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Strategic Customers in a Transportation Station: When Is It Optimal to Wait?

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We consider a transportation station, where customers arrive according to a Poisson process. A transportation facility visits the station according to a renewal process and serves at each visit a random number of customers according to its capacity. We assume that the arriving customers decide whether to join the station or balk, based on a natural reward-cost structure. We study the strategic behavior of the customers and determine their symmetric Nash equilibrium strategies under two levels of information.

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1. Introduction

The objective of the present paper is to study the strategic behavior of customers that arrive at a certain transportation station and face the dilemma of whether to wait for the next transportation facility or balk. An arriving customer that encounters a small number of waiting customers in the station anticipates a large waiting time for the next transportation facility. Indeed, the presence of few customers gives a signal that the previous visit of the facility occurred recently and therefore, under some natural conditions, the time till the next visit may be long. On the other hand, the presence of only a few customers implies that the next transportation facility will have enough space to accommodate him with high probability (assuming that the transportation facility accepts the customers on a first-come, first-served basis till its capacity is exhausted). On the contrary, an arriving customer that finds a lot of customers in the system expects the time till the next arrival of the facility to be relatively short, but he assumes a high risk of not being served by it. Therefore, it is not clear which decision is preferable for an arriving customer, to wait or balk, given the number of customers that he finds in the system. As the customers want to maximize their individual benefits, taking into account that the other customers have the same objective, we can consider this situation as a game among them.

The study of service (queueing) systems under a game-theoretic perspective was initiated by Naor (1969) who studied the strategic behavior of customers in the basic $M/M/1$ queue with a linear reward-cost structure. More specifically, Naor (1969) assumed that an arriving customer observes the number of customers and then makes the decision whether to join or balk (observable case). Subsequently, Edelson and Hildebrand (1975) complemented this study by considering the same queueing system, but assuming that the customers make their decisions without being informed about the state of the system. Since then, there is a growing number of papers that deal with the strategic behavior of customers in variants of the $M/M/1$ queue.

One important family of papers that deals with strategic customer behavior in $M/M/1$-type queues is the literature on vacation queueing systems and strategic customers; see, e.g., Burnetas and Economou (2007) (for a passive server) or Guo and Hassin (2011) and Guo and Zhang (2013) (for proactive servers). There are also diverse studies that analyze the effect of the information on the strategic behavior of the customers (see, e.g., Whitt 1999, Guo and Zipkin 2007, Hassin 2007, Armony et al. 2009, and Guo and Hassin 2011). On the other hand, results on the strategic customer behavior in queueing systems with general service times are very scarce (see, e.g., Altman and Hassin 2002, Economou et al. 2011, and Kerner 2011). The books of Hassin and Haviv (2003) and Stidham (2009) present the main approaches and several results in this area of the economic analysis of queueing systems.

Most of the queueing models that have been studied under this game-theoretic point of view assume that the facility serves the customers one by one. However, the situation of a transportation facility that visits a certain station periodically is represented by stochastic models with batch services and frequently by stochastic clearing systems. Indeed, it is reasonable to assume (at least as a first approximation) that the station is left empty after each visit of the facility, since the waiting customers are normally unwilling to wait for its next visit. Stochastic clearing systems have been studied extensively in the literature. More specifically, in the majority of the corresponding studies, the interest of the researchers lies in the performance descriptors of the underlying processes. Moreover, the dynamic control of such systems by a central decision maker has also been studied. However, the only work that we are aware of on the strategic behavior of customers in such systems is the recent paper by Economou and Manou (2012), which considers a different model under more restrictive assumptions with respect to this paper.

In the present paper, we study the customer strategic behavior at a clearing system that models a transportation station. The model is substantially different from existing work in two ways. First, we assume that the transportation facility has varying capacity in its successive visits, that is modeled by a sequence of independent identically distributed random variables. This is more realistic, as the capacity of the facility upon arrival to the station of interest is the remaining space that is conditioned by its visits to other stations of its itinerary. Second, we assume generally distributed intervisit times, a fact that allows representing more practical situations. We have to stress here that the majority of studies of a game-theoretic nature in queueing assume a Markovian framework. From a technical point of view, the departure from the Markovian assumption in the present paper makes the analysis nontrivial and more interesting. Furthermore, the insights are considerably richer with respect to the Markovian case. Indeed, it turns out that, avoid-the-crowd (ATC), follow-the-crowd (FTC), or mixed strategic behavior of the customers can be observed, according to the type of the intervisit time and the capacity distributions. On the contrary, in the Markovian framework, the analysis of the observable case is trivial: The expected net benefit function of a tagged customer does not depend on the strategy of the other customers and there exist dominant strategies.

The main contributions of the paper can be summarized as follows:

- The analysis of strategic customer behavior in observable queueing systems with general service times is a quite new endeavor. Indeed, the only papers we are aware of that deal with this problem is Altman and Hassin (2002) and Kerner (2011). As for clearing systems, to our knowledge, this paper is the first to consider general intervisit times. The basic difficulty in these type of problems lies in the computation of the conditional distributions of the residual service time at an arrival instant of a tagged customer, given the various possible states of the system and a strategy followed by the other customers. Kerner (2008) provided an analytic approach to compute these distributions, which relies on the computation of the joint stationary mass-density function of the queue length and the remaining service time using the method of supplementary variables.

In the present paper, we present a novel alternative approach that establishes recursive equations for the sojourn times of customers that find n and $n-1$ customers in the system in terms of random variables. This approach is probabilistic and relies on sample-path arguments that compare the sojourn times of a customer that finds n customers in the system with a customer that finds $n-1$ customers. This new approach seems to have three advantages: First, it is direct and economical, since it does not require the computation of the joint stationary mass-density function of the queue length and the remaining service time. Second, it does not assume the existence of a density for the service times, so it can also deal with discrete/mixed distributions for the service times. Third, it enables one to use closure (preservation) properties of stochastic orders to formally establish monotonicity properties of the best responses and uniqueness of equilibrium strategies for some classes of service time distributions.

- On the modeling side, we consider a fairly general model of a transportation facility with non-Markovian intervisit times and random capacity and demonstrate that this model can be analyzed in the strategic customers case. This model corresponds to a clearing queueing system with general service times, in which all customers are served simultaneously. Up to now, the analysis of strategic customer behavior in observable queueing systems with general service times has been carried out in the framework of the $M/G/1$ queueing system, which is substantially different, since the customers are served there sequentially (one by one).
- On the technical side, we have also provided a short and direct derivation of the stationary queue length distribution under any joining strategy of the customers, using Little's law and the Poisson arrivals see time averages (PASTA) property.
- In the Markovian case, both the equilibrium and socially optimal strategies have been identified. Moreover, it has been proven that the equilibrium and socially optimal strategies coincide in the observable case. In the literature, the coincidence of equilibrium and socially optimal strategies is encountered in situations where the customers do not impose externalities to other customers. The present paper provides a rare example, where the customers impose negative externalities to the other customers, but nevertheless the equilibrium and socially optimal strategies do coincide. This is because the negative externalities are shown to be always smaller than the positive value for the tagged customer.
- Using the recursive relationships for the random variables that represent sojourn times of customers that find n and $n-1$ customers in the system and closure properties of the classes of decreasing mean residual life (DMRL) and increasing mean residual life (IMRL) distributions, we establish structural properties and uniqueness of the equilibrium strategies under natural conditions on the service time (intervisit time) distributions. This also enables conclusion on the ATC or FTC joining behavior of the customers.

The paper is organized as follows. In §2, we describe the dynamics of the model, the reward-cost structure, and the information-decision framework. In §3, we deal with the performance analysis of the observable case, using probabilistic arguments. Subsequently, in §4, we use the results of this analysis and proceed to determine the equilibrium joining strategies in the observable case. The analysis of the unobservable case is carried out in §5. Some special cases, where the analysis can advance further, are provided in §§6 and 7. Finally, we conclude with §§8 and 9, where we provide the findings of several numerical experiments, discuss the theoretical results, and point to some remaining open issues. There is also an online appendix (available as supplemental material at http://dx.doi.org/10.1287/opre.2014.1280) that includes several technical proofs, including some alternative analytic proofs for various results inspired by the methodology developed by Kerner (2011).

2. The Model

We consider a transportation station with infinite waiting space, where potential customers (passengers) arrive according to a Poisson process $\{P(t)\}$ at rate λ. Let $I_1, I_2, \ldots$ denote the successive customer interarrival times. A transportation facility visits the station according to a renewal process $\{M(t)\}$. The times $X_1, X_2, \ldots$ between the successive visits of the facility have an absolutely continuous distribution with finite moments, distribution function $F(x)$, probability density function $f(x)$, and Laplace-Stieltjes transform (LST) $\tilde{F}(s) = \int_0^\infty e^{-sx}\, dF(x)$. Moreover, we assume that the successive capacities $C_1, C_2, \ldots$ of the facility at the moments of its visits to the station are discrete independent identically distributed random variables with finite moments, probability mass function (g_k: $k = 1, 2, \ldots$), and probability generating function (PGF) $G(z)$. When a transportation facility with capacity k visits the station, it serves at most k customers instantaneously and the waiting customers that cannot be served, abandon the system. In other words, the facility serves all present customers, if their number does not exceed its capacity. Otherwise, it serves as many customers as its capacity. In any case, the station is left empty after the departure of the facility. Finally, we assume that all interarrival times of the customers, the intervisit times, and the successive capacities of the facility are mutually independent.

The state of the station at a given time t can be represented by a pair $(N(t), R(t))$, where $N(t)$ records the number of customers at the station and $R(t)$ denotes the residual service time (i.e., the time till the next visit of the facility). The stochastic process $\{(N(t), R(t))\colon t \geqslant 0\}$ is a continuous time Markov process with state space $S = \{(n, r)\colon n \in \mathbb{N},\ r \in [0, +\infty)\}$.

We are interested in the behavior of the customers, when they have the option to decide whether to join or balk. We assume that a customer receives a reward of R units, if he gets the service (i.e., if he joins the system and the next arriving facility accomodates him). Moreover, a customer accumulates costs at a rate of K units per time unit that he remains in the system. We also assume that customers are risk neutral and wish to maximize their net benefit. Finally, their decisions are assumed irrevocable, in the sense that neither reneging of entering customers nor retrials of balking customers are allowed.

Since all customers are assumed indistinguishable, we can consider the situation as a symmetric game among them. Denote the common set of strategies (set of available actions) and the utility (payoff) function by $\mathcal{S}$ and U, respectively. More concretely, let $U(s_{\text{tagged}}, s_{\text{others}})$ be the utility for a tagged customer who follows strategy s_{tagged}, when all other customers follow s_{others}. A strategy s_1 is said to dominate strategy s_2 if $U(s_1, s) \geqslant U(s_2, s)$, for every $s \in \mathcal{S}$ and the inequality is strict for at least one s. A strategy s_* is said to be dominant if it dominates all other strategies in $\mathcal{S}$. A strategy $\tilde{s}$ is said to be a best response against a strategy s_{others}, if $U(\tilde{s}, s_{\text{others}}) \geqslant U(s, s_{\text{others}})$, for every $s \in \mathcal{S}$. Finally, a strategy s_e is said to be a (symmetric) Nash equilibrium, if it is a best response against itself, i.e., $U(s_e, s_e) \geqslant U(s, s_e)$, for every $s \in \mathcal{S}$. The intuitive interpretation of a Nash equilibrium is that it is a stable point of the game in the sense that if all customers follow it, then no one has an incentive to deviate from it. We remark that the notion of a dominant strategy is stronger than the notion of a Nash equilibrium strategy. In fact, every dominant strategy is a Nash equilibrium strategy, but the converse is not true. Moreover, Nash equilibrium strategies do exist in most situations, whereas dominant strategies rarely do.

In the next sections we will determine the customer equilibrium joining strategies. We assume that the customers do not have information on $R(t)$. However, the information about $N(t)$ may be available, so we distinguish two cases depending on the information that the customers receive at their arrival instants, before the decisions are made:

- Observable case: Customers observe $N(t)$.
- Unobservable case: Customers do not observe $N(t)$.

