A REVENUE MANAGEMENT PROBLEM WITH A CHOICE MODEL OF CONSUMER BEHAVIOUR IN A RANDOM ENVIRONMENT

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A Revenue Management Problem with a Choice Model of Consumer Behaviour in a Random Environment

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Abstract

Modeling consumer behavior is a relevant and growing research area in revenue management. Single-resource (single-leg) capacity control problems comprising consumer choice modeling constitute the backbone of more complicated models. In existing models, the distribution of demand is assumed to be independent of external factors. However, in reality demand may depend on the current external environment which represents the prevailing economic, financial or other factors that affect customer behavior. We formulate a stochastic dynamic program that comprises a discrete choice model of consumer behavior in a randomly fluctuating demand environment with a Markovian structure. We derive some structural results on the optimal policy for capacity control. The model and the results generalize earlier work of Talluri and van Ryzin (2004 a). In particular, the concept of an efficient set of products plays an important rule but such sets may depend on the particular external environment. We also present some computational results which illustrate the structural properties and explore the benefits of explicitly modeling the external environment.

Key words: Revenue Management, Dynamic Programming, Markov Modulation, Choice Behavior

1. Introduction

Modeling the behavior of consumers is one of the widely studied topics of the revenue management in recent years. Talluri and van Ryzin (2004 a) cover the whole field of revenue management and present relevant consumer choice models in this context. Shen and Su (2007) provide a more recent overview of consumer behavior modeling in revenue management. A

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standard consumer behavior model involves choosing from a given set of products offered to the consumer. In the context of revenue management, such a model, referred to as the general discrete choice model of consumer behavior, is first presented and investigated by Talluri and van Ryzin (2004 b). Gallego et al. (2004) study the network case and propose a choice-based linear program which is later refined by Liu and van Ryzin (2008). Kunnumkal and Topaloglu (2008), Zhang and Adelman (2009), and Meissner and Strauss (2012a) provide dynamic programming approximations of such network models and Kunnumkal and Topaloğlu (2010) present a new decomposition approach. Chen and Homem-de-Mello (2010) propose and investigate a preference order based choice model. Vulcano et al. (2010) address estimation issues for this model using real data and assess the potential benefits in practical applications. Chaneton and Vulcano (2011) and Meissner and Strauss (2012b) propose approximate and heuristic methods to compute optimal bid prices (which can be used to construct a capacity control policy) under consumer choice.

Despite the recent emphasis on consumer behavior modeling, most of the above models assume that the arrival process of fare classes is independent of external factors. On the other hand, there are situations where the demand rates change according to some external process, which is referred to as the environmental process. We model this environmental process by a Markov chain and consider the general discrete choice model of consumer behavior in such a randomly fluctuating demand environment. To our knowledge, such a model has not been studied in a revenue management setting. The only studies that seem to be related to our work are Barz (2007) and Özkan et al. (2013). These papers both model the classical single resource allocation problem in a randomly fluctuating demand environment but do not consider consumer behavior. Barz (2007) models an infinite horizon problem and shows that an environment-dependent threshold policy is optimal. Özkan et al. (2013) investigate further structural results of a finite-horizon environment-based model and perform a sensitivity analysis on the optimal policy in terms of the problem parameters such as arrival probability and revenue of each fare-class.

We observe that modeling a fluctuating demand environment by a Markov modulated demand process is well-established in inventory systems. For example, Song and Zipkin (1993), Özekici and Parlar (1999) motivate their models by arguing that current state of the demand environment can be described by economic and financial conditions which have an impact on the demand. A fluctuating demand environment is also considered in joint dynamic pricing and replenishment problems for inventory systems. Gayon et al. (2009) investigate possible pricing strategies in such a setting. Such demand models are more sophisticated in terms of capturing the dependence of demand on certain factors than standard models that ignore such dependence. For revenue management models and applications, van Ryzin (2005) make a case for better demand modeling. The environment-based framework is an improvement in this direction. In particular, van Ryzin (2005) emphasizes that short term market conditions such as competitors' availabilities and prices and weather conditions are determining factors for final consumer demand for a product. There is also considerable evidence that the aggregate demand is affected by external market forces such as currency exchange rates and fuel prices. These external factors are each of a different nature but the all have an influence on the realized demand. Therefore, capturing the effects of these factors should lead to improved capacity control models. In this paper, we present such a model for a consumer-choice based demand structure and assess its benefits. The optimal policy structure for the single-leg revenue management problem under the standard demand model (without consumer choice) is well understood. Talluri and van Ryzin (2004 a), and Aydın et al. (2009)) present results for environment independent demand case and Barz (2007) and Özkan et al. (2013) present corresponding results for environment-dependent demand. In addition, Aviv and Pazgal (2005) investigate a dynamic pricing model where demand is partially observed. They also show that optimal pricing policy depends on the belief vector which represents the probability distribution of true state of the current environment. Under consumer choice modeling, Talluri and van Ryzin (2004 b) explore the structure of the optimal policy and obtain a relatively simple characterization despite the complexity of the problem. This paper can be considered an extension of Talluri and van Ryzin (2004 b) and complements the former results by establishing the optimal policy under a consumer choice model with randomly fluctuating demand.

