified exogenously. In particular, we deal with the multiple product newsvendor problem with satisfaction probability constraint which enables us to incorporate the VaR concept into a classical inventory problem. Another difference in our model with respect to Tapiero's (2005) model is the uncertainty in our newsvendor profit (cost) function is dependent on the demand uncertainty. This implies that even if the probability distribution of demand is specified and easy to handle, the newsvendor cost function may have a much more complicated probability distribution. This makes our problem difficult.

Gan et al. (2004) incorporate the VaR concept to a newsvendor problem with a downside risk constraint for a single product. This is a decision making problem of a risk-averse newsvendor subject to a downside risk constraint which is characterized as the probability that the newsvendor's realized profit is less than or equal to his specified target profit. In this paper, we present results for a similar problem in the multi-product case. What makes the multi-product case considerably more difficult is the nature of the downside risk constraint. Because this constraint creates a dependency between each product, we cannot solve multi-product problem separately for each product.

Finally, Gotoh and Takano (2007) and Zhou et al. (2008) independently consider the CVaR minimization in a multi-product newsvendor setting. It is shown that the CVaR minimization problem in this setting can be represented as a linear program. From an optimization point of view, the CVaR formulation has certain nice features (see for example Artzner et al. 1999) whereas the VaR problem appears to be more challenging. We are nevertheless able to develop a non-linear mathematical programming formulation which is tractable for a small number of products and an effective approximation that is appropriate when the number of products is large.

In light of the related literature, the contribution of our study is two-fold. First, we present a compact representation of the total profit distribution for the two-product newsvendor problem. This derivation yields the total profit distribution based on the given demand distributions. This approach allows us to handle a wide range of cases including the correlated demand case. Consequently, the optimal order quantities are determined by using a mathematical program that incorporates the profit distribution. Second, we present an approximation method for the multi-product case and evaluate the accuracy of this approximation numerically. Since the numerical results show that this approximation is quite accurate, the proposed method can be used effectively to analyze multi-product newsvendor problems.

3. Model

We consider a single-period multi-product stochastic inventory control problem where a retailer determines the optimal order quantities for N different products that have stochastic demand with the objective of maximizing the expected profit subject to a downside risk constraint. The downside risk constraint is defined as the probability of earning less than the target profit value π_0 is less than or equal to the threshold probability value β. The demand for product i, D_i, is a random variable with a distribution function $F_i(x_i)$ and density $f_i(x_i)$. The joint distribution function of $D_1, D_2,\ldots,D_N$ is $F(x_1, x_2,\ldots,x_N)$ and the joint density is $f(x_1, x_2,\ldots,x_N)$. The sales price of product i is p_i, purchasing cost is c_i, and the salvage value is s_i. We assume that the natural relationship between the cost and revenue parameters $p_i > c_i > s_i$ holds. The order quantity for product i is Q_i. The profit obtained from the sales of product i is π_i and the total profit obtained from the sales of all the products is π.

Our main interest is integrating risk considerations into the single-period multi-product stochastic inventory problem through a VaR approach. This could be performed in several ways but our main focus is on the following formulation which parallels the single-product formulation of Gan et al. (2004). Consider the problem of determining the order quantities that maximize the expected total profit while satisfying a VaR constraint.

$$\operatorname{Max} E[\pi] = \sum_{i=1}^{N} E[\pi_i] \tag{1}$$

subject to

$$P\left[\sum_{i=1}^{N} \pi_i \leqslant \pi_0\right] \leqslant \beta \tag{2}$$

The objective function of the above optimization problem maximizes the total expected profit just like in the standard formulation of the newsvendor problem. The VaR constraint turns out to be the main challenge in the above formulation since it is expressed in terms of the probability distribution of the profit. For this reason, we first turn our attention to obtaining or approximating this probability distribution and then determine the solution by using a mathematical programming approach.

3.1. Results for the single-product newsvendor problem with a VaR constraint

In order to illustrate the approach, we first summarize the results for the case with a single product. When there is only one product, the profit can be expressed as

$$\pi_i = (p_i - c_i)Q_i - (p_i - s_i)[Q_i - D_i]^+ \tag{3}$$

The distribution of π_i has a probability mass at the maximum possible profit of $(p_i - c_i)Q_i$ with probability $P[Q_i \leqslant D_i]$, i.e.,

$$P[\pi_i = (p_i - c_i)Q_i] = 1 - F_i(Q_i) \tag{4}$$

When $Q_i > D_i$, equivalently when $\pi_i < (p_i - c_i)Q_i$

$$\begin{aligned} P[\pi_i < x] &= P[(s_i - c_i)Q_i + (p_i - s_i)D_i < x] \\ &= P\left[D_i < \frac{x - (s_i - c_i)Q_i}{p_i - s_i}\right] \end{aligned} \tag{5}$$

Since $D_i>0$, Eq. (5) can be summarized as

$$P[\pi_i<x]=\begin{cases} F_i\left(\dfrac{x-(s_i-c_i)Q_i}{p_i-s_i}\right) & (s_i-c_i)Q_i\leqslant x<(p_i-c_i)Q_i \\ 0 & x<(s_i-c_i)Q_i \end{cases} \tag{6}$$

Eqs. (4) and (6) completely define the distribution of the profit for one product. Then the newsvendor problem with a VaR constraint can be written in terms of the probability density and distribution function of the demand as

$$\operatorname{Max} E[\pi_i]=(p_i-c_i)Q_i-(p_i-s_i)Q_iF(Q_i)+(p_i-s_i)\int_0^{Q_i} xf_i(x)\,\mathrm{d}x \tag{7}$$

subject to

$$F_i\left(\frac{\pi_0-(s_i-c_i)Q_i}{p_i-s_i}\right)\leqslant\beta. \tag{8}$$

where $(s_i-c_i)Q_i\leqslant\pi_0\leqslant(p_i-c_i)Q_i$.

The unconstrained solution of (7) is determined by the critical ratio

$$Q_i=F^{-1}\left(\frac{p_i-c_i}{p_i-s_i}\right). \tag{9}$$

If the above quantity satisfies the condition (8), then the solution of the problem given in (7) subject to (8) is the same. Otherwise, the VaR constraint must be binding. Note that if $F_i(\pi_0/p_i-s_i)>\beta$, or equivalently, if $\pi_0>F^{-1}(\beta)(p_i-s_i)$ the above problem is infeasible.

Given that the problem is feasible, the optimal order quantity is

$$Q_i=\frac{F^{-1}(\beta)(p_i-s_i)-\pi_0}{c_i-s_i} \tag{10}$$

as given in Gan et al. (2004).

4. Two-product newsvendor problem with a VaR constraint

When there are two products, the solution cannot be determined easily as the single product case. Similar to the one-product newsvendor problem with a VaR constraint, the multi-product newsvendor problem with a VaR constraint can be infeasible depending on the problem parameters.

Given that there is a feasible solution, if the VaR constraint is not binding at the unconstrained solution, the optimal values are obtained by the critical ratios for both products as given in Eq. (9). However, if it is binding, there are infinitely many Q_1 and Q_2 values that satisfy the VaR constraint and the ones that maximize the objective function need to be determined.