Note that in the observable case, an arriving customer bases his join/balk decision on the number of waiting customers $N(t)$, which serves as a signal for the time elapsed since the last service instant (similar to Whitt 1986, Altman and Hassin 2002, Haviv and Kerner 2007, and Kerner 2011). We study the observable case in §§3 and 4, whereas §5 is dedicated to the unobservable case.

3. The Observable Case: Performance Analysis

In this section we consider the observable case of our model. In this case, the customers, upon arrival and before making their decisions about whether to join or balk, observe the number of present customers in the system. Thus, a general joining strategy is specified by a vector of joining probabilities $\mathbf{q} = (q_0, q_1, q_2, \ldots)$, where q_i denotes the joining probability when a customer finds i customers in the system upon arrival (excluding himself). Moreover, we denote by

$\mathbf{q}_n$ the vector $(q_0, q_1, q_2, \ldots, q_n)$, which includes the $n+1$ initial components of $\mathbf{q}$. Thus, the vector $\mathbf{q}_n$ describes a customer's strategic behavior when he sees up to n present customers.

The first step in the search for the equilibrium joining strategies of the customers is the study of the best response of a tagged customer against a given strategy of the others. However, for determining a customer's best response against a strategy $\mathbf{q}$ followed by the other customers, it is first necessary to compute the conditional mean sojourn time of the customer, given that he finds n present customers in the system, for all possible values of n. Of course, such a conditional mean sojourn time depends both on the number of customers n and the strategy $\mathbf{q}$. The study of the social benefit function per time unit, under a given strategy $\mathbf{q}$ of the customers requires also to compute the equilibrium distribution of the number of customers in the system at arbitrary instants, given that the customers follow the strategy $\mathbf{q}$.

In this section, we compute the conditional mean sojourn times of a customer and the equilibrium distribution of the number of customers in the system, using a probabilistic approach. The results can be alternatively derived using the analytic approach introduced by Kerner (2008), who determined the conditional distributions of the residual service time in an $M_n/G/1$ queue. As the latter approach is completely different and has independent interest, we provide the detailed derivations of the results using it in the online appendix.

Let $R_{\mathbf{q}}(t)$ be the residual service time at time t, when the customers follow a strategy $\mathbf{q}$ and denote by $R_{\mathbf{q}}$ its equilibrium (stationary, limiting) version. Moreover, let $R_{n,\mathbf{q}}$ represent the equilibrium conditional residual service time (at arbitrary instants), given that there are n present customers in the system, when the customers follow the strategy $\mathbf{q}$. We also consider the equilibrium conditional residual service time at arrivals that find n customers in the system and at arrivals that find n customers in the system and decide to join. We denote them by $R^a_{n,\mathbf{q}}$ and $R^j_{n,\mathbf{q}}$, respectively. Similarly, let $N_q(t)$ denote the number of customers in the system at time t, given that the customers follow a strategy $\mathbf{q}$ and denote by $N_{\mathbf{q}}$, $N^a_{\mathbf{q}}$ and $N^j_{\mathbf{q}}$, respectively, the equilibrium versions at arbitrary instants, at arrivals, and at arrivals that decide to join.

Suppose, now, that the customers follow a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$ and denote by $\bar{n}(\mathbf{q})$ the first index for which q_n becomes 0, i.e.,

$$\bar{n}(\mathbf{q}) = \inf\{n \geqslant 0\colon\ q_i > 0 \text{ for } i < n \text{ and } q_n = 0\}. \tag{1}$$

Then, $R^j_{\bar{n}(\mathbf{q}),\mathbf{q}}$ is not defined, since, whenever $N_{\mathbf{q}} = \bar{n}(\mathbf{q})$, there is no stream of arrivals who join the system. On the other hand, $R_{\bar{n}(\mathbf{q}),\mathbf{q}}$ and $R^a_{\bar{n}(\mathbf{q}),\mathbf{q}}$ are defined, since it is possible to observe $\bar{n}(\mathbf{q})$ customers at an arbitrary or an arrival instant, under strategy $\mathbf{q}$. Note, also, that for $n > \bar{n}(\mathbf{q})$, $R^j_{n,\mathbf{q}}$, $R^a_{n,\mathbf{q}}$, and $R_{n,\mathbf{q}}$ are not defined, since the system can never have more than $\bar{n}(\mathbf{q})$ customers. We can now easily argue that $R^j_{n,\mathbf{q}}$, $R^a_{n,\mathbf{q}}$, and $R_{n,\mathbf{q}}$ are identically distributed, whenever they are defined, i.e.,

$$R_{n,\mathbf{q}} \stackrel{\mathrm{d}}{=} R^a_{n,\mathbf{q}}, \quad 0 \leqslant n \leqslant \bar{n}(\mathbf{q}), \tag{2}$$

$$R_{n,\mathbf{q}} \stackrel{\mathrm{d}}{=} R^j_{n,\mathbf{q}}, \quad 0 \leqslant n < \bar{n}(\mathbf{q}). \tag{3}$$

Indeed, if $\{P(t),\ t \geqslant 0\}$ denotes the Poisson process, at rate λ, that generates the customer arrivals in the system, we have that $\{P(t+u) - P(t),\ u \geqslant 0\}$ and $\{(N_{\mathbf{q}}(u), R_{\mathbf{q}}(u)),\ 0 \leqslant u \leqslant t\}$ are independent. So, the lack of anticipation assumption is satisfied and the PASTA property is applicable (see, e.g., Tijms 1994, Section 1.7) and we deduce (2). For justifying (3), let $\{P_n(t),\ t \geqslant 0\}$, be independent Poisson processes with respective rates λq_n, $n = 0, 1, \ldots, \bar{n}(\mathbf{q}) - 1$. We can think of $\{P_n(t),\ t \geqslant 0\}$ as the process that generates the arrivals of customers who join the transportation station, whenever $N_{\mathbf{q}}(t) = n$. We have that $\{P_n(t+u) - P_n(t),\ u \geqslant 0\}$ and $\{(N_{\mathbf{q}}(u), R_{\mathbf{q}}(u)),\ 0 \leqslant u \leqslant t\}$ are independent for $n = 0, 1, \ldots, \bar{n}(\mathbf{q}) - 1$. Again the lack of anticipation assumption is valid and we obtain (3), using the conditional PASTA property of van Doorn and Regterschot (1988) (see also the same type of argument in the proof of Theorem 2.2.2 in Kerner 2008).

Since, $R_{n,\mathbf{q}}$, $R^a_{n,\mathbf{q}}$ and $R^j_{n,\mathbf{q}}$ are equidistributed, whenever they are defined, we can determine their common distribution by studying any one of them. We will refer to this distribution as the (equilibrium) conditional residual service time distribution, given that there are n customers in the system.

Proposition 3.1. *Consider the observable model of a transportation station, where the customers join the system according to a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$. Then, we have the following cases for a distributional representation of $R_{n,\mathbf{q}}$ in terms of $R_{n-1,\mathbf{q}}$ (for $n = 0$ we have a representation of $R_{n,\mathbf{q}}$ in terms of a service time random variable X, as $R_{n-1,\mathbf{q}}$ is not defined).*

Case 1. $0 = n = \bar{n}(\mathbf{q})$. *Let $R(X)$ be a random variable having the equilibrium residual renewal time distribution of a renewal process with interrenewal times distributed as X, a generic intervisit time of the transportation facility. Then, the distribution of the conditional residual service time $R_{n,\mathbf{q}} = R_{0,\mathbf{q}}$ coincides with the distribution of $R(X)$. Symbolically, we have*

$$R_{n,\mathbf{q}} = R_{0,\mathbf{q}} \stackrel{d}{=} R(X), \quad 0 = n = \bar{n}(\mathbf{q}). \tag{4}$$

Case 2. $0 = n < \bar{n}(\mathbf{q})$. *Let X and $T_{\lambda q_0}$ be independent random variables, where X has the distribution of the intervisit times of the transportation facility and $T_{\lambda q_0}$ has an exponential distribution with parameter λq_0. Then, the distribution of the conditional residual service time $R_{n,\mathbf{q}} = R_{0,\mathbf{q}}$ coincides with the conditional distribution of the difference $X - T_{\lambda q_0}$, given that $X \geqslant T_{\lambda q_0}$. Symbolically, we have*

$$R_{n,\mathbf{q}} = R_{0,\mathbf{q}} \stackrel{d}{=} (X - T_{\lambda q_0} \mid X \geqslant T_{\lambda q_0}), \quad 0 = n < \bar{n}(\mathbf{q}). \tag{5}$$

Case 3. $1 \leqslant n = \bar{n}(\mathbf{q})$. *Let $R(R_{n-1,\mathbf{q}})$ be a random variable having the equilibrium residual renewal time distribution of a renewal process with interrenewal times distributed as $R_{n-1,\mathbf{q}}$. Then, the distribution of the conditional residual service time $R_{n,\mathbf{q}}$ coincides with the distribution of $R(R_{n-1,\mathbf{q}})$. Symbolically, we have*

$$R_{n,\mathbf{q}} \stackrel{d}{=} R(R_{n-1,\mathbf{q}}), \quad 1 \leqslant n = \bar{n}(\mathbf{q}). \tag{6}$$

Case 4. $1 \leqslant n < \bar{n}(\mathbf{q})$. *Let $R_{n-1,\mathbf{q}}$ and $T_{\lambda q_n}$ be independent random variables, where $R_{n-1,\mathbf{q}}$ has the conditional residual service time distribution given that there are $n-1$ customers in the system and $T_{\lambda q_n}$ has an exponential distribution with parameter λq_n. Then, the distribution of the conditional residual service time $R_{n,\mathbf{q}}$ coincides with the conditional distribution of the difference $R_{n-1,\mathbf{q}} - T_{\lambda q_n}$, given that $R_{n-1,\mathbf{q}} \geqslant T_{\lambda q_n}$. Symbolically, we have*

$$R_{n,\mathbf{q}} \stackrel{d}{=} (R_{n-1,\mathbf{q}} - T_{\lambda q_n} \mid R_{n-1,\mathbf{q}} \geqslant T_{\lambda q_n}), \quad 1 \leqslant n < \bar{n}(\mathbf{q}). \tag{7}$$

Proof. *Case* 1. Suppose that the customers follow a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$ with $\bar{n}(\mathbf{q}) = 0$. Then no customer joins the station. Consider now a tagged customer that arrives to the system. This customer will necessarily find $n = 0$ customers in it and, because of the PASTA property, his residual service time coincides with the residual renewal time at an arbitrary epoch of the renewal process that generates the visits of the facility to the station. Therefore, we obtain (4).

Case 2. Suppose that the customers follow a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$ with $\bar{n}(\mathbf{q}) \geqslant 1$. We consider the transportation station just after a visit of the facility. Then, the station is empty and the time till the arrival of the first customer who decides to join is exponentially distributed with parameter λq_0. Denote this time by $T_{\lambda q_0}$. Then, the residual service time of that customer will be the current intervisit time X of the transportation facility minus $T_{\lambda q_0}$, given that the intervisit time X exceeds $T_{\lambda q_0}$ (so that such a customer exists). Therefore, we obtain (5).

Case 3. Suppose that the customers follow a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$ with $\bar{n}(\mathbf{q}) \geqslant 1$. Then, the arrivals of customers that find in the system $n = \bar{n}(\mathbf{q})$ customers and so do not join (since $q_n = 0$ by the definition of $\bar{n}(\mathbf{q})$) occur only during the residual service times of customers that find $n-1$ customers in the system and decide to join. By "gluing" all time intervals that correspond to residual service times of customers that find $n-1$ customers in the system and decide to join, we construct a renewal process $\{\hat{M}(t)\}$. Moreover, we observe that the arrivals of customers that find in the system n customers occur only during the time intervals of $\{\hat{M}(t)\}$ and constitute a Poisson process. Then, because of the PASTA property, the residual service time of a customer that finds n customers upon arrival coincides with the residual renewal time at an arbitrary epoch of the renewal process $\{\hat{M}(t)\}$. But the process $\{\hat{M}(t)\}$ has interrenewal times distributed as $R_{n-1,\mathbf{q}}$. Therefore, we obtain (6).