The paper is structured as follows. We first provide the notation and the model formulation in Section 2. After presenting the model, we identify some structural properties of the optimal admission control policy in Section 3. In Section 4, we provide an example to illustrate our structural results and quantify the improvement achieved by using an environment based model. Concluding remarks are provided in Section 5.

2. Model Formulation

We formulate a discrete time, finite horizon (T periods) Markov decision process (MDP) model of the general discrete choice model of consumer behavior under a fluctuating demand environment. Let $X_t \in \{1, 2, \cdots, M\}$ denote the environment. $X = \{X_0, X_1, \cdots, X_T\}$ is assumed to be a Markov chain with transition matrix P where $p_{ij} = P\{X_{t+1} = j | X_t = i\}$. We assume that there is at most one arrival during each time interval. We denote the probability of arrival in environment j by λ_j.

Each customer makes a decision according to the current environment and set of products offered by the firm. Therefore, firm's objective is to choose the optimal set of products to offer to maximize its expected revenue. Let N be the finite set of all products that can be offered by the firm. Let $P_a^j(S)$ denote the probability that a product of type $a \in S$ has been chosen in environment j given that the set of offered products is S. Similarly, we define $P_0^j(S)$ as the probability of no purchase in environment j when the firm offers product set $S \subseteq N$. Please note that the model is general in that both the demand rate and the purchase probabilities are allowed to depend on the environment. The situations where only the demand rate or purchase probabilities are environment dependent are special cases of the general model.

For each product a that is sold the reward is r_a. The transition probabilities and reward function are assumed to be stationary and we suppose without loss of generality that the fare classes are ordered so that $r_{a_2} \leq r_{a_1}$ when $a_1 \leq a_2$. We let $\mathbb{R} = (-\infty, +\infty)$ denote the set of real numbers and $\mathbb{R}_+ = [0, +\infty)$ denote the set of positive real numbers.

We also use the following notations:

$v_t(x,j) =$ expected maximum revenue from period t until period T given that the current inventory level is x and the environment is j.

$\Delta v_t(x,k) = v_t(x,k) - v_t(x-1,k)$

The optimal solution to this problem can be obtained by solving the following Bellman optimality equation which is a modification of Talluri and van Ryzin (2004 b) to be used in an environment dependent model:

$$\begin{aligned} v_t(x,j) &= \max_{S\subseteq N}\{\sum_{a\in S}\lambda_j P_a^j(S)\left(r_a + \sum_{k=1}^{M} P_{jk} v_{t+1}(x-1,k)\right) \\ &\quad + \left(\lambda_j P_0^j(S) + 1 - \lambda_j\right)\sum_{k=1}^{M} P_{jk} v_{t+1}(x,k)\} \\ &= \max_{S\subseteq N}\left\{\sum_{a\in S}\lambda_j P_a^j(S)\left(r_a - \sum_{k=1}^{M} P_{jk}\Delta v_{t+1}(x,k)\right)\right\} + \sum_{k=1}^{M} P_{jk} v_{t+1}(x,k) \end{aligned} \tag{1}$$

with the following boundary conditions

$$\begin{aligned} v_t(0,j) &= 0 \quad \text{for any environment } j,\ t = 1, \cdots, T, \\ v_T(x,j) &= 0 \quad \text{for any inventory level } x, \text{ environment } j. \end{aligned}$$

Talluri and van Ryzin (2004 b) suggest the reformulation for the choice model in a way that uses the total probability of purchase and the total expected revenue when the offered product set is S. We adapt this reformulation to the environment based model and write:

$$v_t(x,j) = \lambda_j \max_{S \subseteq N} \{(R^j(S) - Q^j(S) \sum_{k=1}^{M} P_{jk} \Delta v_{t+1}(x,k)\} + \sum_{k=1}^{M} P_{jk} v_{t+1}(x,k) \tag{2}$$

where

$$Q^j(S) = \sum_{a \in S} P_a^j(S) = 1 - P_0^j(S) \tag{3}$$

and

$$R^j(S) = \sum_{a \in S} r_a P_a^j(S). \tag{4}$$

Note that $Q^j(S)$ and $R^j(S)$ respectively denote the probability that a product will be purchased and the expected revenue if S is offered in environment j. To understand the structure of the optimal sets, we now define environment based efficient sets.

Definition 1. *A set T is $j-$inefficient if there exists probabilities $\alpha(S)$ for any $S \subseteq N$ with $\sum_{S \subseteq N} \alpha(S) = 1$ such that*

$$Q^j(T) \geq \sum_{S \subseteq N} \alpha(S) Q^j(S) \quad \textit{and} \quad R^j(T) < \sum_{S \subseteq N} \alpha(S) R^j(S).$$

Otherwise, T is $j-$efficient.

