Newsvendor Models with Dependent Random Supply and Demand

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Abstract

The newsvendor model is perhaps the most widely analyzed model in inventory management. In this single-period model, the only source of randomness is the demand during the period and one tries to determine the optimal order quantity in view of various cost factors. We consider an extention where supply is also random so that the quantity ordered is not necessarily received in full at the beginning of the period. Such models have been well-received in the literature with the assumption of independence between demand and supply. In this setting, we suppose that the random demand and supply are not necessarily independent. We focus on the resulting optimization problem and provide interesting characterizations on the optimal order quantity.

Keywords. Newsvendor model, random capacity, random yield, quasi-concavity

1 Introduction

The major source of randomness in inventory models is the demand. If the demand exceeds or falls short of expectations, the inventory manager will face shortage or lost sales. Moreover, the uncertainty of demand is not necessarily the only source of randomness. In fact, in recent years, there has been a lot of emphasis on models with supply uncertainty as well. The combined randomness of demand and supply enhances the level of uncertainty, thus leading to increased complexity. In this paper, we provide an example in the form of the well-known newsvendor model. Although this is a rather simple single-period model, it often forms the building block of many multi-period dynamic inventory, capacity-planning, and contract design problems.

The main theme of this exposure concerns randomness in supply. This is an issue that should not be neglected or underestimated. There are a lot of tragic examples concerning losses incurred due to the randomness in supply caused by uncertainties in production and transportation processes. Among many others, long machine downtimes due to unplanned maintenance, strikes, seconds and scraps in a production run, lack of raw material and rework are some reasons which leads to uncertainty during the production stage. In addition, uncertainty during transportation is another cause for supply randomness. This is due to accidents, deficiencies in the quality of transportation and various environmental factors. Chopra and Sodhi [2] and Serel [17] discuss some of the issues related to randomness in supply and mention a number of real cases. For example, as reported in Norrman and Jansson [15], a fire at a supplier's plant disrupted the supply of radio-frequency chips to Ericsson in 2001 resulting in a loss of $400 million. Juttner [11] reports that in the same year, the continuity of production at Land Rover was threatened due to financial problems faced by the UK chassis manufacturer UPF Thompson. Kharif [13] states that Motorola failed to ship the phones promised to its major customers during the holiday season in 2003 due to component shortages.

In random supply models, the quantity ordered by the inventory manager is not received in full with certainty. Instead, the manager receives a random amount that depends on the order quantity. An earlier review is provided in Yano and Lee [22] and more recent developments are summarized in Grosfeld-Nir and Gerchak [9]. The earliest model of a random supply in inventory model was developed by Karlin [12] who assumes that the only decision available is whether to order, and that if an order is placed, a random quantity is delivered. It is also shown that if the inventory holding and shortage cost functions are convex increasing, then there is a single critical initial on-hand inventory below which an order should be placed; otherwise, it is optimal not to order. Shih [19] assumes that inventory holding and shortage costs are linear and that the distribution of the fraction defective is invariant with the production level. The optimal production/order quantity can be found using a variant of the newsvendor model. Noori and Keller [14] extend Shih's study by providing closed form solutions for the optimal order quantity for various distributions of the quantity received. Gerchak et al. [8] obtain the same result for the profit maximization objective by assuming continuous demand and yield. According to their work, there is a critical level of initial stock above which no order will be placed, and this level is the same as the certain yield case. Henig and Gerchak [10] discuss single and multi-period models with more general assumptions about the random replenishment distribution and the cost structure. They prove that for a single-period model there exists an optimal order point that is independent of replenishment randomness.

Most of the literature on supply randomness considers the following models based on randomness in the capacity of the supplier and in the yield. An implicit assumption in almost all of the papers is the independence of demand and supply. Let y be the amount ordered and $Q(y)$ be the amount received.

- Random Yield: The amount ordered could be different from the amount received so that only a fraction enters the stockpile and

  $$Q(y) = yU \tag{1}$$

  where U represents the proportion of nondefective items received. Henig and Gerchak [10] show that a non-order-up-to policy is optimal in this case. However, in a simplified version where U is either 0 or 1, also called the random availability model, Özekici and Parlar [16] establish the optimality of base-stock policies.

- Random Capacity: The supplier has some random replenishment capacity K so that

  $$Q(y) = \min\{K, y\}. \tag{2}$$

  When an order is placed for y units, the suppliers will ship y if the total amount K of on hand inventory that they poses is greater than y. Or else, they will send all the inventory they poses, which is K. Erdem and Özekici [5] consider a periodically reviewed single-item inventory model in a random environment with random capacity and show that a base-stock policy is optimal.