Case 4. Suppose that the customers follow a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$ with $\bar{n}(\mathbf{q}) \geqslant 2$ and consider a tagged customer that finds n present customers at his arrival instant, with $1 \leqslant n < \bar{n}(\mathbf{q})$ and decides to join the system. Then, his residual service time $R_{n,\mathbf{q}}$ equals the residual service time of the customer who joined the system just before him minus the time between their arrivals, given that the facility has not visited the station during this interarrival time. Note that the customer who joined the system just before the tagged customer found $n-1$ customers upon arrival and that the interarrival time between his arrival and the arrival of the tagged customer is an exponentially distributed random variable $T_{\lambda q_n}$ with parameter λq_n, independent of $R_{n-1,\mathbf{q}}$. Therefore, we obtain (7). □

To translate the recursive scheme (4)–(7) for the random variables $R_{n,\mathbf{q}}$ in a scheme for the corresponding LSTs we will use the following Lemma 3.1.

Lemma 3.1. *Let T_1, T_2, and Y be independent random variables, with T_1 and T_2 being exponentially distributed with parameters λ_1 and λ_2, respectively, and Y being a nonnegative generally distributed random variable with LST $\tilde{F}_Y(s)$. Then we have the following formulas*:

$$\Pr[Y \leqslant T_1] = \tilde{F}_Y(\lambda_1), \tag{8}$$

$$\Pr[Y \leqslant T_1 + T_2] = \frac{\lambda_2}{\lambda_2 - \lambda_1}\tilde{F}_Y(\lambda_1) + \frac{\lambda_1}{\lambda_1 - \lambda_2}\tilde{F}_Y(\lambda_2), \quad \lambda_1 \neq \lambda_2, \tag{9}$$

$$\Pr[Y \leqslant T_1 + T_2] = \tilde{F}_Y(\lambda_1) - \lambda_1 \tilde{F}'_Y(\lambda_1), \quad \lambda_1 = \lambda_2. \tag{10}$$

Proof. Let $F_Y(y)$ be the distribution function of Y. Considering the left side of (8) and conditioning on Y, we obtain

$$\Pr[Y \leqslant T_1] = \int_0^\infty \Pr[T_1 \geqslant y]\, dF_Y(y) = \int_0^\infty e^{-\lambda_1 y}\, dF_Y(y) = \tilde{F}_Y(\lambda_1).$$

Equations (9) and (10) are proved similarly by using the formulas

$$\Pr[T_1 + T_2 \geqslant y] = \frac{\lambda_2}{\lambda_2 - \lambda_1}e^{-\lambda_1 y} + \frac{\lambda_1}{\lambda_1 - \lambda_2}e^{-\lambda_2 y}, \quad y \geqslant 0,\ \lambda_1 \neq \lambda_2,$$

$$\Pr[T_1 + T_2 \geqslant y] = e^{-\lambda_1 y} + \lambda_1 y e^{-\lambda_1 y}, \quad y \geqslant 0,\ \lambda_1 = \lambda_2,$$

respectively. □

In Proposition 3.2 we provide the recursive scheme for the LSTs of the conditional residual service times.

Proposition 3.2. *Consider the observable model of a transportation station, where the customers join the system according to a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$. Then, the LSTs*

$\tilde{F}_{n,\mathbf{q}}(s) = E[e^{-sR_{n,\mathbf{q}}}]$ *of the conditional residual service times are given by the recursive scheme*

$$\tilde{F}_{n,\mathbf{q}}(s) = \frac{-(1-\tilde{F}_{n-1,\mathbf{q}}(s))}{s\tilde{F}'_{n-1,\mathbf{q}}(0)}, \quad 1 \leqslant n = \bar{n}(\mathbf{q}), \tag{11}$$

$$\tilde{F}_{n,\mathbf{q}}(s) = \frac{\lambda q_n(\tilde{F}_{n-1,\mathbf{q}}(\lambda q_n) - \tilde{F}_{n-1,\mathbf{q}}(s))}{(s-\lambda q_n)(1-\tilde{F}_{n-1,\mathbf{q}}(\lambda q_n))}, \quad 1 \leqslant n < \bar{n}(\mathbf{q}),\ s \neq \lambda q_n, \tag{12}$$

$$\tilde{F}_{n,\mathbf{q}}(\lambda q_n) = \frac{-\lambda q_n \tilde{F}'_{n-1,\mathbf{q}}(\lambda q_n)}{1-\tilde{F}_{n-1,\mathbf{q}}(\lambda q_n)}, \quad 1 \leqslant n < \bar{n}(\mathbf{q}), \tag{13}$$

with initial conditions

$$\tilde{F}_{0,\mathbf{q}}(s) = \frac{-(1-\tilde{F}(s))}{s\tilde{F}'(0)}, \quad \bar{n}(\mathbf{q}) = 0, \tag{14}$$

$$\tilde{F}_{0,\mathbf{q}}(s) = \frac{\lambda q_0(\tilde{F}(\lambda q_0) - \tilde{F}(s))}{(s-\lambda q_0)(1-\tilde{F}(\lambda q_0))}, \quad \bar{n}(\mathbf{q}) > 0,\ s \neq \lambda q_0, \tag{15}$$

$$\tilde{F}_{0,\mathbf{q}}(\lambda q_0) = \frac{-\lambda q_0 \tilde{F}'(\lambda q_0)}{1-\tilde{F}(\lambda q_0)}, \quad \bar{n}(\mathbf{q}) > 0. \tag{16}$$

Proof. It is known that if the LST of the interrenewal time distribution of a renewal process is $\tilde{F}_Y(s)$, then the LST of the equilibrium residual renewal time distribution $\tilde{F}_{R(Y)}(s)$ is given by

$$\tilde{F}_{R(Y)}(s) = \frac{-(1-\tilde{F}_Y(s))}{s\tilde{F}'_Y(0)}. \tag{17}$$

Therefore, (4) and (6) imply immediately (14) and (11). Now, let T_s be an exponentially distributed random variable with parameter s. Then, using (8) and (7) we have

$$\begin{aligned}\tilde{F}_{n,\mathbf{q}}(s) &= \Pr[T_s \geqslant R_{n,\mathbf{q}}] \\ &= \Pr[T_s \geqslant R_{n-1,\mathbf{q}} - T_{\lambda q_n} \mid R_{n-1,\mathbf{q}} \geqslant T_{\lambda q_n}] \\ &= \frac{\Pr[R_{n-1,\mathbf{q}} \leqslant T_{\lambda q_n} + T_s] - \Pr[R_{n-1,\mathbf{q}} < T_{\lambda q_n}]}{\Pr[R_{n-1,\mathbf{q}} \geqslant T_{\lambda q_n}]}. \end{aligned} \tag{18}$$

If $s \neq \lambda q_n$, then we can use (8) and (9) in (18) and we obtain that

$$\tilde{F}_{n,\mathbf{q}}(s) = \left(\frac{\lambda q_n}{\lambda q_n - s}\tilde{F}_{n-1,\mathbf{q}}(s) + \frac{s}{s-\lambda q_n}\tilde{F}_{n-1,\mathbf{q}}(\lambda q_n) - \tilde{F}_{n-1,\mathbf{q}}(\lambda q_n)\right) \cdot (1-\tilde{F}_{n-1,\mathbf{q}}(\lambda q_n))^{-1},$$

which reduces, after some simplifications, to (12). On the other hand, if $s = \lambda q_n$, we can use (8) and (10) in (18) and we obtain that

$$\tilde{F}_{n,\mathbf{q}}(s) = \frac{\tilde{F}_{n-1,\mathbf{q}}(\lambda q_n) - \lambda q_n \tilde{F}'_{n-1,\mathbf{q}}(\lambda q_n) - \tilde{F}_{n-1,\mathbf{q}}(\lambda q_n)}{1-\tilde{F}_{n-1,\mathbf{q}}(\lambda q_n)},$$

which yields (13). The Equations (15) and (16) are proved similarly starting from (5) and using (8)–(10). □

Remark 3.1. Because of (14)–(16), we have that $\tilde{F}_{0,\mathbf{q}}(s)$ depends on $\mathbf{q}$ only through q_0. Therefore, we can write $\tilde{F}_{0,\mathbf{q}}(s) = \tilde{F}_{0,q_0}(s)$. Also, for $n \geqslant 1$, Equations (11)–(13) show that $\tilde{F}_{n,\mathbf{q}}(s)$ is a function of $\tilde{F}_{n-1,\mathbf{q}}(s)$ and q_n. Inductively, we deduce that $\tilde{F}_{n,\mathbf{q}}(s)$ depends on $\mathbf{q}$ only through $\mathbf{q}_n$ and we can write $\tilde{F}_{n,\mathbf{q}}(s) = \tilde{F}_{n,\mathbf{q}_n}(s)$. We will also write $R_{n,\mathbf{q}}$ as $R_{n,\mathbf{q}_n}$.

Note also that (11) can be seen as a limiting case of (12). Indeed, taking the limit in the right side of (12) as $q_n \to 0^+$ gives (11). Similarly, taking the limit in the right side of (15) as $q_0 \to 0^+$ yields (14).

Now, by differentiating (11)–(12) and (14)–(15) with respect to s and evaluating at $s = 0$, we obtain recursive formulas for the expected conditional residual service times. We state the result as Corollary 3.1.

Corollary 3.1. *Consider the observable model of a transportation station, where customers join the system according to a strategy* $\mathbf{q} = (q_0, q_1, q_2, \ldots)$. *For the expected conditional residual service times,* $E[R_{n,\mathbf{q}_n}]$, *we have the following recursive scheme*:

$$E[R_{n,\mathbf{q}_n}] = \frac{E[R^2_{n-1,\mathbf{q}_{n-1}}]}{2E[R_{n-1,\mathbf{q}_{n-1}}]}, \quad q_i \neq 0,\ 0 \leqslant i \leqslant n-1,\ q_n = 0,\ n \geqslant 1, \tag{19}$$

$$E[R_{n,\mathbf{q}_n}] = \frac{E[R_{n-1,\mathbf{q}_{n-1}}]}{1-\tilde{F}_{n-1,\mathbf{q}_{n-1}}(\lambda q_n)} - \frac{1}{\lambda q_n}, \quad q_i \neq 0,\ 0 \leqslant i \leqslant n,\ n \geqslant 1, \tag{20}$$

with initial conditions

$$E[R_{0,q_0}] = \frac{E[X^2]}{2E[X]}, \quad q_0 = 0, \tag{21}$$

$$E[R_{0,q_0}] = \frac{E[X]}{1-\tilde{F}(\lambda q_0)} - \frac{1}{\lambda q_0}, \quad q_0 \neq 0. \tag{22}$$

We will now determine the equilibrium distribution of the number of customers in the transportation station, when the customers follow a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$. We begin with Proposition 3.3, where we provide some recursive formulas for the equilibrium probabilities.

Proposition 3.3. *Consider the observable model of a transportation station, where the customers join the system according to a strategy* $\mathbf{q} = (q_0, q_1, q_2, \ldots)$. *Then, the equilibrium probabilities* $\pi_{n,\mathbf{q}} = \Pr[N_{\mathbf{q}} = n]$, $n \geqslant 0$, *for the number of customers in the station are given by the recursive scheme*

$$\pi_{n,\mathbf{q}} = \lambda q_{n-1} E[R_{n-1,\mathbf{q}_{n-1}}]\pi_{n-1,\mathbf{q}}, \quad q_i \neq 0,\ 0 \leqslant i \leqslant n-1,\ q_n = 0,\ n \geqslant 1, \tag{23}$$

$$\pi_{n,\mathbf{q}} = \frac{q_{n-1}(1-\tilde{F}_{n-1,\mathbf{q}_{n-1}}(\lambda q_n))}{q_n}\pi_{n-1,\mathbf{q}}, \quad q_i \neq 0,\ 0 \leqslant i \leqslant n,\ n \geqslant 1, \tag{24}$$

with initial conditions

$$\pi_{0,\mathbf{q}} = 1, \quad q_0 = 0, \tag{25}$$

$$\pi_{0,\mathbf{q}} = \frac{1-\tilde{F}(\lambda q_0)}{\lambda q_0 E[X]}, \quad q_0 \neq 0. \tag{26}$$

PROOF. The proof can be found in the online appendix. □

Using the recursive relations of Proposition 3.3, we obtain a product-form formula for the equilibrium distribution of the number of customers in the station. More specifically we have the following Proposition 3.4.