In order to utilize a mathematical programming approach, the VaR constraint needs to be derived explicitly in terms of the decision variables Q_1 and Q_2. We first derive the distribution of profit for two products with a joint demand distribution and then analyze the conditions for the feasibility of the problem. Finally, we present a mathematical programming formulation that yields the optimal order quantities.

4.1. Distribution of the profit for two products with a joint demand distribution

The total profit for two product, $\pi=\pi_1+\pi_2$ can be written as:

$$\pi=\begin{cases} (p_1-c_1)Q_1+(p_2-c_2)Q_2 & Q_1\leqslant D_1 \quad Q_2\leqslant D_2 \\ (p_1-c_1)Q_1+(s_2-c_2)Q_2+(p_2-s_2)D_2 & Q_1\leqslant D_1 \quad Q_2>D_2 \\ (s_1-c_1)Q_1+(p_1-s_1)D_1+(p_2-c_2)Q_2 & Q_1>D_1 \quad Q_2\leqslant D_2 \\ (s_1-c_1)Q_1+(p_1-s_1)D_1+(s_2-c_2)Q_2+(p_2-s_2)D_2 & Q_1>D_1 \quad Q_2>D_2 \end{cases} \tag{11}$$

The distribution of π has a probability mass at the maximum possible profit given below:

$$P[\pi=(p_1-c_1)Q_1+(p_2-c_2)Q_2]=P[Q_1\leqslant D_1,Q_2\leqslant D_2] \tag{12}$$

When $\pi<(p_1-c_1)Q_1+(p_2-c_2)Q_2$

$$P[\pi<x|D_1,D_2]=\begin{cases} P\left[D_2<\dfrac{x-(p_1-c_1)Q_1-(s_2-c_2)Q_2}{p_2-s_2}\right] & Q_1\leqslant D_1 \quad Q_2>D_2 \\ P\left[D_1<\dfrac{x-(p_2-c_2)Q_2-(s_1-c_1)Q_1}{p_1-s_1}\right] & Q_1>D_1 \quad Q_2\leqslant D_2 \\ P[(p_1-s_1)D_1+(p_2-s_2)D_2<x-(s_1-c_1)Q_1-(s_2-c_2)Q_2] & Q_1>D_1 \quad Q_2>D_2 \end{cases}$$

Equivalently, the above equation can be written as

$$P[\pi<x]=P\left[D_1\geqslant Q_1,D_2<\frac{x-\theta_a}{p_2-s_2}\right]+P\left[D_1<\frac{x-\theta_b}{p_1-s_1},D_2\geqslant Q_2\right]+g(x) \tag{13}$$

where $g(x)=P[(p_1-s_1)D_1+(p_2-s_2)D_2<x-\theta_c,Q_1>D_1,Q_2>D_2]$, $\theta_a=(p_1-c_1)Q_1+(s_2-c_2)Q_2$, $\theta_b=(p_2-c_2)Q_2+(s_1-c_1)Q_1$ and $\theta_c=(s_1-c_1)Q_1+(s_2-c_2)Q_2$.

Note that condition $D_2<(x-\theta_a)/(p_2-s_2)$ implies $Q_2>D_2$ and condition $D_1<(x-\theta_b)/(p_1-s_1)$ implies $Q_1>D_1$.

Let Products 1 and 2 be indexed such a way that $(p_1-s_1)Q_1\geqslant(p_2-s_2)Q_2$, i.e., $\theta_a\geqslant\theta_b$. Since $D_1>0$ and $D_2>0$, Eq. (13) can be expressed as

$$P[\pi<x]=\begin{cases} \int_{Q_1}^{\infty}\int_0^{x-\theta_a/p_2-s_2} f(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1+\int_0^{x-\theta_b/p_1-s_1}\int_{Q_2}^{\infty} f(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1+g(x) & \theta_m\geqslant x\geqslant\theta_a \\ \int_0^{x-\theta_b/p_1-s_1}\int_{Q_2}^{\infty} f(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1+g(x) & \theta_a\geqslant x\geqslant\theta_b \\ g(x) & \theta_b\geqslant x\geqslant\theta_c \\ 0 & \theta_c>x \end{cases} \tag{14}$$

where $\theta_m=(p_1-c_1)Q_1+(p_2-c_2)Q_2$.

The function $g(x)=P[(p_1-s_1)D_1+(p_2-s_2)D_2<x-\theta_c,Q_1>D_1,Q_2>D_2]$ is determined by integrating the probability density functions of Products 1 and 2 in an

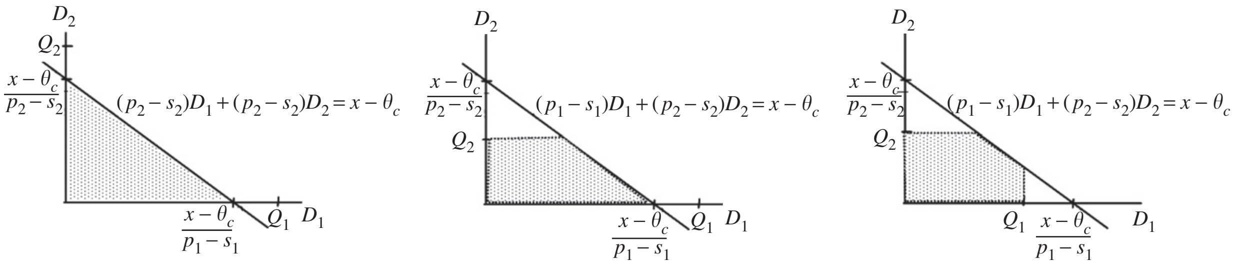

Fig. 1. Different cases for $g(x)$.

appropriate region depending on the system parameters. Fig. 1 depicts these different regions.

Consequently, when $Q_1 \geqslant (x-\theta_c)/(p_1-s_1)$, or equivalently when $\theta_a \geqslant x$ and $Q_2 \geqslant (x-\theta_c)/(p_2-s_2)$ that is when $\theta_b \geqslant x$, $g(x)$ is evaluated as

$$g(x) = \int_0^{x-\theta_c/p_1-s_1} \int_0^{x-\theta_c-(p_1-s_1)x_1/p_2-s_2} f(x_1,x_2)\,dx_2\,dx_1. \tag{15}$$

When $\theta_a \geqslant x$ and $\theta_b < x$, $g(x)$ is evaluated as

$$g(x) = \int_0^{x-\theta_c-(p_2-s_2)Q_2/p_1-s_1} \int_0^{Q_2} f(x_1,x_2)\,dx_2\,dx_1 + \int_{x-\theta_c-(p_2-s_2)Q_2/p_1-s_1}^{x-\theta_c/p_1-s_1} \int_0^{x-\theta_c-(p_1-s_1)x_1/p_2-s_2} \times f(x_1,x_2)\,dx_2\,dx_1. \tag{16}$$