- Random Yield and Capacity: This is another model that combines the previous two so that

  $$Q(y) = U \min\{K, y\}. \tag{3}$$

  Once y units are ordered, the supplier can ship at most K and only a proportion U is received in good shape. In a recent article, Arifoğlu and Özekici [1] consider a model with random yield and fixed capacity operating in a hidden Markov environment. A state-dependent modified inflated base-stock policy is shown to be optimal.

In this paper, we discuss variations of the standard newsvendor model with random demand as well as random supply. The main contribution of the paper is that the demand and supply are dependent. Moreover, this dependence has an arbitrary structure and no special restrictions are imposed. Although there is now abundant literature on inventory models with random supply, only a few consider the dependence between demand and supply. This is often achieved by supposing that the inventory model operates in a randomly changing "environment". As the state of the environment changes so do all stochastic and deterministic parameters. Because the marginal distributions of the demand and supply both depend on the state of the environment, there is stochastic dependence between them. For example, Song and Zipkin [20] modeled the "state-of-the-world" as a Markov chain and assumed that demand in successive periods are dependent on the state of this Markov chain. In a later study, Sethi and Cheng [18] also incorporated fluctuating demand environment into their model using Markov chains, and found the most general setting under which an environment-dependent (s, S) policy is optimal. Later, starting with mid-1990s, researchers introduced models where not only the demand, but the supply is also affected by the fluctuating environment. For example, Song and Zipkin [21] analyzed the effect of the fluctuating environment on supply by using a Markov chain approach. They show that the optimal policy has the same structure as in standard models, but that its parameters change dynamically to reflect current supply conditions. Another paper which considers the possible effect of the fluctuating environment on supply as well as on demand and cost parameters is Özekici and Parlar [16]. They assume that the supplier is either available or unavailable when the order is placed so that the order is either totally satisfied or in the other extreme, remains entirely unfulfilled. Erdem and Özekici [5] extend this line of research by assuming that the supplier is always available, but that its capacity is random and dependent on the state of the environment. A survey of articles on supply randomness can be found in Gallego and Hu [6]. The available literature on inventory models modulated by an external environmental process provides sufficient justification for the dependence between demand, supply and other possible sources of randomness. They are dependent because the system as a whole operates in a randomly changing environment that affects them. This leads to a number of interesting and practically relevant situations. For instance, supply and demand may be positively correlated due to similar environmental conditions. On the other hand, during periods where demand surges, suppliers may have to ration production capacity between multiple customers leading to negative correlation between the supply quantity and the demand. Although there is dependence between demand and supply due to the randomly changing environment, these models usually suppose that in any "fixed" environmental state the demand and supply are independent. Their joint distribution is the product of the marginal distributions in that state. Therefore, there is conditional independence between demand and supply given the environment. This, of course, provides a computationally tractable procedure to deal with dependence. In this paper, we do not suppose any special structure of dependence. The demand and supply have an arbitrary joint distribution function.

In recent years, there has been growing interest in managing risks for the newsvendor model using financial instruments like futures and options. Gaur and Seshadri [7] provides statistical justification and motivation by showing that an index that measures sales/demand (Redbook) is highly correlated with a financial index (SP500). This correlation provides an opportunity to hedge risks associated with random demand by investing in a portfolio of derivatives of the financial index. We refer the reader to Chu et al. [3], Ding et al. [4], and Okyay et al. [**?**] for some developments along this direction. The current literature deals with managing risks associated with demand uncertainty. Our model involves supply uncertainty as well and similar risk hedging portfolios may be constructed to control the uncertainty in the cash flows.

In addition to its practical relevance, the non-independence assumption leads to more challenging and interesting optimization problems and characterizations of the optimal order quantity. We first review the classic newsvendor problem briefly in Section 2. The random supply extensions with random yield, random capacity, and their combination are discussed in Section 3, Section 4, and Section 5 respectively. Concluding remarks and ideas for future work are presented in Section 6.

2 The Newsvendor Problem

The newsvendor problem is a well-known single-item, single-period inventory problem in which the decision maker (or newsvendor) has to decide on how much to order. The replenishment decision is critical because if he orders too many, purchase cost will be unnecessarily high; on the contrary, there will be a missed opportunity for additional profit if he orders too few. In daily life, it is very common to encounter examples of newsvendor models, that's the foremost reason why these models are studied extensively. There is random demand D with a known distribution function that has a probability density function. Throughout this paper, we assume that all marginal, joint and conditional distributions corresponding to random demand D, random yield U and random capacity K have marginal, joint and conditional probability density functions. Moreover, we suppose that there is a fixed sale price p, a fixed purchase cost c, a fixed shortage penalty h, and a fixed salvage value s which satisfy $p > c > s > 0$ and $p + h > c$ to avoid trivial situations.