PROPOSITION 3.4. *Consider the observable model of a transportation station, where the customers join the system according to a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$. Then, the equilibrium probabilities $\pi_{n,\mathbf{q}}$ are given by the formulas*

$$\pi_{0,\mathbf{q}} = 1, \quad q_0 = 0, \tag{27}$$

$$\pi_{n,\mathbf{q}} = \frac{(1-\tilde{F}(\lambda q_0))E[R_{n-1,\mathbf{q}_{n-1}}]}{E[X]} \prod_{i=1}^{n-1}(1-\tilde{F}_{i-1,\mathbf{q}_{i-1}}(\lambda q_i)), \quad q_i \neq 0,\ 0 \leqslant i \leqslant n-1,\ q_n = 0, \tag{28}$$

$$\pi_{n,\mathbf{q}} = \frac{1-\tilde{F}(\lambda q_0)}{\lambda q_n E[X]} \prod_{i=1}^{n}(1-\tilde{F}_{i-1,\mathbf{q}_{i-1}}(\lambda q_i)), \quad q_i \neq 0,\ 0 \leqslant i \leqslant n, \tag{29}$$

$$\pi_{n,\mathbf{q}} = 0, \quad \text{if } q_i = 0 \text{ for some } i \leqslant n-1. \tag{30}$$

REMARK 3.2. In light of (28)–(30) it is obvious that $\pi_{n,\mathbf{q}}$ depends on $\mathbf{q}$ only through $\mathbf{q}_n$. So, we can write $\pi_{n,\mathbf{q}}$ as $\pi_{n,\mathbf{q}_n}$.

4. The Observable Case: Equilibrium Strategies

In this section we will determine the equilibrium joining strategies in the observable case. We have already mentioned that a strategy is said to be an equilibrium if it is a best response against itself. So, in order to find the best responses of a tagged customer against a strategy of the other customers we have to compute his expected net benefit given that the other customers follow a given strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$. Hence, we begin the study by determining the expected net benefit functions.

PROPOSITION 4.1. *Consider the observable model of a transportation station, where the customers join the system according to a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$. Then, the expected net benefit $S_n^{\text{obs}}(\mathbf{q})$ of an arriving customer, who finds n present customers in the system and decides to join, is given by the formulas*

$$S_0^{\text{obs}}(\mathbf{q}) = R - K\frac{E[X^2]}{2E[X]}, \quad q_0 = 0, \tag{31}$$

$$S_0^{\text{obs}}(\mathbf{q}) = R - K\left[\frac{E[X]}{1-\tilde{F}(\lambda q_0)} - \frac{1}{\lambda q_0}\right], \quad q_0 \neq 0, \tag{32}$$

$$S_n^{\text{obs}}(\mathbf{q}) = R\sum_{k=n+1}^{\infty} g_k - K\frac{E[R^2_{n-1,\mathbf{q}_{n-1}}]}{2E[R_{n-1,\mathbf{q}_{n-1}}]}, \quad q_i \neq 0,\ 0 \leqslant i \leqslant n-1,\ q_n = 0,\ n \geqslant 1, \tag{33}$$

$$S_n^{\text{obs}}(\mathbf{q}) = R\sum_{k=n+1}^{\infty} g_k - K\left[\frac{E[R_{n-1,\mathbf{q}_{n-1}}]}{1-\tilde{F}_{n-1,\mathbf{q}_{n-1}}(\lambda q_n)} - \frac{1}{\lambda q_n}\right], \quad q_i \neq 0,\ 0 \leqslant i \leqslant n,\ n \geqslant 1. \tag{34}$$

PROOF. We assume that the customers follow a strategy $\mathbf{q} = (q_0, q_1, q_2, \ldots)$ and we consider a tagged customer who finds n present customers at his arrival instant and decides to join. Then, his expected net benefit will be equal to the difference of his expected reward from service and his expected waiting cost. So, we have

$$S_n^{\text{obs}}(\mathbf{q}) = RP_n^{\text{obs}} - KE[R_{n,\mathbf{q}_n}], \tag{35}$$

where P_n^{obs} is the probability that the tagged customer receives service, given that there are n customers in front of him. We have clearly that

$$P_n^{\text{obs}} = \sum_{k=n+1}^{\infty} g_k, \tag{36}$$

since a tagged customer who occupies the $n+1$th waiting position of the station will be accommodated by the next transportation facility, only if its capacity is at least $n+1$. By plugging (36) and the formulas for $E[R_{n,\mathbf{q}_n}]$ of Corollary 3.1 in (35) we obtain immediately formulas (31)–(34). □

REMARK 4.1. It is obvious that $S_n^{\text{obs}}(\mathbf{q})$ depends on $\mathbf{q}$ only through $\mathbf{q}_n$. So, we can write $S_n^{\text{obs}}(\mathbf{q}) = S_n^{\text{obs}}(\mathbf{q}_n)$.

REMARK 4.2. Using Hospital's rule, we can prove that $\lim_{q_n\to 0} S_n^{\text{obs}}(q_0, q_1, q_2, \ldots, q_{n-1}, q_n) = S_n^{\text{obs}}(q_0, q_1, q_2, \ldots, q_{n-1}, 0)$ and that $\lim_{q_0\to 0} S_0^{\text{obs}}(q_0) = S_0^{\text{obs}}(0)$. We can then easily see that the functions $S_n^{\text{obs}}(\mathbf{q})$ are continuous.

We are now ready to determine the equilibrium joining strategies $\mathbf{q}^e = (q_0^e, q_1^e, q_2^e, \ldots)$ for the customers. More specifically, we will see that the equilibrium joining probabilities q_n^e can be computed recursively, using an idea inspired by Kerner (2011). In Theorem 4.1 we determine all possible equilibrium joining probabilities q_0^e . Subsequently, in Theorem 4.2, assuming that we have an equilibrium joining probability vector $\mathbf{q}_{n-1}^e$ at hand, for some $n \geqslant 1$, we determine all possible equilibrium joining probabilities q_n^e.

THEOREM 4.1. *Consider the observable model of a transportation station. Then, an equilibrium probability q_0^e for joining when finding the system empty exists. Specifically, we have the following comprehensive (but not necessarily mutually exclusive) cases:*

Case I. $R/K \leqslant E[X^2]/(2E[X])$. *Then, 0 is an equilibrium joining probability q_0^e.*

Case II. $E[X^2]/(2E[X]) < R/K < E[X]/(1-\tilde{F}(\lambda))-1/\lambda$. *Then, the equation* $E[X]/(1-\tilde{F}(\lambda x))-1/(\lambda x)=R/K$ *has a solution in* $(0,1)$. *Every such solution is an equilibrium joining probability* q_0^e.

Case III. $R/K \geqslant E[X]/(1-\tilde{F}(\lambda))-1/\lambda$. *Then,* 1 *is an equilibrium joining probability* q_0^e.

Proof. *Case* I. Assume that all customers balk and consider a tagged customer at his arrival instant. Then, his expected net benefit, if he decides to join, is $S_0^{\text{obs}}(0) = R-K(E[X^2]/(2E[X])) \leqslant 0$. So, a best response is to balk. Thus, 0 is an equilibrium joining probability q_0^e.

Case II. In this case, we have that $S_0^{\text{obs}}(0) > 0$ and $S_0^{\text{obs}}(1) < 0$. Since $S_0^{\text{obs}}(q_0)$ is continuous in q_0, using Bolzano's Theorem we have that there exists an x, such that $0<x<1$ and $S_0^{\text{obs}}(x)=0$. If the customers join the system with probability x, when it is empty, then the expected net benefit of a tagged customer, who finds the system empty at his arrival instant and decides to join, is $S_0^{\text{obs}}(x)=0$. So, he is indifferent between joining and balking. In particular, x is a best response. More generally, any solution $x\in(0,1)$ of the equation $S_0^{\text{obs}}(x)=0$ (which is written equivalently as $E[X]/(1-\tilde{F}(\lambda x))-1/(\lambda x)=R/K$) is an equilibrium joining probability q_0^e.

Case III. Now, consider a tagged customer at his arrival instant, who finds the system empty, and assume that all other customers join the system when they find it empty. Then, the expected net benefit of the tagged customer, if he decides to join is $S_0^{\text{obs}}(1)=R-K[E[X]/(1-\tilde{F}(\lambda))-1/\lambda] \geqslant 0$. In this case, joining the system is a best response for the tagged customer. Thus, 1 is an equilibrium joining probability q_0^e. □

Theorem 4.2. *Consider the observable model of a transportation station. Then, assuming that an equilibrium joining probability vector* $\mathbf{q}_{n-1}^e$ *is known, an equilibrium probability* q_n^e *for joining when finding n present customers in the system exists. Specifically, we have the following cases*:

Case I. $(R\sum_{k=n+1}^{\infty} g_k)/K \leqslant E[R^2_{n-1,\mathbf{q}_{n-1}^e}]/(2E[R_{n-1,\mathbf{q}_{n-1}^e}])$. *Then,* 0 *is an equilibrium joining probability* q_n^e.

Case II. $E[R^2_{n-1,\mathbf{q}_{n-1}^e}]/(2E[R_{n-1,\mathbf{q}_{n-1}^e}]) < (R\sum_{k=n+1}^{\infty} g_k)/K < E[R_{n-1,\mathbf{q}_{n-1}^e}]/(1-\tilde{F}_{n-1,\mathbf{q}_{n-1}^e}(\lambda))-1/\lambda$. *Then, the equation* $E[R_{n-1,\mathbf{q}_{n-1}^e}]/(1-\tilde{F}_{n-1,\mathbf{q}_{n-1}^e}(\lambda x))-1/(\lambda x) = (R\sum_{k=n+1}^{\infty} g_k)/K$ *has a solution in* $(0,1)$. *Every such solution is an equilibrium joining probability* q_n^e.

Case III. $(R\sum_{k=n+1}^{\infty} g_k)/K \geqslant E[R_{n-1,\mathbf{q}_{n-1}^e}]/(1-\tilde{F}_{n-1,\mathbf{q}_{n-1}^e}(\lambda)) -1/\lambda$. *Then,* 1 *is an equilibrium joining probability* q_n^e.

Proof. *Case* I. Assume that the customers follow a strategy $\mathbf{q}^e$ with initial part $(\mathbf{q}_{n-1}^e,0)=(q_0^e,q_1^e,q_2^e,\ldots,q_{n-1}^e,0)$ and consider a tagged customer, who finds n present customers at his arrival instant. Then, his expected net benefit, if he decides to join, is

$$S_n^{\text{obs}}(q_0^e,q_1^e,q_2^e,\ldots,q_{n-1}^e,0) = R\sum_{k=n+1}^{\infty} g_k - K\frac{E[R^2_{n-1,\mathbf{q}_{n-1}^e}]}{2E[R_{n-1,\mathbf{q}_{n-1}^e}]} \leqslant 0.$$

Then, a best response is to balk and 0 is an equilibrium joining probability q_n^e.

Case II. Now, we have that $S_n^{\text{obs}}(q_0^e,q_1^e,q_2^e,\ldots,q_{n-1}^e,0)>0$ and $S_n^{\text{obs}}(q_0^e,q_1^e,q_2^e,\ldots,q_{n-1}^e,1)<0$. Since $S_n^{\text{obs}}(q_0^e,q_1^e,q_2^e,\ldots,q_{n-1}^e,q_n)$ is continuous in q_n, we apply Bolzano's Theorem and we have that there exists an $x\in(0,1)$, such that $S_n^{\text{obs}}(q_0^e,q_1^e,q_2^e,\ldots,q_{n-1}^e,x)=0$. If the customers join the system with a strategy $\mathbf{q}^e$ with initial part $(\mathbf{q}_{n-1}^e,x)$, then the expected net benefit of a tagged customer, who finds n present customers at his arrival instant and decides to join, is $S_n^{\text{obs}}(q_0^e,q_1^e,q_2^e,\ldots,q_{n-1}^e,x)=0$. Therefore, the tagged customer is indifferent between joining and balking. In particular, x is a best response, i.e., it is an equilibrium joining probability q_n^e.