Similarly, when $\theta_a < x$ and $\theta_b < x$, $g(x)$ is evaluated as

$$g(x) = \int_0^{x-\theta_c-(p_2-s_2)Q_2/p_1-s_1} \int_0^{Q_2} f(x_1,x_2)\,dx_2\,dx_1 + \int_{x-\theta_c-(p_2-s_2)Q_2/p_1-s_1}^{Q_1} \int_0^{x-\theta_c-(p_1-s_1)x_1/p_2-s_2} \times f(x_1,x_2)\,dx_2\,dx_1. \tag{17}$$

Since $\theta_c + (p_2-s_2)Q_2 = \theta_b$, $g(x)$ is completely defined by

$$g(x) = \begin{cases} \int_0^{x-\theta_b/p_1-s_1} \int_0^{Q_2} f(x_1,x_2)\,dx_2\,dx_1 + \int_{x-\theta_b/p_1-s_1}^{Q_1} \times \int_0^{x-\theta_c-(p_1-s_1)x_1/p_2-s_2} f(x_1,x_2)\,dx_2\,dx_1 & \theta_m \geqslant x \geqslant \theta_a \\ \int_0^{x-\theta_b/p_1-s_1} \int_0^{Q_2} f(x_1,x_2)\,dx_2\,dx_1 + \int_{x-\theta_b/p_1-s_1}^{x-\theta_c/p_1-s_1} \times \int_0^{x-\theta_c-(p_1-s_1)x_1/p_2-s_2} f(x_1,x_2)\,dx_2\,dx_1 & \theta_a > x > \theta_b \\ \int_0^{x-\theta_c/p_1-s_1} \int_0^{x-\theta_c-(p_1-s_1)x_1/p_2-s_2} f(x_1,x_2)\,dx_2\,dx_1 & \theta_b \geqslant x \geqslant \theta_c \end{cases}. \tag{18}$$

Now Eqs. (14) and (18) allow one to evaluate the probability distribution of the total profit by using the joint demand distribution. If the integrals in Eqs. (14) and (18) can be evaluated in closed form, the distribution of the total profit can be written in closed form. For example, when the demands for Products 1 and 2 are independent exponential random variables with averages λ_1 and λ_2, the distribution of the total profit is given in closed form in Appendix B.

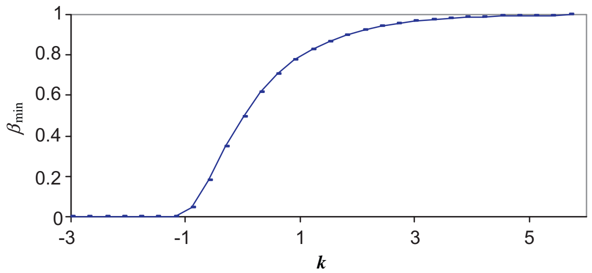

Fig. 2. Minimum β values for the feasibility of the 2-product VaR problem $x = E[\pi_{uc}] + k\sigma[\pi_{uc}]$, $p_i = 1$, $c_i = 0.5$, $s_i = 0.3$, $\lambda_i = 1$, $i = 1{:}2$.

4.2. Feasibility of the two-product newsvendor problem with a VaR constraint

As in the single-product case, there may not be a feasible solution in the multi-product case. More specifically, if $\min_{Q_1,Q_2} P[\pi < x] > \beta$, there will be no feasible solution to this problem.

For two products, the minimum β value that ensures the feasibility of the problem can be determined by minimizing the probability distribution function derived in Eqs. (14) and (18) for a given value of π_0. Fig. 2 depicts the minimum β value for different values of π_0 for a system with two identical products and exponential demand distribution where $E[\pi_{uc}]$ and $\sigma[\pi_{uc}]$ are the expectation and the standard deviation of the total profit without a VaR constraint and k is a constant that shows how many $\sigma[\pi_{uc}]$'s x is away from $E[\pi_{uc}]$.

4.3. A mathematical programming approach for the newsvendor problem with a VaR constraint

In the two-product case, the probability distribution function is determined by Eq. (14). Since the distribution function is different depending on the region, the problem can be written as:

$$\operatorname{Max} E[\pi] = \sum_{i=1}^{2} (p_i - c_i)Q_i - (p_i - s_i)Q_i F(Q_i) + (p_i - s_i)\int_0^{Q_i} x f_i(x)\,dx \tag{19}$$

subject to

$$\begin{array}{ll} \int_{Q_1}^{\infty}\int_0^{x-\theta_a/p_2-s_2} f(x_1,x_2)\,dx_2\,dx_1 & \\ \quad +\int_0^{x-\theta_b/p_1-s_1}\int_{Q_2}^{\infty} f(x_1,x_2)\,dx_2\,dx_1 + g(\pi_0)\leqslant\beta & \text{if } \theta_m>\pi_0>\theta_a \\ \int_0^{x-\theta_b/p_1-s_1}\int_{Q_2}^{\infty} f(x_1,x_2)\,dx_2\,dx_1 + g(\pi_0)\leqslant\beta & \text{if } \theta_a>\pi_0>\theta_b \\ g(\pi_0)\leqslant\beta & \text{if } \theta_b>\pi_0>\theta_c \end{array} \tag{20}$$

where $g(x)$ was given in Eq. (18). Since $\theta_a = (p_1-c_1)Q_1+(s_2-c_2)Q_2$, $\theta_b=(p_2-c_2)Q_2+(s_1-c_1)Q_1$, and $\theta_c=(s_1-c_1)Q_1+(s_2-c_2)Q_2$, the regions also depend on the decision variables.

It is possible to include the boundary constraints of the type if $f(x)<0$ then $g(x)<0$ by adding two different constraints $g(x)\leqslant M(1-I)$ and $f(x)\leqslant M(1-I)$ where M is a big number and I is binary decision variable in the formulation.

Combining all the definitions in the formulation yields

$$\operatorname{Max} E[\pi] = \sum_{i=1}^{2}(p_i-c_i)Q_i-(p_i-s_i)Q_iF(Q_i) + (p_i-s_i)\int_0^{Q_1} xf_i(x)\,dx \tag{21}$$

subject to

$$\pi_0-(p_1-c_1)Q_1+(p_2-c_2)Q_2\leqslant M(1-I_1)(1-I_2)(1-I_3),$$

$$(p_1-c_1)Q_1+(s_2-c_2)Q_2-\pi_0\leqslant M(1-I_1)$$

$$\pi_0-(p_1-c_1)Q_1-(s_2-c_2)Q_2\leqslant M(1-I_2)(1-I_3)$$

$$(p_2-c_2)Q_2+(s_1-c_1)Q_1-\pi_0\leqslant M(1-I_2)$$

$$\pi_0-(p_2-c_2)Q_2-(s_1-c_1)Q_1\leqslant M(1-I_3)$$

$$(s_2-c_2)Q_2+(s_1-c_1)Q_1-\pi_0\leqslant M(1-I_3)$$

$$\int_{Q_1}^{\infty}\int_0^{x-\theta_a/p_2-s_2} f(x_1,x_2)\,dx_2\,dx_1 + \int_0^{x-\theta_b/p_1-s_1}\int_{Q_2}^{\infty} f(x_1,x_2)\,dx_2\,dx_1 + \int_0^{x-\theta_b/p_1-s_1}\int_0^{Q_2} f(x_1,x_2)\,dx_2\,dx_1 + \int_{x-\theta_b/p_1-s_1}^{Q_1}\int_0^{x-\theta_c-(p_1-s_1)x_1/p_2-s_2} f(x_1,x_2)\,dx_2\,dx_1 - \beta\leqslant M(1-I_1)$$