The aim of the newsvendor is to maximize the expected cash flow at the end of the period by choosing an ordering quantity y, or

$$\max_y E\left[CF\left(D, y\right)\right] \tag{4}$$

where $CF\left(D, y\right)$ is the random cash flow which can be written as

$$\begin{aligned} CF\left(D, y\right) &= -cy + p\min\left\{D, y\right\} + s\max\left\{y - D, 0\right\} - h\max\left\{D - y, 0\right\} \\ &= \left(s - c\right)y + \left(p + h - s\right)\min\left\{D, y\right\} - hD \end{aligned} \tag{5}$$

so that

$$E\left[CF\left(D, y\right)\right] = \left(s - c\right)y + \left(p + h - s\right)E\left[\min\left\{D, y\right\}\right] - hE\left[D\right]. \tag{6}$$

Note that for any random variable X with a probability density function f_X, we can write

$$E\left[\min\left\{X, y\right\}\right] = \int_0^y x f_X(x)dx + y\int_y^{+\infty} f_X(x)dx$$

and one can easily show that

$$\frac{dE\left[\min\left\{X, y\right\}\right]}{dy} = \int_y^{+\infty} f_X(x)dx = P\left\{X > y\right\}. \tag{7}$$

In our analysis, we will use (7) repeatedly.

Particularly, in order to solve (4), we take the derivative of (6) with respect to y and set it equal to 0 where X is D. Hence, we obtain the first order condition

$$g(y) = \frac{d}{dy}E\left[CF\left(D, y\right)\right] = \left(s - c\right) + \left(p + h - s\right)P\left\{D > y\right\} = 0. \tag{8}$$

It can easily be verified that the second order condition is satisfied because the objective function (6) is concave since

$$\frac{dg(y)}{dy} = \frac{d^2E\left[CF\left(D, y\right)\right]}{dy^2} = -\left(p + h - s\right)f_D(y) \le 0 \tag{9}$$

where f_D is the probability density function of D. Recall that the randomness in (5) is generated by D only and the newsvendor makes replenishment decisions based on his expectation of $CF\left(D, y\right)$. It then follows from (8) that the optimal order quantity y^* satisfies

$$P\left\{D \le y^*\right\} = \frac{p + h - c}{p + h - s} = \hat{p}. \tag{10}$$

Note that (10) is the optimality condition and $\hat{p}$ denotes the critical ratio which clearly satisfies $0 \le \hat{p} \le 1$. It is possible that (10) does not have a solution. If $P\left\{D = 0\right\} > \hat{p}$, then there is no y^* that satisfies (10) and the optimal solution of (4) is $y^* = 0$ trivially. This follows by noting that $g(0) < 0$ so that the objective function (6) is clearly decreasing by (9). Similarly, if $P\{D < +\infty\} < \hat{p}$, then there also is no y^* that satisfies (10) and now the optimal solution of (4) is $y^* = +\infty$ trivially. This follows by noting that $g(+\infty) > 0$ which now implies that the objective function (6) is increasing by (9). If $P\{D = 0\} \leq \hat{p} \leq P\{D < +\infty\}$, then there is $0 \leq y^* \leq +\infty$ which satisfies (10). Moreover, the solution is unique if $P\{D \leq y\}$ is strictly increasing in y.

3 Random Yield

Let $U \geq 0$ be a random variable representing the proportion of the ordered quantity that will be received in good condition so that the model is given by (1). Note that U and D are not necessarily independent. If we rewrite our payoff function in (5) in this case, we get

$$CF(D, U, y) = (s - c)Uy + (p + h - s)\min\{D, Uy\} - hD. \tag{11}$$

Hence, the objective function becomes

$$E[CF(D, U, y)] = (s - c)yE[U] + (p + h - s)E[\min\{D, Uy\}] - hE[D]. \tag{12}$$

Note that for any random variables X and Z with probability density functions f_X and f_Z, we can write

$$E[\min\{X, Zy\}] = \int_0^{+\infty} f_Z(z)\,dz \left(\int_0^{zy} x f_{X|z}(x)\,dx + zy\int_{zy}^{+\infty} f_{X|z}(x)\,dx\right)$$

where $f_{X|z}$ is the conditional probability density function of X given $Z = z$. One can show that

$$\frac{dE[\min\{X, Zy\}]}{dy} = \int_0^{+\infty} z f_Z(z)\,dz \int_{zy}^{+\infty} f_{X|z}(x)\,dx = E\left[Z1_{\{X>Zy\}}\right]. \tag{13}$$

By using (13), we take the derivative of (12) and set it equal to zero to obtain the first order optimality condition

$$g(y) = \frac{d}{dy}E[CF(D, U, y)] = (p + h - c)E[U] - (p + h - s)E\left[U1_{\{D\leq Uy\}}\right] = 0$$

and this optimality condition can be written as

$$\frac{E\left[U1_{\{D\leq Uy^*\}}\right]}{E[U]} = \frac{p + h - c}{p + h - s} = \hat{p}. \tag{14}$$