Case III. Consider a tagged customer at his arrival instant, who finds n present customers, and assume that the other customers join the system according to a strategy $\mathbf{q}^e$ with initial part $(\mathbf{q}_{n-1}^e,1)$. Then, the expected net benefit of the tagged customer, if he decides to join, is

$$S_n^{\text{obs}}(q_0^e,q_1^e,q_2^e,\ldots,q_{n-1}^e,1) = R\sum_{k=n+1}^{\infty} g_k - K\left[\frac{E[R_{n-1,\mathbf{q}_{n-1}^e}]}{1-\tilde{F}_{n-1,\mathbf{q}_{n-1}^e}(\lambda)}-\frac{1}{\lambda}\right] \geqslant 0.$$

In this case, joining the system is a best response for the tagged customer. Thus, 1 is an equilibrium joining probability q_n^e. □

We comment now on the associated social optimization problem.

Remark 4.3. In the observable model of a transportation station, where the customers join the system according to a strategy $\mathbf{q}=(q_0,q_1,\ldots)$, the expected social benefit per time unit is given by

$$S_{\text{soc}}^{\text{obs}}(\mathbf{q}) = \lambda\sum_{n=0}^{\infty}\pi_{n,\mathbf{q}} q_n S_n^{\text{obs}}(\mathbf{q}) \tag{37}$$

$$= R\lambda\sum_{n=0}^{\infty}\pi_{n,\mathbf{q}} q_n \sum_{k=n+1}^{\infty} g_k - KE[N_{\mathbf{q}}]. \tag{38}$$

The complexity of the terms $\pi_{n,\mathbf{q}}$ and $E[N_{\mathbf{q}}]=\sum_{n=0}^{\infty} n\pi_{n,\mathbf{q}}$ does not allow the analytic determination of a strategy $\mathbf{q}$ that maximizes $S_{\text{obs}}^{\text{soc}}(\mathbf{q})$. However, in §6, we determine such a socially optimal strategy in the case where the times between successive visits of the transportation facility follow the exponential distribution.

5. The Unobservable Case

In this section we consider the unobservable model of a transportation station. First, we compute the expected net benefit of a customer, if he decides to join, and then we determine the equilibrium joining strategies. A general joining strategy in this case is specified by a single joining probability q. In Proposition 5.1 we provide the expected net benefit of a customer that decides to join.

PROPOSITION 5.1. *Consider the unobservable model of a transportation station, where the customers join the system according to a strategy q. Then, the expected net benefit $S^{un}(q)$ of an arriving customer who decides to join is given by the formula*

$$S^{un}(q)=R\frac{E[C1\{C\leqslant I_q\}+I_q1\{C>I_q\}]}{\lambda qE[X]}-K\frac{E[X^2]}{2E[X]} \tag{39}$$

$$=R\left[\sum_{k=1}^{\infty}g_k\sum_{j=0}^{k-1}\int_0^{\infty}e^{-\lambda qt}\frac{(\lambda qt)^j}{j!}\frac{1-F(t)}{E[X]}dt\right]-K\frac{E[X^2]}{2E[X]}, \tag{40}$$

where C, X, and I_q are independent random variables; C is a discrete random variable with probability mass function (g_k: $k=1,2,\ldots$) (the capacity probability mass function); X is a continuous random variable with probability density function $f(x)$ (the service time density); and I_q is a discrete random variable with probability mass function

$$\Pr[I_q=i]=\int_0^{\infty}e^{-\lambda qx}\frac{(\lambda qx)^i}{i!}dF(x),\quad i\geqslant 0. \tag{41}$$

The function $S^{un}(q)$ is decreasing in q, so the customers adopt an ATC behavior.

PROOF. We assume that the customers follow a strategy q and we consider a tagged customer at his arrival instant who decides to join. Then, his expected net benefit is given by

$$S^{un}(q)=RP^{un}(q)-KE[R^{un}], \tag{42}$$

where $P^{un}(q)$ is the probability that the tagged customer receives service, given that the other customers follow the strategy q and R^{un} is the sojourn time of the tagged customer in the system, which coincides with the residual service time at his arrival instant. Using the PASTA property, we have that it is also equal to the residual service time at an arbitrary instant. Therefore,

$$E[R^{un}]=\frac{E[X^2]}{2E[X]}, \tag{43}$$

where X represents a generic service time. Now, because of the regenerative nature of the process, the elementary renewal theorem is applicable and therefore, the probability $P^{un}(q)$ equals to the ratio of the expected number of customers served in a service cycle over the expected number of customers that join in a service cycle.

The number of customers that get service in a service cycle equals to the capacity of the bus, if the number of the customers who join is equal to or exceeds the capacity C. Otherwise, it is equal to the number of customers who join the station. Denoting by I_q the number of customers that decide to join in a service cycle, we have that the probability mass function of I_q is given by (41), since the conditional distribution of I_q given that the service time has length x is Poisson with rate λqx. It is now clear that the expected number of customers that get service in a service cycle equals to $E[C1\{C\leqslant I_q\}+I_q1\{C>I_q\}]$. On the other hand, the expected number of arrivals who decide to join in a service cycle is $E[I_q]=\lambda qE[X]$. Hence,

$$P^{un}(q)=\frac{E[C1\{C\leqslant I_q\}+I1\{C>I_q\}]}{\lambda qE[X]}. \tag{44}$$

Plugging (43) and (44) into (42), we obtain (39). Using (44) and conditioning successively on X, I_q, and C yields

$$P^{un}(q)=\frac{\int_0^{\infty}\sum_{i=1}^{\infty}e^{-\lambda qu}\frac{(\lambda qu)^i}{i!}\sum_{k=1}^{\infty}g_k\min(k,i)\,dF(u)}{\lambda qE[X]}. \tag{45}$$

After a bit of algebraic manipulations, (45) reduces to

$$P^{un}(q)=\sum_{k=1}^{\infty}g_k\sum_{j=0}^{k-1}\int_0^{\infty}e^{-\lambda qt}\frac{(\lambda qt)^j}{j!}\frac{1-F(t)}{E[X]}dt. \tag{46}$$

Plugging (43) and (46) into (42) yields (40).

We can now see that

$$\frac{d(\sum_{j=0}^{k-1}e^{-\lambda qt}(\lambda qt)^j/(j!))}{dq}=-\lambda te^{-\lambda qt}\frac{(\lambda qt)^{k-1}}{(k-1)!}<0.$$

Therefore, differentiating (40) with respect to q yields

$$\frac{dS^{un}(q)}{dq}=R\sum_{k=1}^{\infty}g_k\int_0^{\infty}\left(-\lambda te^{-\lambda qt}\frac{(\lambda qt)^{k-1}}{(k-1)!}\right)\frac{1-F(t)}{E[X]}dt<0.$$

Thus, $S^{un}(q)$ is seen to be decreasing in q. □

REMARK 5.1. Formula (46) can be derived alternatively by noting that $P^{un}(q)=\Pr[N_q^j<C]$, where N_q^j has the equilibrium distribution of the number of customers in the system at arrival instants of customers who join the system, given that the customers follow the strategy q and C has the capacity distribution (g_k). Indeed, a joining customer gets served, if and only if the capacity of the next transportation facility exceeds the number of customers that finds upon arrival. Using the PASTA property, we have that the elapsed time from the most recent arrival of the transportation facility till the arrival of a customer that joins has the equilibrium distribution of the age of a renewal process with interarrival times distributed according to $F(t)$; thus, its probability density function is $(1-F(t))/E[X]$, $t\geqslant 0$. Now, by conditioning, we have that the probability mass function of N_q^j is given by

$$\Pr[N_q^j=i]=\int_0^{\infty}e^{-\lambda qx}\frac{(\lambda qx)^i}{i!}\frac{1-F(x)}{E[X]}dx,\quad i\geqslant 0, \tag{47}$$

and we easily obtain (46).

Now, we can determine the equilibrium joining strategies in the unobservable case. We have the following Theorem 5.1.

THEOREM 5.1. *Consider the unobservable model of a transportation station. Then a unique equilibrium joining strategy exists. In particular, we have the following cases*:

Case I. $R/K \leqslant E[X^2]/(2E[X])$. *Then, a unique equilibrium joining strategy exists that prescribes to balk. Furthermore, it is a dominant strategy.*

Case II. $E[X^2]/(2E[X]) < R/K < (E[X^2]/(2E[X]))/(\sum_{k=1}^{\infty} g_k \sum_{j=0}^{k-1} \int_0^{\infty} e^{-\lambda t}((\lambda t)^j/(j!))((1-F(t))/E[X])\,dt)$. *Then, a unique equilibrium joining strategy exists that prescribes to join with probability* q_e, *where* q_e *is the unique root of* $S^{un}(q)$ *in* $(0,1)$.

Case III. $R/K \geqslant (E[X^2]/(2E[X]))/(\sum_{k=1}^{\infty} g_k \sum_{j=0}^{k-1} \int_0^{\infty} e^{-\lambda t} \cdot ((\lambda t)^j/(j!))((1-F(t))/E[X])\,dt)$. *Then, a unique equilibrium joining strategy exists that prescribes to join. Furthermore, it is a dominant strategy.*

PROOF. *Case* I. Consider a tagged customer at his arrival instant and assume that the other customers follow a strategy q. If $q=0$, the expected net benefit of the tagged customer, if he decides to join is $S^{un}(0)=R-K(E[X^2]/(2E[X]))\leqslant 0$. Moreover, the monotonicity of $S^{un}(q)$ implies that $S^{un}(q)<0$, $q\in(0,1]$. Thus, the best response of the tagged customer is to balk, against any strategy of the other customers. Hence, the strategy of balking is the unique dominant strategy.

Case II. In this case, we have that $S^{un}(0)>0$, whereas $S^{un}(1)<0$. Thus, by appealing to Bolzano's theorem and to the monotonicity of $S^{un}(q)$, we have that $S^{un}(q)$ has a unique root $q^e\in(0,1)$. Now, if the customers follow a strategy q, with $q\in[0,q^e)$, the expected net benefit of a tagged customer is positive, so his best response is to join the system. Thus, no $q\in[0,q^e)$ can be an equilibrium strategy. Similarly for any $q\in(q^e,1]$, the only best response is to balk so such a q cannot be an equilibrium strategy. Finally, if the customers follow the strategy q^e, the expected net benefit of a tagged customer is $S^{un}(q^e)=0$, so he is indifferent between joining and balking. In particular, the strategy q^e is a best response against itself. We deduce that q^e constitutes the unique equilibrium joining strategy.

Case III. Suppose that the customers follow the strategy of joining, $q=1$. Then the expected net benefit of a tagged customer, if he decides to join, is

$$S^{un}(1)=R\left(\sum_{k=1}^{\infty} g_k \sum_{j=0}^{k-1}\int_0^{\infty} e^{-\lambda t}\frac{(\lambda t)^j}{j!}\frac{1-F(t)}{E[X]}\,dt\right) -K\frac{E[X^2]}{2E[X]}\geqslant 0.$$

Moreover, the monotonicity of $S^{un}(q)$ implies that $S^{un}(q)>0$, $q\in[0,1)$. So, if $q\in[0,1)$, the best response of the tagged customer is to join. Thus, the best response of the tagged customer is to join, against any strategy of the other customers. Hence, the strategy of joining is the unique dominant strategy. □

We comment now on the associated social optimization problem.