$$\int_0^{x-\theta_b/p_1-s_1}\int_{Q_2}^{\infty} f(x_1,x_2)\,dx_2\,dx_1 + \int_0^{x-\theta_b/p_1-s_1}\int_0^{Q_2} f(x_1,x_2)\,dx_2\,dx_1 + \int_{x-\theta_b/p_1-s_1}^{x-\theta_c/p_1-s_1}\int_0^{x-\theta_c-(p_1-s_1)x_1/p_2-s_2} f(x_1,x_2)\,dx_2\,dx_1 - \beta\leqslant M(1-I_2)$$

$$\int_0^{x-\theta_c/p_1-s_1}\int_0^{x-\theta_c-(p_1-s_1)x_1/p_2-s_2} \times f(x_1,x_2)\,dx_2\,dx_1-\beta\leqslant M(1-I_3)$$

$$(p_1-s_1)Q_1-(p_2-s_2)Q_2>0$$

$$I_1+I_2+I_3=1$$

$$Q_1\geqslant 0,\quad Q_2\geqslant 0,\ I_1,I_2,I_3\in\{0,1\} \tag{22}$$

Note that the above formulation is a mixed-integer programming formulation with a nonlinear objective function and mixed linear and nonlinear constraints. Alternatively, it is also possible to solve a number of different problems assuming a certain region in each of them. Then the feasible results obtained for each sub-problem can be compared to determine the global optimal of the problem.

4.4. Results for independent exponential demand distribution

Appendix A gives the distribution of the total profit when the demands for Products 1 and 2 are independent exponential random variables. Using this distribution and solving the resulting mathematical program yield the optimal order quantities as shown in Fig. 3. The dotted lines show the unconstrained solutions that follow Eq. (9). Similarly Fig. 4 shows the optimal expected total profit for the same case.

4.5. Results for a bivariate exponential demand distribution

Since the distribution of the total profit is derived by using a joint demand distribution, our methodology allows us to analyze two-product newsvendor problem

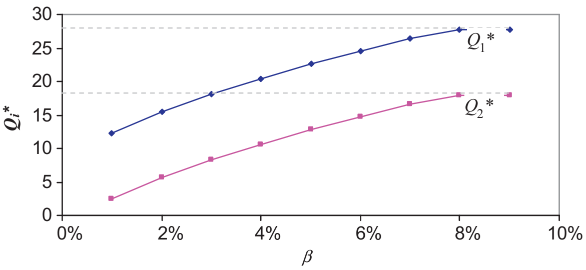

Fig. 3. Optimal order quantity for the two-product VaR problem with exponential demand $p_1=2$, $p_2=1$, $c_1=0.5$, $c_2=0.5$, $s_1=0.4$, $s_2=0.4$, $\lambda_1=10$, $\lambda_2=10$, $\pi_0=0$.

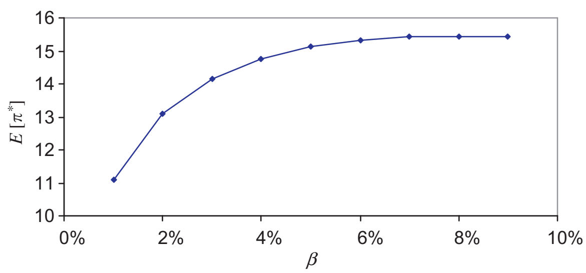

Fig. 4. Optimal expected total profit for the two-product VaR problem with exponential demand. $p_1=2$, $p_2=1$, $c_1=0.5$, $c_2=0.5$, $s_1=0.4$, $s_2=0.4$, $\lambda_1=10$, $\lambda_2=10$, $\pi_0=0$.

with a VaR constraint and with correlated demand distributions.

For example, let us consider a case where the demands for Products 1 and 2 have a bivariate exponential distribution. That is

$$P[D_1 \leqslant x_1, D_2 \leqslant x_2] = F_1(x_1)F_2(x_2)(1 + \alpha[1 - F_1(x_1)][1 - F_2(x_2)]) \quad |\alpha| \leqslant 1$$

and $F_i(x_i) = 1 - e^{-1/\lambda_i x_i}$. The correlation coefficient of D_1 and D_2 is $\rho = \alpha/4$. There are many forms of bivariate exponential distributions in the literature. We use this particular form Gumbel (1960) because it is especially convenient to model correlated exponential random variables and experimenting by varying the correlation coefficient systematically. This particular form has the limitation that the correlation coefficient is restricted $|\rho| \leqslant 1/4$ which is sufficient for illustration purposes. Of course, similar results can be generated for other tractable bivariate forms using the results in Section 4.1. For this setting, the probability distribution function for the demand is derived in closed form in Appendix B. Figs. 5 and 6 show the cumulative distribution and probability density function of the total profit for different values of ρ for a specific case.

Once the probability function is available, the solution of the VaR problem is obtained by using the mathematical programming approach. Fig. 7 shows the optimal order quantities for different values of ρ and β for a specific case. Fig. 8 shows the optimal expected profit and Fig. 9 shows the percentage change in the expected total profit with respect to the independent demand, i.e., $\rho = 0$ case for the same example. As the figures show, when two products are negatively correlated, it is possible to increase the expected profit up to 25% with respect to the independent case. This is due to the reduction in variability when the demands are negatively correlated. Similarly, when the demands are positively correlated, the expected profit is lower.

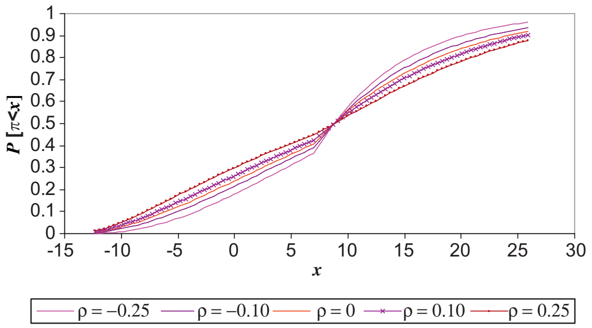

Fig. 5. Cumulative distribution function of the profit $p_1 = 5$, $p_2 = 5$, $c_1 = 3$, $c_2 = 3$, $s_1 = 2$, $s_2 = 2$, $\lambda_1 = 5$, $\lambda_2 = 5$, $Q_1 = 7$, $Q_2 = 7$.