Note that the objective function (11) is also concave in y since

$$\frac{dg(y)}{dy} = \frac{d^2E[CF(D, U, y)]}{dy^2} = -(p + h - s)\int_0^{+\infty} u^2 f_U(u) f_{D|u}(uy)du \leq 0$$

where f_U is the probability density function of U and $f_{D|u}$ is the conditional probability density function of D given $U = u$. Therefore, there is an optimal y^* which satisfies (14) provided that $g(0) \geq 0$ and $g(+\infty) \leq 0$ or, equivalently, $E\left[U1_{\{D=0\}}\right] \leq \hat{p}E[U] \leq \lim_{y\to+\infty} E\left[U1_{\{D\leq Uy\}}\right]$). If $E\left[U1_{\{D\leq Uy\}}\right]$ is strictly increasing in y, then this solution is unique. A similar argument as in previous section can be made to show that $y^* = 0$ if $E\left[U1_{\{D=0\}}\right] > \hat{p}E[U]$, and $y^* = +\infty$ if $\lim_{y\to+\infty} E\left[U1_{\{D\leq Uy\}}\right] < \hat{p}E[U]$.

As a special case, if the yield is certain so that $U = 1$, then the optimality condition (14) reduces to (10) and the problem is identical to the standard newsvendor problem. We now consider a numerical illustration to demonstrate our results. The supplier's facility has demand-dependent yield. This is plausible when the supplier satisfies demands from multiple customers and when the market demand is high. So, he has to produce faster using excess capacity and human resources which may decrease yield rates. In particular, assume that the buyer's demand has a (truncated) normal distribution with mean 50 and standard deviation 15. If demand is less than 50 then the supplier has perfect yield ($U = 1$); but when demand is more than 50, the supplier's yield is uniformly distributed between u_l and u_h. The fixed financial parameters are assumed to be: $c = \$2$, $s = \$1$ and $h = \$0$.

The optimality condition defined by (14) can easily be solved by numerical integration to obtain the optimal order quantity y^*. As a benchmark, we also compute and report the optimal order quantity under independent yield and demand referred to as y^*_{ind}. For fairness, it is assumed, in this case, that yield is random with the identical marginal probability density function (i.e., U takes the value of 1 with probability $P\{D < 50\}$ and is uniformly distributed between u_l and u_h otherwise).

Table 1 reports the optimal ordering quantities under dependent demand and yield (y^*) versus if demand and yield were assumed to be independent (y^*_{ind}) for varying yield distributions and two different sale prices $p = \$5$ and $p = \$10$. It can be observed that as the yield when demand is low stochastically decreases, y^* increases much faster than y^*_{ind} for both values of p. For instance, when $(u_l, u_h) = (0.3, 0.5)$, it is optimal to order 35 additional units (when $p = 5$) and 37 additional units (when $p = 10$) under dependent yield and demand.

	$p = 5$		$p = 10$	
(u_l, u_h)	y^*	y^*_{ind}	y^*	y^*_{ind}
(0.7,0.9)	73	67	85	78
(0.6.0.8)	82	72	96	86
(0.5,0.7)	93	76	111	97
(0.4,0.6)	106	82	131	112
(0.3,0.5)	119	84	159	132

Table 1: Optimal ordering quantities for dependent and independent demand and yield when average yield changes

Next, we investigate the effect if increasing yield variability. To this end, we fix $p = 10$ and vary the intervals (u_l, u_h) while keeping the midpoint of the interval at 0.5. Table 2 reports that the optimal order quantities are relatively insensitive to increasing yield variability.

(u_l, u_h)	y^*	y^*_{ind}
(0.5,0.5)	130	113
(0.4, 0.6)	131	112
(0.3,0.7)	132	112
(0.2,0.8)	134	107
(0.1,0.9)	120	102

Table 2: Optimal ordering quantities for dependent and and independent demand and yield when variability of yield changes

Finally, we investigate the effects of increasing demand variability on the optimal order quantities. The parameters are: $p = 10$ and $(u_l, u_h) = (0.3, 0.5)$ and the standard deviation of the truncated normal demand distribution is varied between 1 and 50. Figure 1 depicts the optimal order quantities under dependent and independent demand assumptions. It is observed that the rate of increase in the optimal order quantity is much higher under the dependent model.