REMARK 5.2. In the unobservable model of a transportation station, where the customers join the system according to a strategy q, the expected social benefit per time unit is given by

$$S_{\text{soc}}^{un}(q)=\lambda q S^{un}(q)=\lambda q(RP^{un}(q)-KE[R^{un}]), \tag{48}$$

where $P^{un}(q)$ and $E[R^{un}]$ are given by (46) and (43), respectively. The complexity of the term $P^{un}(q)$ as a function of q does not allow the analytic derivation of a strategy q that maximizes $S_{\text{soc}}^{un}(q)$. However, in §6 we determine the socially optimal strategy in the case where the times between successive visits of the transportation facility follow the exponential distribution.

6. The Exponential Case

In this section, we study in some more detail the special case where the distribution $F(x)$ is exponential, i.e., when the visits of the transportation facility occur according to a Poisson process. More specifically, we assume that $F(x)=1-e^{-\mu x}$, for $x>0$. This special case is amenable to further analysis and we can determine the equilibrium and socially optimal strategies in both information cases (observable and unobservable).

First, we consider the observable model. Because of the memoryless property of the exponential distribution, we can easily see that the conditional residual service times $R_{n,\mathbf{q}}$ are all exponentially distributed. Therefore, $\tilde{F}_{n,\mathbf{q}}(s)=\tilde{F}(s)=\mu/(\mu+s)$ and $E[R_{n,\mathbf{q}}]=E[X]=1/\mu$. Using (35) and (36), we can easily see that the expected net benefit function $S_n^{\text{obs}}(\mathbf{q})$ assumes the form

$$S_n^{\text{obs}}(\mathbf{q})=R\sum_{k=n+1}^{\infty} g_k-K\frac{1}{\mu}. \tag{49}$$

Therefore, we observe that $S_n^{\text{obs}}(\mathbf{q})$ is decreasing in n and does not depend on $\mathbf{q}$. We conclude that a dominant threshold equilibrium strategy exists. More concretely, we have the following Theorem 6.1.

THEOREM 6.1. *Consider the exponential observable case of a transportation station. Then, a dominant threshold strategy exists that prescribes to join when you see upon arrival less than* n^e *present customers, with* n^e *given by*

$$n^e=\min\left\{n\in\mathbb{N}_0\colon R\sum_{k=n+1}^{\infty} g_k-K\frac{1}{\mu}<0\right\}. \tag{50}$$

We now move to the social optimization problem. Again, because of the memoryless property of the exponential distribution, substituting $\tilde{F}_{n,\mathbf{q}}(s)=\mu/(\mu+s)$ and $E[R_{n,\mathbf{q}}]=1/\mu$ in Proposition 3.4 yields

$$\pi_{n,\mathbf{q}}=(1-\rho_n)\prod_{i=0}^{n-1}\rho_i, \quad n\geqslant 0, \tag{51}$$

with

$$\rho_i = \frac{\lambda q_i}{\mu + \lambda q_i}, \quad i \geqslant 0. \tag{52}$$

We can then easily see that the mean number of customers in the station is given by

$$E[N_{\mathbf{q}}] = \sum_{n=0}^{\infty} \rho_0 \rho_1 \ldots \rho_n. \tag{53}$$

Plugging (51) and (53) into (38) gives

$$S_{\text{soc}}^{\text{obs}}(\mathbf{q}) = \mu \sum_{n=0}^{\infty} \rho_0 \rho_1 \ldots \rho_n \left[R \sum_{k=n+1}^{\infty} g_k - K \frac{1}{\mu} \right]. \tag{54}$$

Our objective is to find a socially optimal strategy $\mathbf{q}$ that maximizes the right side of (54). To this end, for $n \geqslant 0$, note that ρ_n is increasing in q_n and does not depend on $q_i, i \neq n$. Note also that the quantity $R \sum_{k=n+1}^{\infty} g_k - K(1/\mu)$ is decreasing in n; thus $R \sum_{k=n+1}^{\infty} g_k - K(1/\mu) \geqslant 0$ for $n < n^e - 1$, whereas $R \sum_{k=n+1}^{\infty} g_k - K(1/\mu) < 0$ for $n \geqslant n^e$. It is now clear that a socially optimal strategy $\mathbf{q}$ should assign the maximum possible coefficient $\rho_0 \rho_1 \ldots \rho_n$ to every $n < n^e - 1$ and $\rho_0 \rho_1 \ldots \rho_n = 0$ for $n \geqslant n^e$. This occurs when $q_i = 1$, for $n < n^e - 1$, and $q_i = 0$, for $n \geqslant n^e$. Thus, we have the following Theorem 6.2.

Theorem 6.2. *Consider the exponential observable case of a transportation station. Then, a socially optimal threshold joining strategy exists that prescribes to join when you see less than n^{soc} present customers, with $n^{\text{soc}} = n^e$ given by* (50).

It is interesting to emphasize that by Theorem 6.2 individually and socially optimal policies coincide under exponential intervisit times.

Now, we consider the unobservable model. After some straightforward algebraic manipulations, we can see that (40) reduces to

$$S^{un}(q) = R \left[1 - \sum_{k=1}^{\infty} g_k \left(\frac{\lambda q}{\lambda q + \mu} \right)^k \right] - K \frac{1}{\mu}$$
$$= R \left[1 - G\left(\frac{\lambda q}{\lambda q + \mu} \right) \right] - \frac{K}{\mu}. \tag{55}$$

Now, using Theorem 5.1 and (55), we deduce the following Theorem 6.3.

Theorem 6.3. *Consider the exponential unobservable model of a transportation station. Then, a unique equilibrium joining strategy exists. In particular, we have the following cases*:

Case I. $R/K \leqslant 1/\mu$. *Then, a unique equilibrium joining strategy exists that prescribes to balk. Furthermore, it is a dominant strategy.*

Case II. $1/\mu < R/K < 1/(\mu[1 - G(\lambda/(\lambda + \mu))])$. *Then, a unique equilibrium joining strategy exists that prescribes to join with probability q^e, given by*

$$q^e = \frac{\mu G^{-1}(1 - K/(R\mu))}{\lambda[1 - G^{-1}(1 - K/(R\mu))]}. \tag{56}$$

Case III. $R/K \geqslant 1/(\mu[1 - G(\lambda/(\lambda + \mu))])$. *Then, a unique equilibrium joining strategy exists that prescribes to join. Furthermore, it is a dominant strategy.*

Now, we consider the social optimization problem for the unobservable model. Using (48) and (55), the expected social benefit per time unit, when the customers follow a strategy q, is given by

$$S_{\text{soc}}^{un}(q) = \lambda q \left[R \left(1 - \sum_{k=1}^{\infty} g_k \left(\frac{\lambda q}{\lambda q + \mu} \right)^k \right) - K \frac{1}{\mu} \right]$$
$$= \lambda q \left[R \left[1 - G\left(\frac{\lambda q}{\lambda q + \mu} \right) \right] - \frac{K}{\mu} \right]. \tag{57}$$

We can now determine the socially optimal strategy for the exponential unobservable model. We have the following Theorem 6.4.

Theorem 6.4. *Consider the exponential unobservable model of a transportation station. Then, a unique socially optimal joining strategy exists. In particular, we have the following cases*:

Case I. $R/K \leqslant 1/\mu$. *Then, a unique socially optimal joining strategy exists that prescribes to balk.*

Case II. $1/\mu < R/K < 1/(\mu[1 - G(\lambda/(\lambda + \mu)) - ((\lambda\mu)/(\lambda + \mu)^2) G'(\lambda/(\lambda + \mu))])$. *Then, a unique socially optimal joining strategy exists that prescribes to join with probability q^{soc}, where q^{soc} is the unique root in* $(0, 1)$ *of the equation*

$$G\left(\frac{\lambda q}{\lambda q + \mu} \right) + \frac{\lambda q \mu}{(\lambda q + \mu)^2} G'\left(\frac{\lambda q}{\lambda q + \mu} \right) = 1 - \frac{K}{\mu R}. \tag{58}$$

Case III. $R/K \geqslant 1/(\mu[1 - G(\lambda/(\lambda + \mu)) - ((\lambda\mu)/(\lambda + \mu)^2) \cdot G'(\lambda/(\lambda + \mu))])$. *Then, a unique socially optimal joining strategy exists that prescribes to join.*

Proof. The first and the second derivative of $S_{\text{soc}}^{un}(q)$ with respect to q are

$$\frac{dS_{\text{soc}}^{un}(q)}{dq} = \lambda \left[R - \frac{K}{\mu} \right] - \lambda R \sum_{k=1}^{\infty} g_k \left(1 + \frac{k\mu}{\lambda q + \mu} \right) \left(\frac{\lambda q}{\lambda q + \mu} \right)^k$$
$$= \lambda \left[R - \frac{K}{\mu} \right] - \lambda R \left[G\left(\frac{\lambda q}{\lambda q + \mu} \right) + \frac{\lambda q \mu}{(\lambda q + \mu)^2} \cdot G'\left(\frac{\lambda q}{\lambda q + \mu} \right) \right] \tag{59}$$

and

$$\frac{d^2 S_{\text{soc}}^{un}(q)}{dq^2} = \lambda R \sum_{k=1}^{\infty} g_k \frac{k\mu\lambda}{(\lambda q + \mu)^2} \left(\frac{\lambda q}{\lambda q + \mu} \right)^{k-1} \cdot \frac{[-(k+1)\mu]}{\lambda q + \mu}. \tag{60}$$

It is evident from (60) that $d^2 S_{\text{soc}}^{un}(q)/dq^2 < 0$, for $q \in [0, 1]$, i.e., $dS_{\text{soc}}^{un}(q)/dq$ is strictly decreasing in q. Moreover,

$$\left[\frac{dS_{\text{soc}}^{un}(q)}{dq} \right]_{q=0} = \lambda \left[R - \frac{K}{\mu} \right] \tag{61}$$

and

$$\left[\frac{dS_{\text{soc}}^{un}(q)}{dq}\right]_{q=1} = \lambda\left[R-\frac{K}{\mu}\right]-\lambda R\left[G\left(\frac{\lambda}{\lambda+\mu}\right)\right.$$
$$\left.+\frac{\lambda\mu}{(\lambda+\mu)^2}G'\left(\frac{\lambda}{\lambda+\mu}\right)\right]. \quad (62)$$

Now, we use these quantities in combination with the monotonicity of $dS_{\text{soc}}^{un}(q)/dq$ to identify the socially optimal joining strategies in the three cases of the theorem.

Case I. In this case we have, $[dS_{\text{soc}}^{un}(q)/dq]_{q=0}\leqslant 0$. So, the monotonicity of $dS_{\text{soc}}^{un}(q)/dq$ implies that $dS_{\text{soc}}^{un}(q)/dq<0$, for $q\in(0,1]$. Then, $S_{\text{soc}}^{un}(q)$ is strictly decreasing for $q\in[0,1]$. So, the unique socially optimal joining strategy is to balk.

Case II. Now, the inequality $[dS_{\text{soc}}^{un}(q)/dq]_{q=1}<0<[dS_{\text{soc}}^{un}(q)/dq]_{q=0}$ holds and the monotonicity of $dS_{\text{soc}}^{un}(q)/dq$ implies that there exists a unique $q^{\text{soc}}\in(0,1)$, such that $[dS_{\text{soc}}^{un}(q)/dq]_{q=q^{\text{soc}}}=0$. In this case, the unique socially optimal joining strategy is q^{soc}, which is given as the unique solution in $(0,1)$ of the Equation (58).

Case III. In this case we can easily show that $[dS_{\text{soc}}^{un}(q)/dq]_{q=1}\geqslant 0$. Then, the monotonicity of $dS_{\text{soc}}^{un}(q)/dq$ implies that $dS_{\text{soc}}^{un}(q)/dq>0$ for all $q\in[0,1)$. Then, $S_{\text{soc}}^{un}(q)$ is strictly increasing for all $q\in[0,1]$ and the unique socially optimal joining strategy prescribes to join. □

We now proceed to the comparison of the equilibrium and the socially optimal joining strategies in the exponential unobservable model. By considering the various cases of Theorems 6.3 and 6.4, we have the following four cases:

Case I. $R/K\leqslant 1/\mu$.