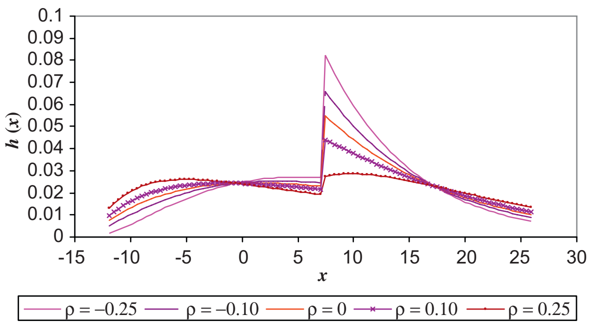

Fig. 6. Probability density function of the profit $p_1 = 5$, $p_2 = 5$, $c_1 = 3$, $c_2 = 3$, $s_1 = 2$, $s_2 = 2$, $\lambda_1 = 5$, $\lambda_2 = 5$, $Q_1 = 7$, $Q_2 = 7$.

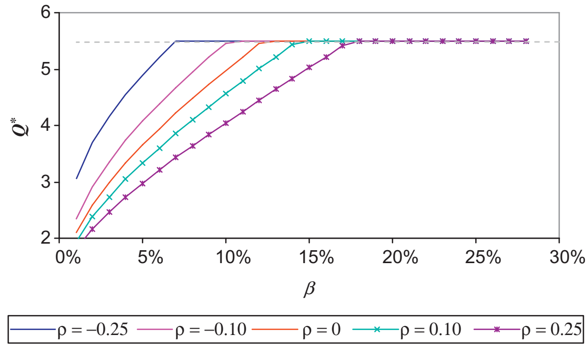

Fig. 7. Optimal order quantity for the VaR problem with bivariate exponential demand $p_1 = 5$, $p_2 = 5$, $c_1 = 3$, $c_2 = 3$, $s_1 = 2$, $s_2 = 2$, $\lambda_1 = 5$, $\lambda_2 = 5$, $Q_1{}^* = Q_2{}^* = Q^*$, $\pi_0 = E[\pi] - \sigma[\pi]$.

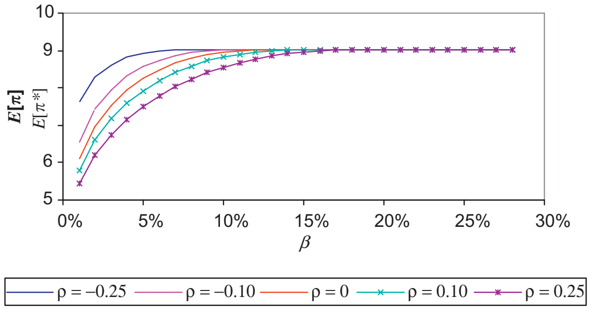

Fig. 8. Optimal expected total profit for the VaR problem with bivariate exponential demand $p_1 = 5$, $p_2 = 5$, $c_1 = 3$, $c_2 = 3$, $s_1 = 2$, $s_2 = 2$, $\lambda_1 = 5$, $\lambda_2 = 5$, $\pi_0 = E[\pi] - \sigma[\pi]$.

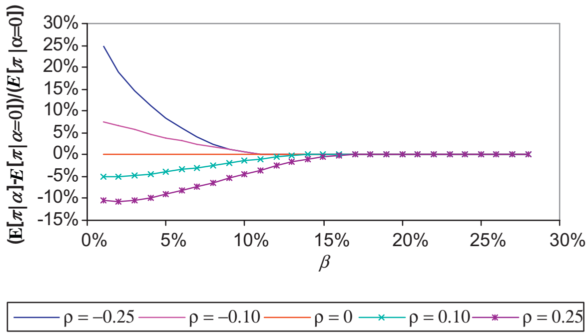

Fig. 9. Effect of the correlation coefficient on the optimal expected total profit with respect to the independent demand case for the VaR Problem with bivariate exponential demand $p_1 = 5$, $p_2 = 5$, $c_1 = 3$, $c_2 = 3$, $s_1 = 2$, $s_2 = 2$, $\lambda_1 = 5$, $\lambda_2 = 5$, $Q_1 = 7$, $Q_2 = 7$, $\pi_0 = E[\pi] - \sigma[\pi]$.

5. Multi-product newsvendor problem with a VaR constraint

If the order quantities that are obtained by the critical ratios given in Eq. (9) for all products satisfy the VaR constraint, in other words, if the VaR constraint is not binding at the unconstrained solution, then these order quantities will be the optimal order quantities for the multi-product newsvendor problem with a VaR constraint.

However, if this is not the case, extending the procedure outlined for the two product case in Section 3 to determine the distribution of π with more than two products is challenging due to the difficulty of determining the probability distribution of the total profit. Furthermore, the multi-product newsvendor problem with a VaR constraint may be infeasible for given system parameters.

This section presents a simple approximation in the case when there are N products with independent demands. Since the total profit is the sum of the profits of individual products that are random with independent distributions, we can utilize the central limit theorem to determine the distribution of the profit approximately.

Once the VaR constraint is expressed by using the normal approximation, we can analyze the feasibility conditions and determine the optimal order quantities by using the mathematical programming approach.

5.1. Approximate distribution of the profit for N products

Under the assumption that the conditions for Central Limit Theorem are satisfied, the distribution of the profit function approaches to a normal distribution in a large product portfolio. Practically, we can write the approximate probability as

$$\tilde{P}(\pi \leqslant x) = \phi\left(\frac{x - E[\pi]}{\sigma[\pi]}\right) \tag{23}$$

where $\phi(x)$ is the standard normal density function.

Since the demands are independent, $E[\pi] = \sum_{i=1}^{N} E[\pi_i]$ and $Var[\pi] = \sum_{i=1}^{N} Var[\pi_i]$ where the expectation and the variance of the profit for a single product can be determined by using the probability distribution of the profit given in Eq. (6). Appendix A presents $E[\pi_i]$ and $Var[\pi_i]$ explicitly for exponential demand.

In this section, we evaluate the accuracy of this approximation by comparing the approximate distribution with simulation results. Fig. 10 depicts the distribution of the total profit as the number of products, N increases from 1 to 20 for a specific case with identical product parameters with $p = 1$, $c = 0.5$, $s = 0.3$, $\lambda = 10$, $Q = 12$. As the figure shows, the distribution of the total profit approaches the normal distribution quite fast. Since the probability mass is at the right tail, using the normal approximation for the probability values closer to the probability mass will be valid when N is large. However, probabilities concerning the left tail, i.e., the downside risk, such as the VaR values can be approximated quite well even when N is small.

Table 1 compares the exact probability obtained by simulation and the normal approximation for a specific system with identical products with exponential demand in a wide range of x values as N increases from 5 to 50. As the table indicates the normal approximation performs quite well especially when x is less than the expected total profit. It is also observed that the average error is very small as the number of products is large. Table 2 presents the same comparison for non-identical products with uniformly distributed price, cost, and salvage values.

These results assure us that the normal approximation can be used to describe the VaR constraint in the multi-product newsvendor problem. Therefore, the same mathematical programming approach will be used to determine the order quantities for the multi-product newsvendor problem with independent demands.

5.2. Feasibility of the multi-product newsvendor problem with a VaR constraint

Similar to the two-product case, multi-product newsvendor problem with a VaR constraint can be infeasible if $\min_{Q_1,\ldots,Q_N} P[\pi < x] > \beta$.

Fig. 11 shows how the distribution function, determined by using the normal approximation, changes as order quantities change for a system with identical products. As the figure shows, for a given value of x, $P[\pi < x]$ has a minimum for a specific value of Q.