The intuition behind the definition of an environment based inefficient set is similar to the interpretation of inefficient sets in Talluri and van Ryzin (2004 b). A set T is j-inefficient if other sets $S \subseteq N$ exist, such that the combination of their corresponding expected revenues is strictly greater than the expected revenue of T (*i.e.*, $R^j(T)$), but the combination of their corresponding probability of purchase is less than $Q^j(T)$.

Talluri and van Ryzin (2004 b) show that it is never optimal to offer an inefficient set. We also have the same property and this can be shown by using the following version of Proposition 1 proven in Talluri and van Ryzin (2004 b).

Proposition 1. *A set T is $j-$efficient if and only if, for some value $v \geq 0$, T is an optimal solution to*

$$\max_{S \subseteq N} \left\{ R^j(S) - vQ^j(S) \right\}.$$

By using this proposition and the fact that $\sum_{k=1}^{M} P_{jk} \Delta v_{t+1}(x,k) \geq 0$, we have the following important result.

Proposition 2. *A $j-$inefficient set cannot be optimal to (2).*

Since N is a finite set, we have finite number of efficient sets in each environment. Talluri and van Ryzin (2004 b) argued that efficient sets can be ordered such that both expected revenues and probabilities of purchase increase such that:

$$Q^j\left(S_1^j\right) \leq Q^j\left(S_2^j\right) \leq ... \leq Q^j\left(S_k^j\right) \Rightarrow R^j\left(S_1^j\right) \leq R^j\left(S_2^j\right) \leq ... \leq R^j\left(S_k^j\right)$$

where S_n^j corresponds to the nth efficient set in environment j and k is the total number of such sets. Talluri and van Ryzin (2004 b) prove this result by using the following version of Lemma 1 which is also valid for our problem.

Lemma 1. *The efficient frontier $\bar{R}^j : [0,1] \rightarrow \mathbb{R}$ defined by*

$$\bar{R}^j(q) = \max \left\{ \begin{array}{c} \sum\limits_{S\subseteq N} \alpha(S) R^j(S) : \sum\limits_{S\subseteq N} \alpha(S) Q^j(S) \leq q, \\ \sum\limits_{S\subseteq N} \alpha(S) = 1, \alpha(S) \geq 0 \end{array} \right\}$$

is concave increasing in q.

3. Structural Properties

In this section, we obtain structural properties of the optimal policy for the choice model of consumer behavior under a Markov modulated demand. We show the monotonicity results corresponding to the structure of the optimal policy. First, we need some preliminary results before stating them.

Lemma 2. *If $R^j\left(S_l^j\right) - Q^j(S_l^j)v_0 \geq R^j\left(S_k^j\right) - Q^j(S_k^j)v_0$ for some $v_0 \geq 0$ and environments $l > k$, then*

$$R^j\left(S_l^j\right) - Q^j(S_l^j)v \geq R^j\left(S_k^j\right) - Q^j(S_k^j)v$$

for any $0 \leq v \leq v_0$.

Proof. $R^j\left(S_l^j\right) - Q^j(S_l^j)v_0 \geq R^j\left(S_k^j\right) - Q^j(S_k^j)v_0$ is equivalent to

$$R^j\left(S_l^j\right) - R^j\left(S_k^j\right) \geq v_0\left(Q^j(S_l^j) - Q^j(S_k^j)\right).$$

Since $l > k$, we have $Q^j(S_l^j) \geq Q^j(S_k^j)$. Then, we have the following inequality by using $0 \leq v \leq v_0$,

$$v_0\left(Q^j(S_l^j) - Q^j(S_k^j)\right) \geq v\left(Q^j(S_l^j) - Q^j(S_k^j)\right).$$

Hence, we have the desired result. $\square$

Let $k_{j,t}^{*}(x)$ be the index of the efficient set that is optimal in environment j at time t with an inventory level x. In case of equivalence, we take the set with the largest index. We have the following proposition to understand the structure of the optimal policy.

Proposition 3. *$k_{j,t}^{*}(x)$ is decreasing as $\sum_{k=1}^{M} P_{jk}\Delta v_{t+1}(x,k)$ is increasing.*

Proof. Let v^j denote $\sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x,k)$. Consider $0 \leq v_1^j \leq v_2^j$, and let k_i be the index among efficient sets such that it solves $\max_k \left\{R^j\left(S_k^j\right) - Q^j\left(S_k^j\right)v_i^j\right\}$ for $i = 1, 2$. Suppose $k_1 \leq k_2$, then we have

$$R^j\left(S_{k_2}^j\right) - Q^j\left(S_{k_2}^j\right)v_2^j \geq R^j\left(S_{k_1}^j\right) - Q^j\left(S_{k_1}^j\right)v_2^j$$

since k_2 is an optimal efficient set for $\max_k \left\{R^j\left(S_k^j\right) - Q^j\left(S_k^j\right)v_2^j\right\}$. By using $0 \leq v_1^j \leq v_2^j$ and the previous lemma

$$R^j\left(S_{k_2}^j\right) - Q^j\left(S_{k_2}^j\right)v_1^j \geq R^j\left(S_{k_1}^j\right) - Q^j\left(S_{k_1}^j\right)v_1^j.$$

However, this inequality contradicts the optimality of k_1 for $\max_k \left\{R^j\left(S_k^j\right) - Q^j\left(S_k^j\right)v_1^j\right\}$. $\square$

Let $S_t^{*}(x,j)$ denote the optimal set that solves (1) so that

$$S^{*} = \left\{S_t^{*}(x,j);\ x = 0,1,\cdots,N,\ j = 1,2,\cdots,M,\ t = 0,1,\cdots,T\right\}$$

is the optimal policy. We have the following proposition to show the effects of the current inventory level on the optimal policy.