Case II. $1/\mu<R/K<1/(\mu[1-G(\lambda/(\lambda+\mu))])$.

Case III. $1/(\mu[1-G(\lambda/(\lambda+\mu))])\leqslant R/K<1/(\mu[1-G(\lambda/(\lambda+\mu))-((\lambda\mu)/(\lambda+\mu)^2)G'(\lambda/(\lambda+\mu))])$.

Case IV. $R/K\geqslant 1/(\mu[1-G(\lambda/(\lambda+\mu))-((\lambda\mu)/(\lambda+\mu)^2)G'(\lambda/(\lambda+\mu))])$.

In Case I, we have that $q^e=q^{\text{soc}}=0$. In Case II, we have that q^e is the unique root of $S^{un}(q)$ in $(0,1)$ and q^{soc} is the unique root of $dS_{\text{soc}}^{un}(q)/dq$ in $(0,1)$. But, by (59), we have that

$$\left[\frac{dS_{\text{soc}}^{un}(q)}{dq}\right]_{q=q^e} = -\lambda R\sum_{k=1}^{\infty} g_k\frac{k\mu}{\lambda q^e+\mu}\left(\frac{\lambda q^e}{\lambda q^e+\mu}\right)^k<0.$$

Since, $dS_{\text{soc}}^{un}(q)/dq$ is decreasing in q and $[dS_{\text{soc}}^{un}(q)/dq]_{q=q^e}<0$, whereas $[dS_{\text{soc}}^{un}(q)/dq]_{q=q^{\text{soc}}}=0$, we have that $q^e>q^{\text{soc}}$ in Case II. In Case III, we have that $q^e=1$, whereas $q^{\text{soc}}\in(0,1)$. In Case IV, we have that $q^e=q^{\text{soc}}=1$. Therefore, we conclude that

$$q^{\text{soc}}\leqslant q^e,$$

i.e., under individual optimization the customers are more willing to join than it is socially desirable. This is in contrast to the observable model, where the equilibrium and socially optimal joining strategies coincide.

Let us elaborate on the intuition behind these results. Indeed, in the unobservable model, the strategies of the customers do not affect the mean sojourn time of a tagged customer, but affect his probability of receiving service. Specifically, as q increases, the probability that a tagged customer receives service decreases. So, in this model the joining decisions of the customers imply negative externalities for the future customers. This explains why $q^e\geqslant q^{\text{soc}}$. The customers ignore the negative externalities that their joining decisions impose on other customers and they tend to overuse the system. On the other hand, in the observable model, a joining decision of a tagged customer negatively affects future customers, but the negative externalities are easily seen to be smaller than the positive value for the tagged customer, for any $n<n^e$. Indeed, suppose that the socially optimal strategy dictates balking to a tagged customer for some $n<n^e$. Then the queue length will never exceed n. But this is suboptimal, because if the customer that finds the system at state n joins, then he has positive utility and hence the social utility increases. Note the difference between the situation of this model and the Naor (1969) case. There, the negative externalities may exceed the value of the tagged customer. In Naor's model, a balking decision implies that future customers will join at smaller queue lengths and have greater benefits, whereas here if the tagged customer balks, future customers (of this service phase) will not benefit from a smaller queue. Therefore, the equilibrium and the socially optimal strategies coincide.

7. Different Mean Residual Time Behaviour: Avoiding or Following the Crowd

Theorems 4.1 and 4.2 guarantee the existence but not the uniqueness of equilibrium joining strategies for the customers. Indeed, depending on the nature of the distribution $F(x)$ and of the probability mass function (g_k: $k=1,2,\ldots$), more than one equilibrium joining strategies may exist. In what follows, we state some sufficient conditions for the uniqueness of equilibrium strategies. These conditions are satisfied, when the distribution $F(x)$ belongs to the family of the decreasing mean residual life distributions. Similar results hold for the increasing mean residual life distributions. However, the latter do not appear frequently in practice, so we summarize the corresponding results in the online appendix.

Definition 7.1. A nonnegative random variable X is said to have a decreasing mean residual life distribution (respectively, an increasing mean residual life distribution), if the function $E[X-x\,|\,X\geqslant x]$ is decreasing (respectively, increasing) in x, for $x\geqslant 0$. When the monotonicity is strict, X is said to have a strictly DMRL (respectively, strictly IMRL) distribution.

The families of DMRL and IMRL distributions have been extensively studied in the literature, since they capture natural notions of aging (see, e.g., Shaked and Shanthikumar 2007). In the context of our transportation model, it seems more reasonable to assume that $F(x)$ follows a DMRL distribution. Indeed, in that case, the longer the time elapsed from the previous visit of the facility, the shorter the expected time till its next visit. On the other hand IMRL models the counterintuitive situation in which the longer the time elapsed from the previous visit of the facility, the longer the expected time till its next visit.

First, we prove that if X is DMRL, then the expected net benefit function $S_n^{\text{obs}}(\mathbf{q}_n)$ is decreasing in q_n, which implies the uniqueness of an equilibrium joining strategy. To prove this monotonicity result, we will need the following Lemmas 7.1 and 7.2.

Lemma 7.1. *Let X be a nonnegative random variable with a DMRL distribution and T_λ be an exponentially distributed random variable with rate λ, independent of X. Then, the conditional distribution of $X-T_\lambda$ given that $X \geqslant T_\lambda$ has also a DMRL distribution.*

Proof. See the online appendix. □

Lemma 7.2. *Let X be a nonnegative random variable with a DMRL distribution and T_λ be an exponentially distributed random variable with rate λ, independent of X. Then, $E[X-T_\lambda \mid X \geqslant T_\lambda]$ is increasing function of λ.*

Proof. See the online appendix. □

In the following Proposition 7.1, we show that the DMRL property for the distribution $F(x)$ of the times between successive visits of the transportation facility implies the monotonicity of $E[R_{n,\mathbf{q}_n}]$ in q_n.

Proposition 7.1. *Consider the observable model of a transportation station. If the distribution $F(x)$ of the times between successive visits of the transportation facility is a DMRL, then $R_{n,\mathbf{q}_n}$ has a DMRL distribution. Moreover, $E[R_{n,\mathbf{q}_n}]$ is increasing in q_n, for $n=0,1,\ldots$.*

Proof. We will prove by induction on n that $R_{n,\mathbf{q}_n}$ has a DMRL distribution and that $E[R_{n,\mathbf{q}_n}]$ is increasing in q_n.

For $n=0$, from (5) we have that $R_{0,\mathbf{q}} \stackrel{\text{d}}{=} (X-T_{\lambda q_0} \mid X \geqslant T_{\lambda q_0})$, where $T_{\lambda q_0}$ is exponentially distributed with parameter λq_0. Since X has a DMRL distribution, we can apply Lemma 7.1 to conclude that $R_{0,\mathbf{q}}$ has also a DMRL distribution. Lemma 7.2 shows that $E[R_{0,\mathbf{q}_0}]$ is increasing in q_0.

Now, let us assume that $R_{k,\mathbf{q}_k}$ has a DMRL distribution and that $E[R_{k,\mathbf{q}_k}]$ is increasing in q_k. From (7) we have that $R_{k+1,\mathbf{q}_{k+1}} \stackrel{\text{d}}{=} (R_{k,\mathbf{q}_k} - T_{\lambda q_{k+1}} \mid R_{k,\mathbf{q}_k} \geqslant T_{\lambda q_{k+1}})$, where $T_{\lambda q_{k+1}}$ is exponentially distributed with parameter λq_{k+1}. Lemma 7.1 and Lemma 7.2 imply, respectively, that $R_{k+1,\mathbf{q}_{k+1}}$ has a DMRL distribution and that $E[R_{k+1,\mathbf{q}_{k+1}}]$ is increasing in q_{k+1}. □

In light of (35) and (36), Proposition 7.1 yields the following Corollary 7.1.

Corollary 7.1. *Consider the observable model of a transportation station. If the distribution $F(x)$ of the times between successive visits of the transportation facility is a strictly DMRL, then the expected net benefit function $S_n^{\text{obs}}(\mathbf{q}_n)$ is strictly decreasing in q_n and consequently there exists a unique equilibrium joining strategy.*

We can go even further and show that, under certain conditions, the unique equilibrium joining strategy, which exists when $F(x)$ is a strictly DMRL distribution, is of reverse-threshold type.

Definition 7.2. Consider the observable model of a transportation station. A joining strategy $\mathbf{q}=(q_0,q_1,q_2,\ldots)$ is said to be of reverse-threshold type, if there exists an n such that $q_i=0$, for $i<n$, $q_n \in [0,1]$ and $q_i=1$, for $i>n$.

When $F(x)$ is a DMRL distribution, we have the following Theorem 7.1.

Theorem 7.1. *Consider the observable model of a transportation station. If the distribution $F(x)$ of the times between successive visits of the transportation facility is a strictly DMRL and the transportation facility has unlimited capacity, then the unique equilibrium joining strategy is of reverse-threshold type.*

Proof. Since $F(x)$ is a strictly DMRL distribution, Proposition 7.1 yields that $R_{n,\mathbf{q}_n}$ is also strictly DMRL, for $n=0,1,\ldots$. Using Lemma 7.2, we have that

$$E[R_{n,\mathbf{q}_n} - T_{\lambda q_{n+1}} \mid R_{n,\mathbf{q}_n} \geqslant T_{\lambda q_{n+1}}] < E[R_{n,\mathbf{q}_n} - T_{\lambda'} \mid R_{n,\mathbf{q}_n} \geqslant T_{\lambda'}], \quad \lambda' > \lambda q_{n+1},$$

where $T_{\lambda q_{n+1}}$ and $T_{\lambda'}$ are exponentially distributed distributions with rates λq_{n+1} and λ', respectively. Taking $\lambda' \to \infty$ and using (7) yields $E[R_{n+1,\mathbf{q}_{n+1}}] < E[R_{n,\mathbf{q}_n}]$, for $\mathbf{q}_{n+1}=(\mathbf{q}_n, q_{n+1})$. Then, because of the unlimited capacity of the transportation facility, we have that

$$S_n^{\text{obs}}(\mathbf{q}_n) = R - KE[R_{n,\mathbf{q}_n}] < R - KE[R_{n+1,\mathbf{q}_{n+1}}] = S_{n+1}^{\text{obs}}(\mathbf{q}_{n+1}). \tag{63}$$

Consider, now, the equilibrium joining strategy $\mathbf{q}^e = (q_0^e, q_1^e, \ldots)$. To prove that it is of reverse-threshold type, it suffices to show that if $q_n^e \in (0,1]$ for some n, then $q_{n+1}^e = 1$. Indeed, $q_n^e \in (0,1]$ implies that $S_n^{\text{obs}}(\mathbf{q}_n^e) \geqslant 0$. Then (63) yields $S_{n+1}^{\text{obs}}(\mathbf{q}_{n+1}) > 0$, for any $\mathbf{q}_{n+1} = (\mathbf{q}_n^e, q_{n+1})$, $q_{n+1} \in [0,1]$ and we conclude that $q_{n+1}^e = 1$. Therefore, $\mathbf{q}^e$ is a reverse-threshold strategy. □

The intuition behind Theorem 7.1 can be described as follows: In the case where $F(x)$ is a DMRL distribution, the presence of a larger number of customers upon arrival of a tagged customer has two opposite effects. On the one hand it makes less probable that the tagged customer will be accommodated by the next transportation facility. On the other hand, it gives a signal that some time has passed since the last visit of the transportation facility, so because of the DMRL nature of $F(x)$, the tagged customer expects a shorter remaining service time. Therefore, the net effect on the tagged customer is dubious. However, under the additional assumption that the facility has unlimited capacity, the tagged customer prefers to see upon arrival a larger number of customers, since this implies less waiting costs for him. Therefore, the larger the number of customers upon arrival, the more willing is the tagged customer to join and therefore, an equilibrium joining strategy is of reverse-threshold type.