In this case, the condition for the existence of a feasible solution can be determined by finding the values of order quantities Q'_i $i = 1,\ldots,N$ that minimize the probability function for a given value of π_0 given in Eq. (23). These order quantities can be determined from the derivative of Eq. (23) with respect to Q_i that yields

$$-\frac{\partial E[\pi]}{\partial Q_i}(E[\pi^2] + \pi_0 E[\pi]) = \frac{1}{2}\frac{\partial E[\pi^2]}{\partial Q_i}(\pi_0 - E[\pi]), \quad i = 1,\ldots,N$$

where $E[\pi] = \sum_{i=1}^{N} E[\pi_i]$ and $E[\pi^2] = Var[\pi] + E^2[\pi]$.

Fig. 12 shows the minimum β values that ensure feasibility of the multi-product VaR problem with identical products with exponential demand distribution.

5.3. Solution of the multi-product newsvendor problem with a VaR constraint

By using the normal approximation for the probability distribution of the total profit as described in Section 4.1, the VaR problem for N products is written as

$$\text{Max} \quad E[\pi] = \sum_{i=1}^{N} (p_i - c_i)Q_i - (p_i - s_i)Q_i F(Q_i) + (p_i - s_i)\int_0^{Q_i} x f_i(x)\,dx \tag{24}$$

Subject to:

$$P(\pi \leqslant \pi_0) = \phi\left(\frac{\pi_0 - E[\pi]}{\sigma[\pi]}\right) \leqslant \beta \tag{25}$$

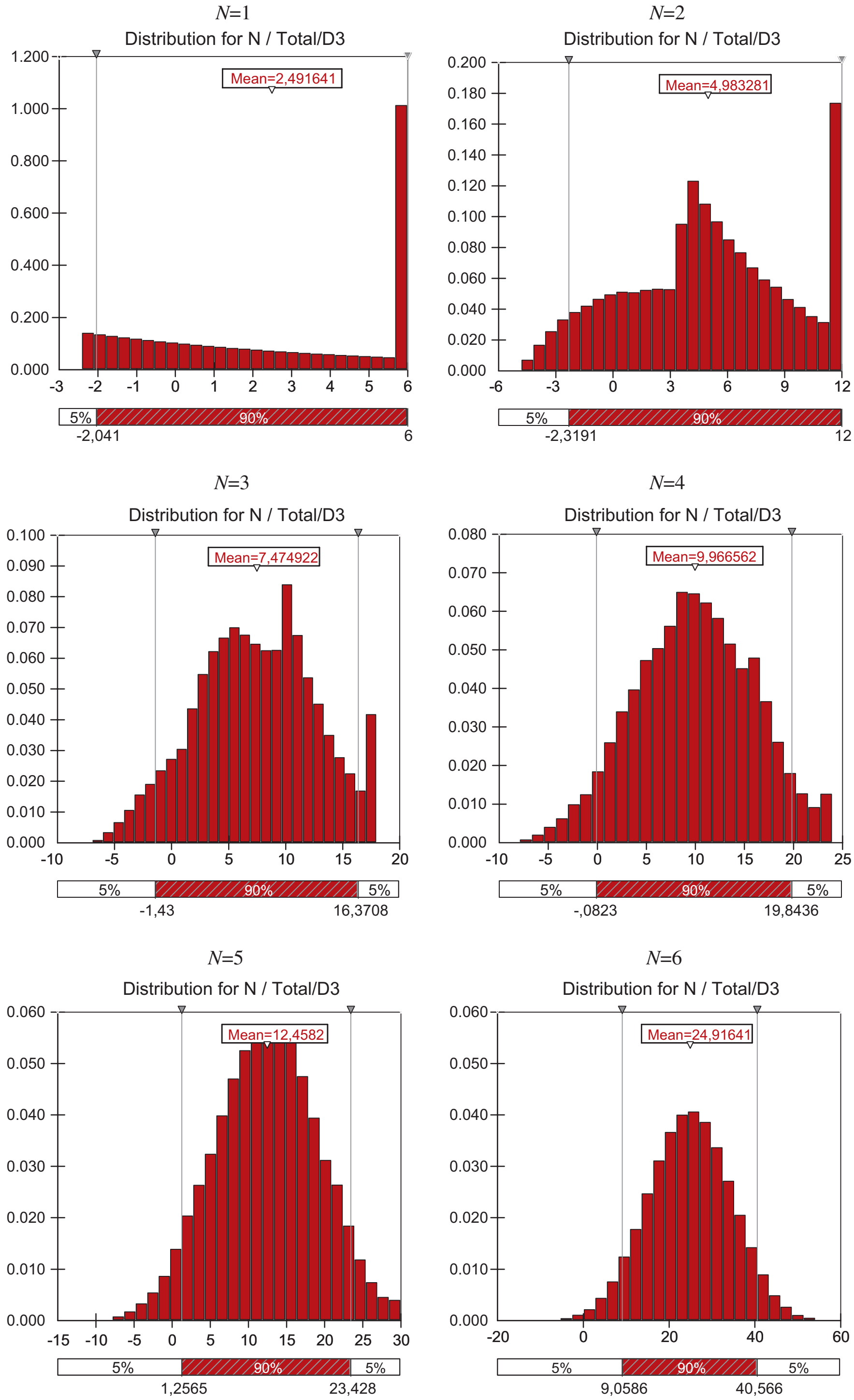

Fig. 10. Distribution of total profit with N products $p_i = 1$, $c_i = 0.5$, $s_i = 0.3$, $\lambda_i = 10$, $Q_i = 12$.

The above constraint can be rewritten as:

$$\pi_0 - E[\pi] \leqslant \phi^{-1}(\beta)\sigma[\pi] \tag{26}$$

where $\phi^{-1}(x)$ is the inverse of the standard normal distribution function.

For each product with possibly different demand distribution, $E[\pi_i]$ and $Var[\pi_i]$ are determined by using the probability distribution of the profit given in Eq. (6). Using these probability distributions in Eqs. (24) and (26) yield a nonlinear optimization problem that can be solved by using standard nonlinear solvers such as BARON in

Table 1
Accuracy of the normal approximation for the total profit $|P(\pi<E[\pi]+k\sigma[\pi])-\tilde{P}(\pi<E[\pi]+k\sigma[\pi])|$ $p_i=1$, $c_i=0.5$, $s_i=0.2$, $\lambda_i=10$, Q_i : optimal for the unconstrained problem, $i=1{:}N$.