Proposition 4. *$v_t(x,j)$ is a concave function of x for any environment j and time t, i.e., $\Delta v_t(x,j) \leq \Delta v_t(x-1,j)$.*

Proof. Clearly $\Delta v_T(x-1,j) = 0$ for any x. By induction, suppose $\Delta v_{t+1}(x,j) \leq \Delta v_{t+1}(x-1,j)$ for any x and j. Then,

$$\begin{aligned}
\Delta v_t(x-1,j) - \Delta v_t(x,j) &= \sum_{k=1}^{M} P_{jk}\Delta v_{t+1}(x-1,k) - \sum_{k=1}^{M} P_{jk}\Delta v_{t+1}(x,k) \\
&+ \sum_{k=1}^{M} P_{jk} \sum_{a\in S_t^*(x-1,j)} \lambda_j P_a^j\left(S_t^*(x-1,j)\right)\left(r_a - \Delta v_{t+1}(x-1,k)\right) \\
&- \sum_{k=1}^{M} P_{jk} \sum_{a\in S_t^*(x-2,j)} \lambda_j P_a^j\left(S_t^*(x-2,j)\right)\left(r_a - \Delta v_{t+1}(x-2,k)\right) \\
&- \sum_{k=1}^{M} P_{jk} \sum_{a\in S_t^*(x,j)} \lambda_j P_a^j\left(S_t^*(x,j)\right)\left(r_a - \Delta v_{t+1}(x,k)\right) \\
&+ \sum_{k=1}^{M} P_{jk} \sum_{a\in S_t^*(x-1,j)} \lambda_j P_a^j\left(S_t^*(x-1,j)\right)\left(r_a - \Delta v_{t+1}(x-1,k)\right)
\end{aligned}$$

Since $S_t^*(x,j)$ is the optimal set when inventory level is x, and the environment is j at time t, any other set will be worse than this set. Hence, we have

$$\begin{aligned}
\Delta v_t(x-1,j) - \Delta v_t(x,j) &\geq \sum_{k=1}^{M} P_{jk}\Delta v_{t+1}(x-1,k) - \sum_{k=1}^{M} P_{jk}\Delta v_{t+1}(x,k) \\
&+ \sum_{k=1}^{M} P_{jk} \sum_{a\in S_t^*(x-2,j)} \lambda_j P_a^j\left(S_t^*(x-2,j)\right)\left(r_a - \Delta v_{t+1}(x-1,k)\right) \\
&- \sum_{k=1}^{M} P_{jk} \sum_{a\in S_t^*(x-2,j)} \lambda_j P_a^j\left(S_t^*(x-2,j)\right)\left(r_a - \Delta v_{t+1}(x-2,k)\right) \\
&- \sum_{k=1}^{M} P_{jk} \sum_{a\in S_t^*(x,j)} \lambda_j P_a^j\left(S_t^*(x,j)\right)\left(r_a - \Delta v_{t+1}(x,k)\right) \\
&+ \sum_{k=1}^{M} P_{jk} \sum_{a\in S_t^*(x,j)} \lambda_j P_a^j\left(S_t^*(x,j)\right)\left(r_a - \Delta v_{t+1}(x-1,k)\right)
\end{aligned}$$

After some cancellations, we obtain

$$\begin{aligned}\Delta v_t(x-1,j)-\Delta v_t(x,j) \;\geq\; & \sum_{k=1}^{M} P_{jk}\Phi(x,k) \\ & +\sum_{k=1}^{M} P_{jk}\sum_{a\in S_t^*(x-2,j)} \lambda_j P_a^j\left(S_t^*(x-2,j)\right)\Phi(x-1,k) \\ & -\sum_{k=1}^{M} P_{jk}\sum_{a\in S_t^*(x,j)} \lambda_j P_a^j\left(S_t^*(x,j)\right)\Phi(x,k)\end{aligned}$$

where $\Phi(x,k)=\Delta v_{t+1}(x-1,k)-\Delta v_{t+1}(x,k)$. This further leads to

$$\begin{aligned}\Delta v_t(x-1,j)-\Delta v_t(x,j) \;\geq\; & \sum_{k=1}^{M} P_{jk}\sum_{a\in S_t^*(x-2,j)} \lambda_j P_a^j\left(S_t^*(x-2,j)\right)\Phi(x-1,k) \\ & +\sum_{k=1}^{M} P_{jk}\left(1-\sum_{a\in S_t^*(x,j)} \lambda_j P_a^j\left(S_t^*(x,j)\right)\right)\Phi(x,k)\end{aligned}$$

after some mathematical manipulations. The right hand side of the inequality is positive by using the induction hypothesis and the fact that $\sum_{a\in S_t^*(x,j)} \lambda_j P_a^j\left(S_t^*(x,j)\right)\leq 1$. □

Proposition 5. $\Delta v_{t+1}(x,j)\leq \Delta v_t(x,j)$ *for any inventory level* x, *environment* j *and time* t.