Regarding the ATC/FTC behavior of the customers, we have seen in Proposition 7.1 that when $F(x)$ is DMRL then $E[R_{n,\mathbf{q}_n}]$ is increasing in q_n. Therefore, if $q_n^1 < q_n^2$, we have that $S_n^{\text{obs}}(\mathbf{q}_{n-1}, q_n^1) \geqslant S_n^{\text{obs}}(\mathbf{q}_{n-1}, q_n^2)$, which implies that the best response of a tagged customer who finds n present customers upon arrival, given that the other customers follow $(\mathbf{q}_{n-1}, q_n^1)$ is greater than or equal to his best response given that the other customers follow $(\mathbf{q}_{n-1}, q_n^2)$. In a sense, we have an ATC situation for every fixed observed state n.

8. Numerical Experiments

In this section, we summarize the findings of several numerical experiments that illustrate the applicability of the theoretical results and shed light on several issues that seem important from a system operator's point of view. We articulate the presentation in three subsections that deal with the number and form of the equilibrium strategies, the effect of the interarrival distribution, and the effect of the capacity of the facility on the behavior of the customers.

8.1. Form and Number of the Equilibrium Strategies

When the reported sufficient conditions for the existence of reverse-threshold strategies fail, then the form of the equilibrium strategies may be very irregular. In particular, this happens when the facility's interarrival distribution $F(x)$ is not unimodal. Below, we present several examples in the case where the interarrival time X assumes one of two values. In such cases, the information on the number of present customers gives a strong signal to a tagged arriving customer regarding the duration of the interarrival time being equal to the low or high possible values of X.

In the first experiment, we see that an equilibrium strategy may have proper probabilities (strictly between 0 and 1) in multiple states. We consider a numerical scenario with arrival rate $\lambda = 0.1$, discrete distribution function $F(x)$ for the times between the successive visits of the facility with possible values 1 and 50 with corresponding probabilities 0.9 and 0.1, $R = 35$, $K = 1$ and discrete uniform distribution on $\{1, 2, \ldots, 5\}$ for the successive capacities of the facility (i.e., $g_j = 0.2$, $j = 1, 2, \ldots, 5$). Then, by applying the algorithmic procedure of Theorems 4.1 and 4.2, we see that an equilibrium joining strategy in this case is $(1, 0.6260, 0.5967, 0.0566, 0, 0, \ldots)$.

In the second experiment, we see that it is not necessary that the probabilities of the equilibrium joining strategy form a monotone sequence. For example, in a numerical scenario with $\lambda = 0.1$, discrete distribution function $F(x)$ with possible values 1 and 52 with corresponding probabilities 0.995 and 0.005, $R = 24$, $K = 1$ and unlimited capacity of the facility at its visits, we obtain an equilibrium joining strategy $(1, 1, 0.6865, 1, 0.9824, 1, 1, \ldots)$. Such a situation can be explained intuitively as follows: When a tagged customer observes "few" customers in the system, he has evidence to believe that the current interarrival time has started quite recently. Then, most probably, it has assumed the low value 1 (whose a priori probability is 0.995) and hence the tagged customer is willing to join. On the other hand, when a customer observes "many" customers in the system, he tends to believe that the interarrival time has assumed the high value 52, but nevertheless the expected remaining time till the next facility's arrival is believed to be small, so he is again willing to join. However, for intermediate values of the number of present customers in the system, the situation is ambiguous for the tagged arriving customer: On the one hand, he has a moderate signal that the interarrival time has assumed the high value, and, on the other hand, there is a considerable probability the remaining time till the next facility's arrival to be long. This is the reason why some intermediate probabilities are less than 1.

In the third experiment, we see that multiple equilibrium joining strategies may exist. Indeed, considering a numerical scenario with $\lambda = 0.2$, discrete distribution function $F(x)$ with possible values 5 and 30 with corresponding probabilities 0.9 and 0.1, $R = 7$, $K = 1$ and unlimited capacity of the facility at its visits, we obtain three equilibrium joining strategies: $(0, 0, \ldots)$, $(0.3076, 0, 0, \ldots)$ and $(1, 0, 0, \ldots)$. Indeed, if no customer joins the system, then the presence of 0 customers gives no information to a tagged arriving customer about the current interarrival time X. If he joins he will receive on average $R - KE[R(X)] < 0$ money units, so he prefers to balk. This yields the equilibrium $(0, 0, \ldots)$. On the other hand, if all customers join when the system is empty, then the presence of 0 customers gives a strong signal to a tagged arriving customer that the facility has visited the station recently. Therefore, as the a priori probability of the length of the current interarrival time is 0.9 for the low value 5, he tends to believe that this is the true value and he becomes willing to join. This yields the equilibrium $(1, 0, 0, \ldots)$. However, if only a portion of the customers join when the system is empty, then we have a mixed situation that yields the equilibrium $(0.3076, 0, 0, \ldots)$.

8.2. The Effect of the Mean and Variability of the Interarrival Times on the Behavior of the Customers and on System's Throughput

In this second series of numerical experiments, we investigate the effect of the mean and variability of the facility's interarrival times on the behavior of the customers. Regarding the effect of the mean, the answer is intuitively clear: The smaller the mean value of the interarrival times, the more willing are the customers to join. Indeed, this fact has been verified by a number of numerical experiments and

Table 1. The effect of the variability of the facility's interarrival distribution on the equilibrium joining strategies of the customers and the throughput.

Distribution	Variance	Equilibrium joining strategy	Throughput
Erlang(1, 0.5)	4.0000	$(0, 0, 0, \ldots)$	0.0000
Erlang(2, 1.0)	2.0000	$(0.2500, 1, 1, 1, \ldots)$	1.1666
Erlang(3, 1.5)	1.3333	$(0.6726, 1, 1, 1, \ldots)$	1.7923
Erlang(4, 2.0)	1.0000	$(0.8562, 1, 1, 1, \ldots)$	1.9231
Erlang(5, 2.5)	0.8000	$(0.9517, 1, 1, 1, \ldots)$	1.9761
Erlang(6, 3.0)	0.6667	$(1, 1, 1, \ldots)$	2.0000

seems to be very robust, independent of the shape of the underlying distribution.

Regarding the effect of the variability, we have again a clear situation: the reduction of the variability of the facility's interarrival times induces the customers to be more willing to join the system. To illustrate this point, we present a numerical scenario in Table 1, for a model with $\lambda = 2$, Erlang distribution function $F(x)$ with n phases and rate $0.5n$, $R = 1.6$, $K = 1$ and unlimited capacity for the facility at its visits. We vary n from 1 to 6, so that the mean of the facility's interarrival time is kept fixed to 2, but the variance, which equals to $4/n$, decreases from 4 to $\frac{4}{6}$. The reduction of the variability is seen to have a very important impact on the behavior of the customers. Indeed, in the case of the exponential distribution ($n = 1$), no customer enters in the system, whereas for an Erlang distribution with six phases and the same mean, all customers do enter. The observations from Table 1 confirm the significant effects of variability in service systems with strategic customers. It is well known that variability has a degrading effect on the performance measures for most queueing systems without strategic customers. Our numerical findings indicate that this negative effect is exacerbated under strategic customer decisions. From a system design point of view, this implies that reducing interarrival time variability must be a target for the system operator.

8.3. The Effect of the Mean and Variability of the Capacity of the Facility on the Behavior of the Customers and on System's Throughput

In this third series of numerical experiments, we discuss the effect of the facility's capacity on the behavior of the customers. It is intuitively plausible and has been verified numerically that the larger the mean value of the capacity, the more willing are the customers to join. So, we concentrate on the effect of the variability of the capacity, which seems less clear. In Table 2, we provide numerical results for a model with $\lambda = 2$, hyperexponential distribution function $F(x)$, which is the mixture of two exponential distributions with rates 1 and 2 and corresponding mixing probabilities 0.2 and 0.8, $R = 2$ and $K = 1$. We consider various probability mass functions (g_k: $k = 1, 2, \ldots$) for the capacity of the facility, all with the same mean $\bar{g} = 3$ and we derive the corresponding equilibrium joining strategies and the throughput. The reduction of the variability is seen again to have a positive impact on the system, i.e., it increases the throughput of served customers. Note, however, that the consideration solely of the equilibrium joining strategies of the customers may be misleading. For example, a look at the lines 1–5 of Table 2 shows that as the variance increases, the customers are willing to enter when they see more customers in the system. But the throughput is decreasing. This happens because the reduction of the variability induces equilibrium distributions for the number of customers in the system that assign more probability mass at lower states.

For an intuitive explanation, consider a tagged arriving customer who sees a few customers waiting, increasing the variance of the capacity implies that there is a probability that he might be served, which provides him an incentive to join. However, at the same time the probability that the capacity is small is also increasing. This causes an undesirable situation with more customers joining the queue but also a larger proportion not receiving service eventually. Again, there seems to be a strong need for providing regular and dependable service when customers are strategic.

9. Discussion—Open Issues

In this paper, we considered the problem of studying customer strategic behavior in a transportation station, where a transportation facility arrives according to a renewal process. We studied two cases with respect to the level of information available to arriving customers and we determined the corresponding equilibrium joining strategies. The study of the customer strategic behavior in observable queueing systems with general service times is a quite new endeavor. Indeed, the paper of Kerner (2011) that determines the equilibrium joining strategies in an observable $M/G/1$ queue seems

Table 2. The effect of the variability of the facility's capacity on the equilibrium joining strategies of the customers and the throughput.

Prob. mass function	Variance	Equilibrium joining strategy	Throughput
$(0, 0, 1, 0, 0, 0, \ldots)$	0.0	$(1, 1, 1, 0, 0, 0, \ldots)$	1.3704
$(0, 0.1, 0.8, 0.1, 0, 0, 0, \ldots)$	0.2	$(1, 1, 1, 0, 0, 0, \ldots)$	1.3488
$(0.2, 0.2, 0.2, 0.2, 0.2, 0, 0, 0, \ldots)$	2.0	$(1, 1, 1, 1, 0, 0, 0, \ldots)$	1.1360
$(0.3, 0.1, 0.2, 0.1, 0.3, 0, 0, 0, \ldots)$	2.6	$(1, 1, 1, 1, 0, 0, 0, \ldots)$	1.0620
$(0.5, 0, 0, 0, 0.5, 0, 0, 0, \ldots)$	4.0	$(1, 1, 1, 1, 1, 0, 0, 0, \ldots)$	0.8925
$(0.8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.2, 0, 0, 0, \ldots)$	16.0	$(1, 0, 0, 0, \ldots)$	0.1778

to be the first one in this direction. In this sense, our first objective was to show that such a study is also possible within the framework of a station with renewal generated visits of a transportation facility.

A key ingredient for the study of customer strategic behavior in a non-Markovian observable queueing system is the computation of the expected conditional residual service times at arrival instants, given the number of waiting customers. Under this perspective, a second objective of our paper was the development of a new probabilistic approach for this computation. This approach may be also applicable in other cases and complements and clarifies intuitively the analytic approach of Kerner (2008).

A third objective of the paper was the study of the existence/uniqueness issue for the equilibrium joining strategies. In the unobservable case, the situation is clear and the existence and uniqueness of an equilibrium joining strategy has been easily justified. On the other hand, in the observable case, an equilibrium joining strategy exists, but there may be multiple equilibrium strategies. However, some natural distributional conditions (e.g., the DMRL property) for the interrenewal times of the visits of the transportation facility are shown to assure the uniqueness of an equilibrium joining strategy. Furthermore, such conditions imply equilibrium joining strategies of reverse-threshold type and are associated with ATC behavior.

Other levels of information would be worth investigating. For example, assuming that a board in the station informs customers about the time for the next visit of the facility, two more levels of information arise. The analytical and numerical comparisons of the equilibrium and socially optimal strategies under the various levels of information will provide further insight on the value of information and the price of anarchy in this class of systems.

Finally, it would also be interesting to determine the equilibrium and socially optimal strategies in the same framework, assuming that the customers who decide to join the system may remain in the station after the next visit of the transportation facility (if they cannot be accommodated). Under this assumption, the model is no longer a clearing system and the computation of the equilibrium joining strategies seems more demanding.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/opre.2014.1280.

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