k	*N*									
	5 (%)	10 (%)	15 (%)	20 (%)	25 (%)	30 (%)	35 (%)	40 (%)	45 (%)	50 (%)
−3.00	0.03	0.01	0.02	0.02	0.02	0.02	0.02	0.02	0.01	0.00
−2.40	0.06	0.13	0.10	0.09	0.10	0.11	0.06	0.06	0.05	0.04
−1.80	0.51	0.30	0.27	0.22	0.19	0.18	0.15	0.13	0.10	0.12
−1.20	0.62	0.38	0.26	0.19	0.19	0.17	0.08	0.12	0.13	0.19
−0.60	0.27	0.07	0.06	0.17	0.16	0.10	0.02	0.11	0.01	0.05
0.00	1.28	0.71	0.60	0.41	0.44	0.35	0.28	0.34	0.32	0.25
0.60	1.15	0.84	0.60	0.46	0.39	0.28	0.33	0.28	0.32	0.30
1.20	0.16	0.10	0.02	0.04	0.02	0.01	0.01	0.06	0.05	0.02
1.80	0.53	0.39	0.35	0.27	0.22	0.19	0.19	0.15	0.14	0.15
2.40	0.82	0.32	0.24	0.19	0.17	0.15	0.11	0.12	0.11	0.12
3.00	0.13	0.11	0.08	0.07	0.05	0.06	0.05	0.04	0.04	0.05
AVG	0.51	0.31	0.24	0.19	0.18	0.15	0.12	0.13	0.12	0.12

Table 2
Accuracy of the normal approximation for the total profit $|P(\pi<E[\pi]+k\sigma[\pi])-\tilde{P}(\pi<E[\pi]+k\sigma[\pi])|$ $c=u_1$, $p_i=c_i\,(1+u_2)$, $s_i=\ s_i\,(1-u_3)$, $\lambda_i=10u_4$, u_1, u_2, u_3, u_4~U(0,1) Q_i : optimal for the unconstrained problem, $i=1{:}N$.

K	N									
	5 (%)	10 (%)	15 (%)	20 (%)	25 (%)	30 (%)	35 (%)	40 (%)	45 (%)	50 (%)
−3.00	0.00	0.13	0.06	0.11	0.02	0.02	0.11	0.00	0.12	0.01
−2.40	0.74	0.80	0.16	0.46	0.21	0.08	0.42	0.06	0.62	0.02
−1.80	1.85	1.95	1.25	0.45	0.77	0.62	0.92	0.29	1.42	0.16
−1.20	5.48	0.41	1.85	1.60	0.93	1.03	0.73	0.24	0.53	0.35
−0.60	0.17	5.13	0.27	3.36	0.22	0.19	0.51	0.16	2.26	0.23
0.00	8.41	2.83	2.46	0.50	1.41	1.48	1.83	0.62	2.85	0.50
0.60	17.42	4.24	4.39	4.55	2.85	1.62	2.00	0.45	0.14	0.77
1.20	11.51	3.46	0.80	1.36	0.45	0.03	0.06	0.03	1.2	0.14
1.80	3.59	0.80	3.16	1.92	1.56	0.96	0.93	0.34	0.74	0.33
2.40	0.82	0.82	0.82	0.82	0.76	0.62	0.81	0.28	0.20	0.30
3.00	0.13	0.13	0.13	0.13	0.13	0.12	0.32	0.10	0.00	0.10
AVG	4.56	1.88	1.40	1.39	0.85	0.62	0.79	0.23	0.92	0.26

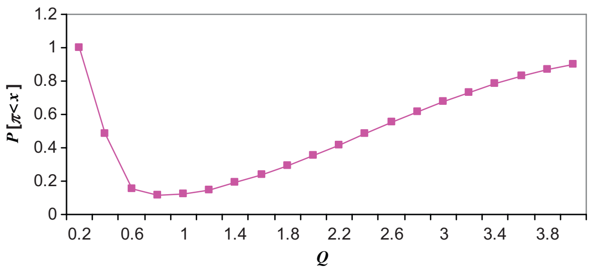

Fig. 11. Effect of the order quantity on $P[\pi<x]$ $p_i=1$, $c_i=0.5$, $s_i=0.2$, $\lambda_i=10$, $Q_i=Q$, $i=1{:}10$, $x=1.4983$.

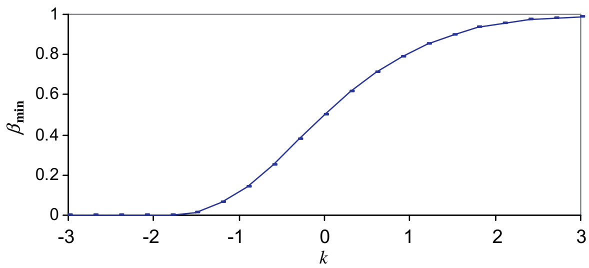

Fig. 12. Minimum β values for the feasibility of the *N*-product VaR problem $x=E[\pi_{uc}]+k\sigma[\pi_{uc}]$, $p_i=1$, $c_i=0.5$, $s_i=0.3$, $\lambda_i=1$, $i=1{:}N$ $(N=10)$.

GAMS. Note that due to the simplicity of the approximate profit probability distribution, the mathematical program for the multi-product case is much simpler than the two-product case that has a mixed integer nonlinear formulation.

5.4. Results for the multi-product case with exponential demand

Fig. 13 shows the optimal order quantities obtained by solving the above problem for a specific case with identical products. The dotted lines show the unconstrained solutions that follow Eq. (9). It is not possible to obtain a feasible solution when β is below the minimum value shown in the figure. Fig. 14 shows the optimal expected total profit for the same case.

Table 3 shows the optimal order quantities with non-identical products with average demand, uniformly distributed price, cost, and salvage values.

6. Conclusions

In this study, we present a mathematical programming approach to solve a multi-product newsvendor problem with a VaR constraint. We express the VaR constraint by deriving the probability distribution of the total profit. For the two-product case, a compact expression that yields the total profit distribution based on the demand distributions is given. This approach allows us to handle a wide range of cases including the correlated demand case that yields new results and insights.

The solution of the two-product newsvendor problem with a VaR constraint and correlated demands illustrate that the existence of the VaR constraint makes the product portfolio selection an important decision. More specifically, we observed that when there is a VaR constraint, the expected profit is higher when two products with negatively correlated demands are used. In contrast, for the multi-product newsvendor problem without a VaR constraint, demand correlations do not affect the expected profit.

We also present an approximation method that is based on approximating the total profit from the sales of different products with independent demand distributions with a normal distribution following the central limit theorem. Our numerical experiments assure us that this approximation is quite accurate and the normal distribution can be used to measure the downside risk even for portfolios with a relatively low number of products.

Extending the same approach to the case with correlated demand structures is not straightforward. Even for the case with multinomial normal distribution, the total profit may not converge to a normal distribution depending on the correlations. For example, if all the products are perfectly correlated, then the total profit will have a probability mass at the maximum profit and the resulting distribution cannot be approximated with a normal distribution. Analysis of the multi-product newsvendor problem with a correlated demand structure is left for future research.

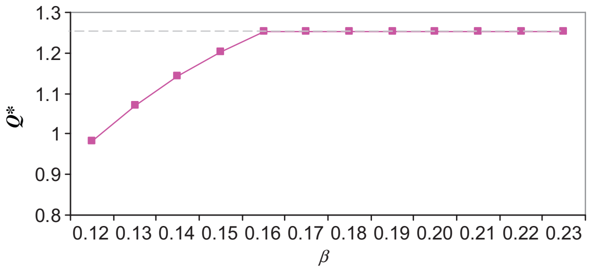

Fig. 13. Optimal expected total profit for the *N*-Product VaR problem $p_i = 1$, $c_i = 0.5$, $s_i = 0.3$, $\lambda_i = 1$, $Q_i = Q^*$, $i = 1{:}10$, $\pi_0 = E[\pi] - \sigma[\pi]$.