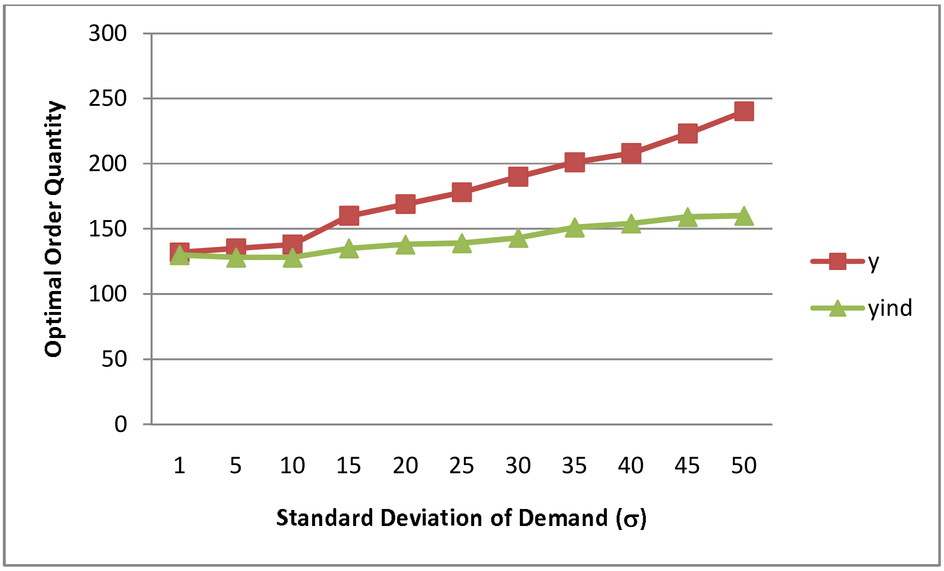

Figure 1: Effect of demand standard deviation

4 Random Capacity

This section deals with the effects of the supply uncertainty when it is caused by some random capacity K as prescribed by (2). Here, K represents the maximum number of units that the supplier can ship. We assume that $P\{K > z\} > 0$ for all z without loss of generality for technical reasons. This assumption implies that there is always some probability that our order will be satisfied in full. Clearly, if $P\{K > z_0\} = 0$ for some z_0, then our order should not exceed this level since we can not possibly receive more than z_0.

Now, (5) should be modified to include the random capacity. The payoff function or the cash flow can now be represented as

$$CF\left(D,K,y\right) = (s-c)\min\left\{K,y\right\} + (p+h-s)\min\left\{D,K,y\right\} - hD. \tag{15}$$

Then, the objective function becomes

$$E\left[CF\left(D,K,y\right)\right] = (s-c)\,E\left[\min\left\{K,y\right\}\right] + (p+h-s)\,E\left[\min\left\{D,K,y\right\}\right] - hE\left[D\right]. \tag{16}$$

Note that for any random variables Y and Z, we can write

$$\frac{dE\left[\min\left\{Y,Z,y\right\}\right]}{dy} = P\left\{X > y, Z > y\right\}. \tag{17}$$

This follows from (7) by taking $X = \min\{Y, Z\}$. In order to obtain the first order optimality condition we need to take the derivative of (16) with respect to y and set it equal to 0.

Using (7) and (17) where Y represents D and Z represents K, we obtain the optimality condition

$$g(y) = \frac{dE\left[CF\left(D,K,y\right)\right]}{dy} = (s-c)\,P\left\{K > y\right\} + (p+h-s)\,P\left\{D > y, K > y\right\} = 0 \tag{18}$$

which can also be written as

$$g\left(y\right) = P\left\{K > y\right\}\left((s-c) + (p+h-s)\,P\left\{D > y \mid K > y\right\}\right) = 0. \tag{19}$$

Since $P\{K > y\} > 0$ for all y by our assumption, (19) can be written as

$$(s - c) + (p + h - s) P\{D > y \mid K > y\} = 0.$$

Rearranging the terms, we finally obtain the optimality condition

$$P\{D \leq y^* \mid K > y^*\} = \frac{p + h - c}{p + h - s} = \hat{p}. \tag{20}$$

Note that we have the same critical ratio on the right-hand side of (10). However, we have a different probability on the left-hand side of (20). For further analysis, let

$$h(y) = P\{D \leq y \mid K > y\} = 1 - P\{D > y \mid K > y\} \tag{21}$$

denote this conditional probability. The existence and uniqueness of a solution of (20) depends on the structure of $h(y)$. In the newsvendor model with random capacity, we do not necessarily end up with a concave objective function since $g(y)$ is not necessarily decreasing. It is clear that we need certain restrictions on the relationship between the two random variables D and K. This can be obtained by reconsidering the optimality conditions (10) and (20). Suppose that the conditional probability $h(y)$ is increasing in y. If $h(0) \leq \hat{p} \leq h(+\infty)$, then there is a $0 \leq y^* \leq +\infty$ that satisfies (20) so that $g(y^*) = 0$ or $h(y^*) = \hat{p}$. It is also unique if $h(y)$ is strictly increasing. Moreover, it follows from (19) and (21) that the derivative $g(y)$ is nonnegative and decreasing on $[0, y^*)$, and it is nonpositive on $[y^*, +\infty)$. This is equivalent to saying that the objective function is concave increasing on $[0, y^*)$ and decreasing on $[y^*, +\infty)$. Therefore, it is quasi-concave and the solution y^* is indeed the optimal solution that maximizes (16). Although the concavity condition does not hold any more in random capacity models, quasi-concavity of the objective function leads to the characterization of the optimal order quantity via (20). We can repeat the arguments made in the previous sections to conclude that $y^* = 0$ if $h(0) \geq \hat{p}$ and the objective function decreases on $[0, +\infty)$. Moreover, $y^* = +\infty$ if $h(+\infty) \leq \hat{p}$ and the objective function is concave increasing on $[0, +\infty)$.