Proof. Clearly $\Delta v_T(x,j)=0\leq \Delta v_{T-1}(x,j)$ for any x. By induction, suppose $\Delta v_{t+2}(x,j)\leq \Delta v_{t+1}(x,j)$ for any x and j. Then,

$$
\begin{aligned}
\Delta v_t(x,j) - \Delta v_{t+1}(x,j) &= \sum_{k=1}^{M} P_{jk} \Delta v_{t+1}(x,k) - \sum_{k=1}^{M} P_{jk} \Delta v_{t+2}(x,k) \\
&\quad + \sum_{k=1}^{M} P_{jk} \sum_{a \in S_t^*(x,j)} \lambda_j P_a^j \left(S_t^*(x,j)\right)\left(r_a - \Delta v_{t+1}(x,k)\right) \\
&\quad - \sum_{k=1}^{M} P_{jk} \sum_{a \in S_t^*(x-1,j)} \lambda_j P_a^j \left(S_t^*(x-1,j)\right)\left(r_a - \Delta v_{t+1}(x-1,k)\right) \\
&\quad - \sum_{k=1}^{M} P_{jk} \sum_{a \in S_{t+1}^*(x,j)} \lambda_j P_a^j \left(S_{t+1}^*(x,j)\right)\left(r_a - \Delta v_{t+2}(x,k)\right) \\
&\quad + \sum_{k=1}^{M} P_{jk} \sum_{a \in S_{t+1}^*(x-1,j)} \lambda_j P_a^j \left(S_t^*(x-1,j)\right)\left(r_a - \Delta v_{t+2}(x-1,k)\right)
\end{aligned}
$$

Since $S_t^*(x,j)$ is the optimal set when inventory level is x, and the environment is j at time t, any other set will be worse than this set. Hence, we have

$$
\begin{aligned}
\Delta v_t(x,j) - \Delta v_{t+1}(x,j) &\geq \sum_{k=1}^{M} P_{jk} \Delta v_{t+1}(x,k) - \sum_{k=1}^{M} P_{jk} \Delta v_{t+2}(x,k) \\
&\quad + \sum_{k=1}^{M} P_{jk} \sum_{a \in S_{t+1}^*(x,j)} \lambda_j P_a^j \left(S_{t+1}^*(x,j)\right)\left(r_a - \Delta v_{t+1}(x,k)\right) \\
&\quad - \sum_{k=1}^{M} P_{jk} \sum_{a \in S_t^*(x-1,j)} \lambda_j P_a^j \left(S_t^*(x-1,j)\right)\left(r_a - \Delta v_{t+1}(x-1,k)\right) \\
&\quad - \sum_{k=1}^{M} P_{jk} \sum_{a \in S_{t+1}^*(x,j)} \lambda_j P_a^j \left(S_{t+1}^*(x,j)\right)\left(r_a - \Delta v_{t+2}(x,k)\right) \\
&\quad + \sum_{k=1}^{M} P_{jk} \sum_{a \in S_t^*(x-1,j)} \lambda_j P_a^j \left(S_t^*(x-1,j)\right)\left(r_a - \Delta v_{t+2}(x-1,k)\right)
\end{aligned}
$$

After some cancellations, we obtain

$$\begin{aligned}\Delta v_t(x,j) - \Delta v_{t+1}(x,j) &\geq \sum_{k=1}^{M} P_{jk}\left(\Delta v_{t+1}(x,k) - \Delta v_{t+2}(x,k)\right) \\ &\quad -\sum_{k=1}^{M} P_{jk} \sum_{a\in S^*_{t+1}(x,j)} \lambda_j P^j_a\left(S^*_{t+1}(x,j)\right)\Phi(x,k) \\ &\quad +\sum_{k=1}^{M} P_{jk} \sum_{a\in S^*_{t}(x-1,j)} \lambda_j P^j_a\left(S^*_{t}(x-1,j)\right)\Phi(x-1,k)\end{aligned}$$

where $\Phi(x,k) = \Delta v_{t+1}(x,k) - \Delta v_{t+2}(x,k)$. This is equivalent to

$$\begin{aligned}\Delta v_t(x,j) - \Delta v_{t+1}(x,j) &\geq \sum_{k=1}^{M} P_{jk}\left(1 - \sum_{a\in S^*_{t+1}(x,j)} \lambda_j P^j_a\left(S^*_{t+1}(x,j)\right)\right)\Phi(x,k) \\ &\quad +\sum_{k=1}^{M} P_{jk} \sum_{a\in S^*_{t}(x-1,j)} \lambda_j P^j_a\left(S^*_{t}(x-1,j)\right)\Phi(x-1,k)\end{aligned}$$

The right hand side of the inequality is positive by using the induction hypothesis and $\sum_{a\in S^*_{t+1}(x,j)} \lambda_j P^j_a\left(S^*_{t+1}(x,j)\right) \leq 1$. □

We have the following corollary for the structure of the optimal policy. This corollary is proven by using Proposition 3, 4, and 5.