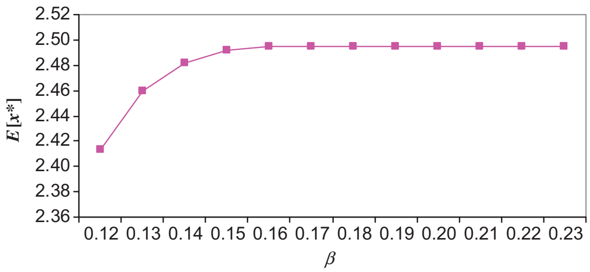

Fig. 14. Optimal expected total profit for the *N*-Product VaR problem. $p_i = 1$, $c_i = 0.5$, $s_i = 0.3$, $\lambda_i = 1$, $Q_i = Q^*$, $i = 1{:}10$, $\pi_0 = E[\pi] - \sigma[\pi]$.

Acknowledgment

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Appendix A. One product exponential demand case

When the demand for product i is an exponential random variable with rate λ_i, the distribution of the profit π_i is given below:

$$P[\pi_i = (p_i - c_i)Q_i] = e^{-Q_i/\lambda_i},$$

$$P[\pi_i < x] = \begin{cases} 1 - e^{-1/\lambda_i x - (s_i - c_i)Q_i/p_i - s_i} & (s_i - c_i)Q_i \leqslant x < (p_i - c_i)Q_i \\ 0 & x < (s_i - c_i)Q_i \end{cases}$$

Table 3
Optimal order quantities as β changes $c_i = u_1$, $p_i = c_i\,(1{+}u_2)$, $s_i = s_i\,(1{-}u_3)$, $\lambda_i = 10u_4$, u_1, u_2, u_3, $u_4 \sim U(0{,}1)$ Q_i : optimal for the optimal for the unconstrained problem, $i = 1{:}10$, $\pi_0 = 0$.

β	Q_1^*	Q_2^*	Q_3^*	Q_4^*	Q_5^*	Q_6^*	Q_7^*	Q_8^*	Q_9^*	Q_{10}^*
0.01	17.835	1.859	0.350	3.880	0.139	4.538	14.901	6.809	0.182	12.157
0.02	17.870	1.863	0.351	4.302	0.139	5.133	16.096	8.414	0.183	15.193
0.03	17.908	1.866	0.335	4.579	0.139	5.580	17.013	9.786	0.183	17.360
0.04	17.918	1.867	0.352	4.857	0.139	5.845	17.441	11.073	0.185	19.146
0.05	17.918	1.867	0.352	4.904	0.139	5.897	17.447	11.205	0.185	19.170
0.06	17.918	1.867	0.352	4.904	0.139	5.897	17.447	11.205	0.185	19.170

Accordingly, the expected profit is written as

$$E[\pi_i] = (p_i - c_i)Q_i e^{-Q_i/\lambda_i} + \int_{(s_i-c_i)Q_i}^{(p_i-c_i)Q_i} x \frac{e^{-1/\lambda_i x-(s_i-c_i)Q_i/p_i-s_i}}{\lambda_i(p_i - s_i)}\, dx \tag{27}$$

which yields

$$E[\pi_i] = (s_i - c_i)Q_i + \lambda_i(p_i - s_i)(1 - e^{-Q_i/\lambda_i}) \tag{28}$$

Similarly,

$$\begin{aligned} E[\pi_i^2] = {} & ((s_i - c_i)Q_i + \lambda_i(p_i - s_i))^2 + \lambda_i^2(p_i - s_i)^2 \\ & + (p_i - c_i)^2 Q_i^2 e^{-Q_i/\lambda_i} - (((p_i - c_i)Q_i \\ & + \lambda_i(p_i - s_i))^2 + \lambda_i^2(p_i - s_i)^2)e^{-Q_i/\lambda_i} \end{aligned} \tag{29}$$

and

$$Var[\pi_i] = \lambda_i(p_i - s_i)^2(\lambda_i(1 - e^{-2Q_i/\lambda_i}) - 2Q_i e^{-Q_i/\lambda_i}).$$

Appendix B. Distribution of the profit with exponential demand—two product case

When the demand for product i is an exponential random variable with rate λ_i, the distribution of the total profit, $\pi_1+\pi_2$ is given below:

$$P[\pi < x] = \begin{cases} 1 - e^{-x-\theta_c/\lambda_1(p_1-s_1)} - \dfrac{1}{\theta_d}(e^{-x-\theta_c/\lambda_1(p_1-s_1)} \\ \quad -e^{-x-\theta_c/\lambda_2(p_2-s_2)}) & \theta_a \geqslant \theta_b \geqslant x \\ 1 - e^{-x-\theta_c/\lambda_1(p_1-s_1)} - \dfrac{e^{-x-\theta_c/\lambda_2(p_2-s_2)}}{\theta_d} \\ \quad \times(e^{x-\theta_c/(p_1-s_1)\theta_d/\lambda_1} - e^{x-\theta_b/(p_1-s_1)\theta_d/\lambda_1}) & \theta_a \geqslant x > \theta_b \\ 1 - e^{-x-\theta_a/\lambda_2(p_2-s_2)-Q_1/\lambda_1} - \dfrac{e^{-x-\theta_c/\lambda_2(p_2-s_2)}}{\theta_d} \\ \quad \times(e^{Q_1\theta_d/\lambda_1} - e^{x-\theta_b/(p_1-s_1)\theta_d/\lambda_1}) & x > \theta_a > \theta_b \end{cases}$$

where $\theta_d = \lambda_1(p_1 - s_1) - \lambda_2(p_2 - s_2)/\lambda_2(p_2 - s_2)$ when $\theta_d \neq 0$ and

$$P[\pi < x] = \begin{cases} 1 - e^{-x-\theta_c/\lambda_1(p_1-s_1)} - \dfrac{e^{-x-\theta_c/\lambda_2(p_2-s_2)}}{\lambda_1}\dfrac{x - \theta_c}{(p_1 - s_1)} & \theta_a \geqslant \theta_b \geqslant x \\ 1 - e^{-x-\theta_c/\lambda_1(p_1-s_1)} - \dfrac{e^{-x-\theta_c/\lambda_2(p_2-s_2)}}{\lambda_1}\dfrac{(p_2 - s_2)Q_2}{(p_1 - s_1)} & \theta_a \geqslant x > \theta_b \\ 1 - e^{-x-\theta_a/\lambda_2(p_2-s_2)-Q_1/\lambda_1} - \dfrac{e^{-x-\theta_c/\lambda_2(p_2-s_2)}}{\lambda_1} \\ \quad \times\left(\dfrac{Q_1(p_1 - c_1) + Q_2(p_2 - c_2) - x}{(p_1 - s_1)}\right) & x > \theta_a > \theta_b \end{cases}$$

when $\theta_d = 0$.

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