The relationship between D and K clearly affects the optimal order quantity. In case $K = +\infty$, $K > y$ is always true for all $y \geq 0$. Hence, the optimality condition (20) reduces to (10) since $P\{D \leq y \mid K > y\} = P\{D \leq y\}$. This is also true if D and K are independent. To see the intuition behind the increasing property imposed on the conditional probability $h(y)$, consider the model where

$$P\{K > y\} = e^{-\mu y}$$

and

$$P\{D > y | K = x\} = e^{-\lambda(x) y} \tag{22}$$

so that K is exponentially distributed with some rate μ, while D is conditionally exponentially distributed with some rate $\lambda(x)$ given $K = x$. We can then obtain

$$h(y) = P\{D \leq y \mid K > y\} = 1 - \int_0^{+\infty} \mu e^{-\mu x} dx e^{-\lambda(x+y) y}. \tag{23}$$

It is clear that if $\lambda(x)$ is increasing in x, then $h(y)$ is also increasing in y and our results are valid. Note that $E[D|K] = 1/\lambda(K)$ and the demand is decreasing in expectation as the capacity increases.

We now consider a special case where the dependence between random demand and capacity are perfect so that $K = A(D)$ for some deterministic, increasing and differentiable function A that is invertible. Then,

$$h(y) = P\{D \leq y \mid K > y\} = \frac{P\{y < K \leq A(y)\}}{P\{K > y\}}.$$

We analyze a couple of interesting cases.

- Case 1: $A(y) \leq y$ for all y. This clearly implies that $h(y) = 0$ and $y^* = +\infty$ since $h(+\infty) = 0 \leq \hat{p}$.
- Case 2: $A(y) > y$ for all y. This implies that

  $$h(y) = \frac{F_K(A(y)) - F_K(y)}{1 - F_K(y)}$$

  where F_K is the cumulative distribution function of K, and one eventually obtains

  $$\frac{dh(y)}{dy} = \frac{f_K(A(y))\, A'(y) - r_K(y)(1 - F_K(A(y)))}{1 - F_K(y)} \tag{24}$$

  where $f_K(y)$ is the probability density function while $r_K(y) = f_K(y)/(1 - F_K(y))$ is the failure rate function of K. Now, it clearly follows from (24) that $h(y)$ is increasing if

  $$A'(y) \geq \frac{r_K(y)}{r_K(A(y))}. \tag{25}$$

  Therefore, the optimality condition is $h(y^*) = \hat{p}$ if the additional restriction (25) is satisfied. This holds, for example, if K has an increasing failure rate distribution and $A'(y) \geq 1$. Although the analysis gets a bit messy, other cases where $A(y)$ intersects y can be analyzed in a similar fashion.

5 Random Yield and Capacity

We now consider the general model (3) that combines the previous two. Then, the cash flow becomes

$$CF(D, K, U, y) = (s - c)\, U \min\{y, K\} + (p + h - s) \min\{D, U \min\{K, y\}\} - hD \tag{26}$$

and the objective function is

$$E[CF(D, K, U, y)] = (s - c)\, E[U \min\{y, K\}] + (p + h - s)\, E[\min\{D, UK, Uy\}] - hE[D]. \tag{27}$$

Note that for random variables X, Z and V with probability density functions f_X, f_Z and f_V

$$\begin{aligned} E[\min\{X, ZV, Vy\}] &= \int_0^{+\infty} f_V(v)\, dv \int_0^{+\infty} f_{Z|v}(z)\, dz \int_0^{v \min\{y,z\}} x f_{X|vz}(x)\, dx \\ &\quad + \int_0^{+\infty} f_V(v)\, dv \int_0^{+\infty} f_{Z|v}(z)\, dz \min\{y, z\} \int_{v \min\{y,z\}}^{+\infty} f_{X|vz}(x)\, dx \end{aligned}$$

where $f_{Z|v}$ is the conditional probability density function of Z given $V = v$, and $f_{X|vz}(x)$ is the conditional probability density function of X given $V = v$ and $Z = z$. One can then show that

$$\begin{aligned} \frac{dE[\min\{X, ZV, Vy\}]}{dy} &= \int_0^{+\infty} v f_V(v)\, dv \int_y^{+\infty} f_{Z|v}(z)\, dz \int_{vy}^{+\infty} f_{X|vz}(x)\, dx \\ &= E\left[V 1_{\{Z>y, X>Vy\}}\right]. \end{aligned} \tag{28}$$