Corollary 1. $k^*_{j,t}(x)$ *is non-decreasing in the inventory level* x *and the time* t*, for any environment* j*.*

Let us discuss the implications of this corollary. As time increases, probability of purchase and expected revenue from the optimal set of products increase. In addition, these values decrease as the inventory level decreases. The system manager should offer products in such a way that the probability of purchase is high when there is more inventory or when there is less time to the end of the selling season.

4. Numerical Illustration

In this section, we present numerical illustrations of some of the properties. Section 4.1 verifies some of the proven structural properties of the optimal policy. Section 4.2 assesses the benefits of using the environment-based approach rather than a simpler approach that ignores the environment dependence.

4.1. Structural Properties

Suppose that there are 2 environments and 3 products which are labeled as K, L and M. Then, we can offer 8 possible sets of products and we label these sets by numbers from 1 to 8. The demand probabilities $P_a^j(S)$ are provided in Table 1.

We further suppose that the prices of the products are:

Product a	K	L	M
Price r_a	100	300	1000

and the arrival probabilities are:

Environment j	1	2
Arrival probability λ_j	0.8	0.9

with planning horizon $T = 11$.

Table 1: Demand Probabilities

Set	Label	1	2	3	4	5	6	7	8
$P_a^1(S)$	a	$\varnothing$	$\{K\}$	$\{L\}$	$\{M\}$	$\{K,L\}$	$\{K,M\}$	$\{L,M\}$	$\{K,L,M\}$
	K	0	0.8	0	0	0.7	0.7	0	0.65
	L	0	0	0.5	0	0.15	0	0.4	0.1
	M	0	0	0	0.2	0	0.1	0.2	0.1
	0	1	0.2	0.5	0.8	0.15	0.2	0.4	0.15
	Label	1	2	3	4	5	6	7	8
$P_a^2(S)$	a	$\varnothing$	$\{K\}$	$\{L\}$	$\{M\}$	$\{K,L\}$	$\{K,M\}$	$\{L,M\}$	$\{K,L,M\}$
	K	0	0.9	0	0	0.55	0.65	0	0.55
	L	0	0	0.6	0	0.4	0	0.5	0.2
	M	0	0	0	0.3	0	0.3	0.2	0.2
	0	1	0.1	0.4	0.7	0.05	0.05	0.3	0.05

To find the efficient sets, we first compute $R^j(S)$ and $Q^j(S)$ for the above parameters using (3) and (4). Hence, we obtain:

Set Label	1	2	3	4	5	6	7	8
	$\varnothing$	$\{K\}$	$\{L\}$	$\{M\}$	$\{K,L\}$	$\{K,M\}$	$\{L,M\}$	$\{K,L,M\}$
$R^1(S)$	0	80	150	200	115	170	320	195
$Q^1(S)$	0	0.8	0.5	0.2	0.85	0.8	0.6	0.85
$R^2(S)$	0	90	180	300	175	365	350	315
$Q^2(S)$	0	0.9	0.6	0.3	0.95	0.95	0.7	0.95

Then, we plot a diagram of the corresponding values $R^j(S)$ and $Q^j(S)$ for any offer set S and $j = 1, 2$ to determine the efficient sets for both environments. We find that $\{M\}$ and $\{L, M\}$ are efficient sets in environment 1 and $\{M\}$, $\{L, M\}$ and $\{K, M\}$ in environment 2 by considering Figure 1.

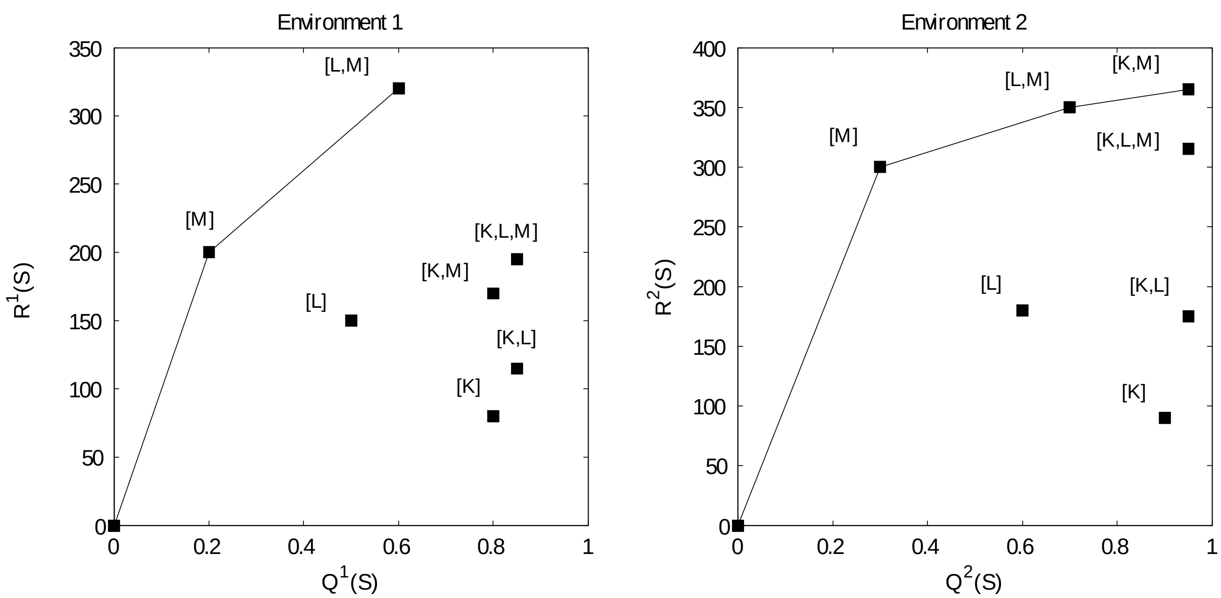


Figure 1: Efficient Sets in Each Environment

Finally, let us suppose that the transition matrix of the environmental process is

$$P = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}.$$

To find the optimal set for any inventory level, environment and time, we use the labels defined for any set in each environment. We search among all sets, in order to verify that inefficient sets cannot be optimal. Also note that efficient sets of environment 1 are $\{M\}$ and $\{L, M\}$ which are labeled 4 and 7 respectively. Since $R^1(\{M\}) \leq R^1(\{L, M\})$, we label

Table 2: Environment 1 - Optimal Sets

Env. 1	$\underrightarrow{\text{Time}}$	1	2	3	4	5	6	7	8	9	10
	1	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$7 \mid 2$	$7 \mid 2$
	2	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$
$\uparrow$	3	$4 \mid 1$	$4 \mid 1$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$
x	4	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$
$\downarrow$	5	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$
	6	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$
	7	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$
	8	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$

$\{M\}$ as the 1st efficient set, and $\{L, M\}$ the 2nd efficient set. Similarly, the efficient sets of environment 2 are $\{M\}, \{L, M\}$ and $\{K, M\}$ which are labeled 4, 7 and 6 respectively. Since $R^1(\{M\}) \leq R^1(\{L, M\}) \leq R^1(\{K, M\})$, we label $\{M\}, \{L, M\}$, and $\{K, M\}$ the 1st, 2nd and 3rd efficient sets respectively. We use the notation $i|j$ to represent the two indices together where i is the index among all sets, j is the index among efficient sets. Table 2 shows the optimal set for a given inventory level and time when the current environment is the first one.

As we expect, only 4th and 7th sets are optimal, and none of the inefficient sets is optimal in environment 1. For a given inventory level, the index of the optimal set increases from 1 to 2 as time increases. Similarly, for a given time, we observe an increase in the index of the optimal set as inventory level increases. The optimal sets for environment 2 are presented in Table 3.

Table 3: Environment 2 - Optimal Sets

Env. 2	$\underrightarrow{\text{Time}}$	1	2	3	4	5	6	7	8	9	10
	1	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$6 \mid 3$
	2	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$6 \mid 3$	$6 \mid 3$
$\uparrow$	3	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$7 \mid 2$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$
x	4	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$4 \mid 1$	$7 \mid 2$	$7 \mid 2$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$
$\downarrow$	5	$4 \mid 1$	$4 \mid 1$	$7 \mid 2$	$7 \mid 2$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$
	6	$7 \mid 2$	$7 \mid 2$	$7 \mid 2$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$
	7	$7 \mid 2$	$7 \mid 2$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$
	8	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$	$6 \mid 3$

Recall that we have 3 efficient sets in environment 2. We observe that only these sets are optimal and monotonicity results corresponding to the structure of the optimal policy also hold in environment 2.

4.2. Effectiveness of the Environment Based Model

In this section we assess the effectiveness of using the environment-based model rather than a simpler approach that ignores the dependence of the problem parameters on the external environment. In order to have a simple but plausible benchmark we assume that the alternative policy is obtained by solving an environment independent problem by averaging over the environment-dependent parameters using a probabilistic mixture. This can be viewed as a certainty equivalent approximation.