By using (13) and (28) we take the derivative of (27) and set it equal to zero. Hence, the optimality condition is

$$g(y) = \frac{dE[CF(D, K, U, y)]}{dy} = (s - c)\, E\left[U 1_{\{K>y\}}\right] + (p + h - s)\, E\left[U 1_{\{K>y, D>Uy\}}\right] = 0$$

which can be written as

$$g(y) = E\left[U 1_{\{K>y\}}\right] \left((s - c) + (p + h - s) \left(\frac{E\left[U 1_{\{K>y, D>Uy\}}\right]}{E\left[U 1_{\{K>y\}}\right]} \right) \right) = 0.$$

Noting that $P\{K > y\} > 0$ for all y by our assumption and supposing that $U \neq 0$ trivially, $E\left[U1_{\{K>y\}}\right] > 0$ and we can equivalently write

$$(s-c) + (p+h-s)\left(\frac{E\left[U1_{\{K>y,D>Uy\}}\right]}{E\left[U1_{\{K>y\}}\right]}\right) = 0$$

so that the optimality condition is

$$\frac{E\left[U1_{\{K>y^*,D\leq Uy^*\}}\right]}{E\left[U1_{\{K>y^*\}}\right]} = \frac{p+h-c}{p+h-s} = \hat{p} \tag{29}$$

since $E\left[U1_{\{K>y,D>Uy\}}\right] = E\left[U1_{\{K>y\}}\right] - E[U1_{\{K>y,D\leq Uy\}}]$.

It is clear that the objective function (27) is not necessarily concave. However, defining

$$h\left(y\right) = \frac{E\left[U1_{\{K>y,D\leq Uy\}}\right]}{E\left[U1_{\{K>y\}}\right]} = 1 - \frac{E\left[U1_{\{K>y,D>Uy\}}\right]}{E\left[U1_{\{K>y\}}\right]} \tag{30}$$

we can obtain similar results as in the previous section. More precisely, if $h(y)$ is increasing in y, there exists $0 \leq y^* \leq +\infty$ which satisfies the optimality condition $h\left(y^*\right) = \hat{p}$ provided that $h\left(0\right) \leq \hat{p} \leq h\left(+\infty\right)$. This solution is unique if $h(y)$ is strictly increasing. Moreover, the objective function is concave increasing on $[0, y^*)$ and decreasing on $[y^*, +\infty)$. Therefore, it is quasi-concave and the solution y^* is indeed the optimal solution that maximizes (27). Finally, $y^* = 0$ if $h\left(0\right) \geq \hat{p}$ and $y^* = +\infty$ if $h\left(+\infty\right) \leq \hat{p}$.

It is clear that (10), (14), and (20) are all special cases of (29) obtained by setting $K = +\infty$ and/or $U = 1$. The optimality condition is expressed in terms of a function $h(y)$ which needs to be determined using the joint distribution of demand, yield, and capacity. In case this function is increasing, the first order condition $h\left(y^*\right) = \hat{p}$ is indeed sufficient for optimality through concavity or quasi-concavity. In all of the cases, it is clear that the optimality condition is stated in terms of a probabilities since we can easily write

$$h\left(y\right) = \frac{E\left[U1_{\{K>y,D\leq Uy\}}\right]}{E\left[U1_{\{K>y\}}\right]} = \frac{E\left[UP\left\{K > y, D \leq Uy|U\right\}\right]}{E\left[UP\left\{K > y|U\right\}\right]} \tag{31}$$

which is indeed a number between 0 and 1.

The optimality conditions for various cases are given explicitly by conditions (10), (14), (20), and (29). It is amazing that all of them involve probabilistic statements with the same critical ratio $\hat{p}$ that is determined by the economic parameters of the model. These conditions clearly require the joint distribution of demand and supply. Although the marginal distribution is sufficient to determine the optimal order quantity using (10) in the standard newsvendor model, we need the joint distributions in the others. This requires probabilistic models as well as statistical analysis of data. We shall not dwell with these issues in detail here since our objective is to obtain optimality conditions. In our models, we have at most 3 random variables and this does not really impose a formidable task. One approach may be to determine the marginal distribution of one of the variables and the conditionals of the others. As a matter of fact, the illustration in Section 4 is an example along this direction which leads to the explicit condition (23). In the random capacity model, the marginal distribution of the capacity and the conditional distribution of the demand given the capacity can be put together to obtain the optimal solution. Of course, this may require the use of numerical methods since it may not be possible to find an exact explicit solution. The numerical illustration in Section 3 also demonstrates how the optimal order quantity can be determined given conditional probabilistic information on demand and yield. Note that our discussion and review on the random environment models in Section 1 also suggest a tractable procedure to obtain the joint distribution. Suppose that the are m possible environmental states and the state during the period will be i with some probability p_i. Moreover, demand D, yield U, and capacity K are conditionally independent with marginal distributions F_D^i, F_U^i, and F_K^i given that environment is i. Then, we can obtain the joint distribution