In the benchmark model, the demand arrival probabilities are obtained by mixing the probabilities of each product demanded so that $P_a(S) = qP_a^1(S) + (1-q)\ P_a^2(S)$ where $q \in [0,1]$ is the mixture probability. Similarly, we mix the arrival probabilities by $\lambda = q\ \lambda_1 + (1-q)\ \lambda_2$. We use the same arrival probabilities as we did in the previous subsection. Please note that the mixing probability q is a parameter of the policy and different mixing probabilities lead to different policies. In the environment based model, the inventory manager makes full use of the information that is available at any time. The optimal product sets that are offered depend on the environment. In the benchmark mixed model, however, the information is either ignored or unavailable. Instead, the manager considers a simplified problem where the parameters for the next period are a probablistic mixture of those in environment 1 with a mixing probability q and those in environment 2 with probability $1-q$.

Finally, we take the horizon length $T = 100$, capacity 50 and the transition matrix of the environmental process to be:

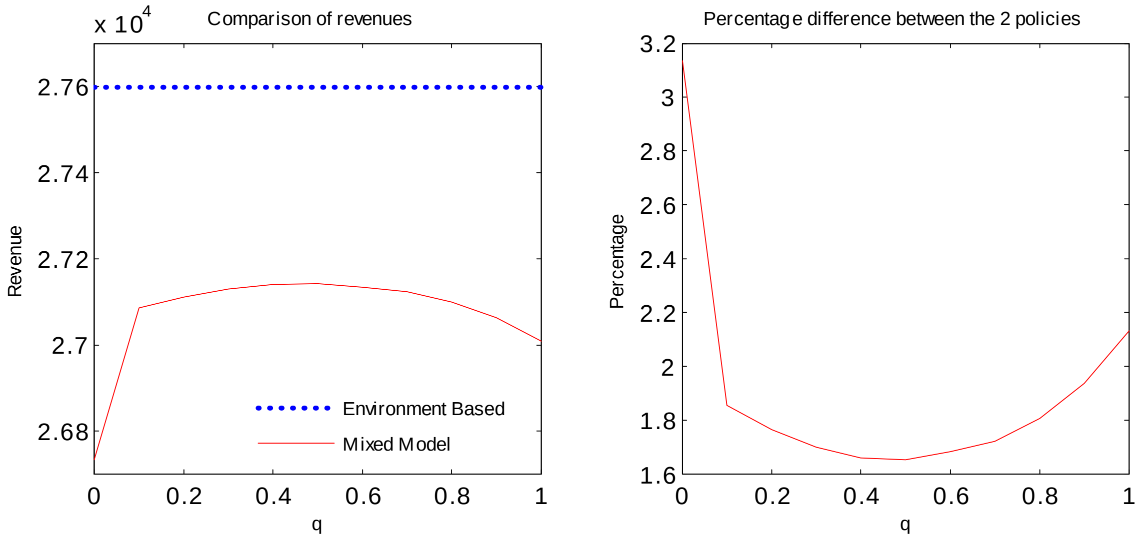
$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

Let us consider the optimal policy of the benchmark problem which is independent of the environment. The optimality equation corresponds to that of a standard problem under consumer choice and is given by:

$$\begin{aligned} w_t(x) \quad &= \quad \max_{S \subseteq N} \{ \sum_{a \in S} \lambda P_a(S)\left(c(a) + w_{t+1}(x-1)\right) \\ &\qquad + \left(\lambda P_0(S) + 1 - \lambda\right) w_{t+1}(x) \} \end{aligned}$$

with boundary conditions $w_T(x) = 0$ and $w_t(0) = 0$ for all x and t.

By using this model, we obtain a policy that does not depend on the environment. This gives an approximate policy for our environment based model. The approximate policy disregards the environments and uses the same product for each environment. We then



evaluate this policy using our environment-based model and we calculate the corresponding expected revenue for time 0 and inventory level 50 starting with environment 1. We compare this value with the expected revenue for time 0, inventory level 50 and environment 1 by using the optimal policy of the environment based model. This comparison and difference is given for different values of q in Figure 2.

Figure 2 clearly demonstrates the benefits of our environment based model. We also observe that the benchmark policy provides the best approximation when the mixing probability is 0.5. This is not surprising because the limiting distribution of environmental process is [0.5 0.5], which means that the proportion of time spent in each environment is 0.5 in the long-run. However, there is still more than 1.6 percent of expected revenue difference between the optimal result and the heuristic when the mixing probability is 0.5. The error becomes more significant if the user of the benchmark policy is not able to correctly identify the best mixing probability and chooses extreme values for these parameters.

5. Conclusion

In this paper, we have established the structural properties of the optimal capacity control policy under a discrete choice model of consumer behavior in a randomly fluctuating demand environment. In particular, we showed that as far as the structure of the optimal policy, the main results of the general discrete choice model of consumer behavior in the standard setting also hold in a randomly fluctuating demand environment with some environment dependence. In other words, we established that only efficient sets can be optimal and the index of the optimal efficient set has a monotonic structure in time and inventory level. Moreover, we illustrated the structural results and assessed the effectiveness of the environment based model. A worthwhile extension of this study would be to consider hidden Markov models of this setting since the environment may not be observed directly. Investigating similar models in network settings or in continuous time are also interesting directions for future research.

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