$$P\{D \leq x, U \leq u, K \leq z\} = \sum_{i=1}^{m} p_i F_D^i(x) F_U^i(u) F_K^i(z) \tag{32}$$

using the marginal distributions.

It is indeed very difficult to claim a specific shape for the distribution (32) as well as the conditional distributions of random demand and supply. It clearly depends on how D, U, and K vary in a given environment. It may be such that D and K are positively correlated because producers increase capacity when demand is high. But, it may very well be the case that the opposite is true if producers do not increase capacity and they ration production capacity between multiple customers. Similarly, one may have negative correlation between D and U since a surge in demand may force excessive production leading to higher defectives. As a matter of fact, the numerical illustration in Section 3 demonstrates such a case. From a mathematical point of view, one may directly use distributions conditional on the value of one of the random variables. For example, the conditional exponential distribution (22) explains such a relationship where the shape of the parameter $\lambda(x)$ identifies the relationship between demand and capacity. Although there is sufficient justification for the dependence of supply and demand, we did not conduct a statistical analysis to determine joint or conditional distributions based on observed data. However, it is clear that one may not have a specific class of distributions that applies to all plausible scenarios.

There are many cases where a single period model is applicable when there is discontinuous seasonal demand and supply for a given duration of time. Moreover, the single period analysis extends to infinite horizon analysis through renewal theory if there are identical periodic conditions and inventory in a given period is used only to satisfy the demand of that period. The reader is referred to Silver et al. [**?**] for examples and applications of single period models. Our discussions and results provide means of determining the optimal order quantity. Furthermore, the analysis of general inventory models in multiple and infinite periods starts with a complete investigation of the single period case. Here, inventory left at the end of a period is carried over to the next to satisfy future demands. Therefore, interest in single period models is also due to its implications and generalizations in multiple periods. This is often accomplished by using dynamic programming through recursive arguments that involve convexity of the objective function and base-stock or (s, S) structures of the optimal ordering policy. The extension to the multiple period is easily made when only demand is random and the optimality of a base-stock policy is well-known. However, this is not necessarily true when supply is also random with an arbitrary dependence structure. Our preliminary investigations show that if there is some initial inventory x at the beginning of the period, then the optimal order-up-to level is not constant and it depends on x. This rules out the optimality of base-stock policies when there is no structured dependence. In the random environment model of Erdem and Özekici [5] with random demand and capacity, an environment-dependent base-stock policy is optimal due to conditional independence of demand and capacity given the state of the environment.

6 Concluding Remarks

In this paper, we considered several variations of the standard newsvendor model that incorporates supply randomness. We analyzed three different types of supply uncertainty: random yield, random capacity, and both. Although the literature includes a variety of such models, to our knowledge, we present the first example where random demand, capacity, and yield are directly dependent. In all cases, we found characterizations of the optimal order quantity through the same critical ratio. These characterizations are based on certain properties and assumptions on the optimality conditions and the structure of the objective function. For the random yield case, the objective function is concave, and we find simple and explicit characterizations for the optimal ordering quantity. Models with random capacity have non-concave objective functions, but we are able to establish quasi-concavity of the objective function and this leads to an explicit characterization of the optimal order quantity through the same critical ratio. However, the existence and uniqueness of the solution require certain assumptions. The most general case that involves both random capacity and yield also leads to non-concave objective functions. Therefore, additional conditions are required to obtain an explicit optimality condition and the existence as well as uniqueness of its solution. A number of special cases are also discussed.

This line of research can be extended in several directions by future research. One of them involves multi-period and infinite-period models. It is indeed a very challenging problem to identify structural properties, if any, of the optimal policy. With an arbitrary and general structure of dependence, the optimal policy does not necessarily have a base-stock structure. The determination of the dependence structure obviously requires statistical analysis based on observed data on demand and supply. We hope future work on statistical issues will be conducted to address this important and interesting issue. Another direction for research involves hedging risks associated with random demand and supply by some financial instruments. It has been shown by Gaur and Seshadri [7] that random demand may be highly correlated by a financial index or variable for which there are spot and derivatives markets. This correlation is exploited to manage the risks associated with uncertainty in demand by holding a portfolio of financial instruments in the market. It makes perfect sense to manage the risks associated with supply uncertainty as well using a similar approach.

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