DECENTRALIZED INVENTORY CONTROL IN A TWO-STAGE CAPACITATED SUPPLY CHAIN

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Abstract

This paper investigates a two-stage supply chain consisting of a capacitated supplier and a retailer that faces a stationary random demand. Both the supplier and the retailer employ base stock policies for inventory replenishment. All unsatisfied demand is backlogged and the customer backorder cost is shared between the supplier and the retailer. We investigate the determination of decentralized inventory decisions when the two parties optimize their individual inventory-related costs in a non-cooperative manner. We explicitly characterize the Nash equilibrium inventory strategies and identify the causes of inefficiency in the decentralized operation. We then study a set of simple linear contracts to see whether these inefficiencies can be overcome. Finally, we investigate Stackelberg games where one of the parties is assumed to be dominating.

1 Introduction

We investigate a simplified model of a supply chain operating in a make-to-stock manner. From the customer satisfaction perspective, keeping buffer inventories in such a system is inevitable. In fact, most of the research work on supply chain inventories is devoted to how much optimal inventory should be kept at different stages of an integrated supply chain. A more recent stream of research recognizes that most real supply chain operations may not be integrated and that decentralized decision making takes place in practice. In this perspective, inventory-replenishment decisions of different players (members of the chain), made in a non-cooperative and decentralized manner, are modeled and analyzed. In this paper, we pursue this analysis for a supply chain consisting of a capacitated supplier and a downstream retailer.

The retailer, in our model, faces a stationary random customer demand and replenishes from its supplier using a base-stock policy. The supplier, who operates a capacitated manufacturing facility also uses a base-stock policy for its internal replenishment. In the decentralized setting, both the supplier and the retailer choose their base stock levels independently in order to minimize their respective inventory-related costs. This decentralized and non-cooperative operation is inefficient in terms of the total supply chain costs with respect to a fully integrated operation. In the first part of the paper, we focus on understanding the causes of this inefficiency and assessing its magnitude. In the second part, we study a number of simple contracts that can be used to overcome the inefficiency of the decentralized system. Although our main focus is on Nash games and equilibria, we also briefly investigate the Stackelberg equilbria where one of the parties is the dominant member of the supply chain.

The paper is structured as follows: Section 2 presents the literature review. The main model is presented in Section 3. Section 4 focuses on the analysis of the decentralized supply chain in the framework of a Nash game. In Section 5 we present and analyze contracts and investigate related coordination issues. Section 6 summarizes our results on Stackelberg games and Section 7 presents the conclusions.

2 Literature Review

The effects of decentralized decision making in supply chains have been investigated in several papers in the recent years. In particular, a number of these papers study supply chains with random demand in a single-period setting based on generalizations of the newsvendor framework. Review papers by Tsay, Nahmias and Agrawal (1999), Lariviere (1999), Cachon (2002) and Cachon and Netessine (2002) provide comprehensive pointers to this literature.

The papers that investigate decentralized supply chains employing stochastic models in an infinite horizon setting are relatively fewer. A number of these papers study the uncapacitated multi-echelon system (the Clark and Scarf model). Lee and Whang (1999) and Chen (1999) focus on coordination mechanisms that use non-linear pricing schemes. Cachon and Zipkin (1999) study the two-stage decentralized supply chain in detail and look into coordination issues through linear transfer payments. Cachon (2001) extends this analysis to the single supplier and multi-retailer system.

There are also a few recent papers that study capacitated supply systems in a decentralized setting. The underlying models in this framework are built on the make-to-stock queue where the supplier's capacity is modeled by the server of a queueing system (see Buzacott and Shanthikumar, 1993). Cachon (1999) studies a supplier-retailer system with lost sales where each party controls its own inventories. Caldentey and Wein (2003) study a similar system with backorders where the customer backorder cost is shared by both parties. In this paper, the supplier sets the capacity level and the retailer controls the inventories in the system. Gupta and Weerawat (2003) study a supplier-manufacturer system in a manufacture-to-order environment and focus on coordination issues through revenue sharing contracts. Elahi, Benjaafar and Donohue (2003) investigate a model where multiple capacitated suppliers compete for the demand coming from a single buyer (manufacturer).

Our paper is closely related to Cachon and Zipkin (1999), Cachon (1999), and Caldentey and Wein (2003). All of these papers study two-stage systems, analyze the decentralized chain and its performance and investigate coordination by linear transfer payments and we follow the same general path. As in Cachon and Zipkin (1999), in our two-stage system, both parties are responsible for their own inventory costs and a portion of the total customer backorder cost. Our cost structure is identical, but the supplier in our case is capacitated. As in Cachon (1999), in our model the transportation times between the supplier's inventories and the retailer are assumed negligible. This makes the analysis of the centralized system tractable and enables us to obtain analytical results even on the decentralized system. In contrast with Cachon (1999) our system experiences backorders and the capacity/queueing effects are manifested in a much sharper manner. Finally, as in Caldentey and Wein (2003) our supplier is capacitated and both parties share the backorder cost but both parties may keep inventory in our setting. We also employ a discrete-state space model (i.e. with integer inventory levels) as opposed to working with a continuous approximation as in Caldentey and Wein (2003).

The positioning of our paper with respect to the above three papers can be summarized as follows: for the two-stage system with a capacitated supplier we obtain explicit analytical results on the decentralized system. The corresponding analysis in Cachon (1999) for the identical system with lost sales mostly relies on numerical calculations essentially due to the difficulty of the centralized system therein (i.e there is no explicit analytical expression for the optimal base stock level, or the optimal supply chain profit in the system with lost sales). This enables us to obtain simple and explicit characterizations of equilibrium behavior in the decentralized system even in the case of unequal holding costs (not treated in Cachon (1999). In this sense, the analytical simplicity and transparency of our results are comparable to those of Caldentey and Wein (2003) whose focus is different. Apart from reaching simple and exact analytical characterizations, there is another important reason for looking at the backorder version of the model in Cachon (1999)). The effect of limited capacity and its consequences in terms of replenishment delays are dampened for the lost sales system because some arrivals are lost. These effects usually appear in a more distinguishing manner in the backorder system where the average cost per unit time goes to infinity as the utilization rate approaches 1.

Finally, it should be noted that our assumption of negligible transportation times (as in Cachon, 1999) between the supplier's inventories and the retailer is crucial for the tractability of the analysis. The corresponding system with two capacitated suppliers in tandem cannot be analyzed exactly even in the centralized case (see Buzacott, Price and Shanthikumar, 1992 for approximations). The other problem for this system is the lack of a complete characterization of the optimal inventory policy. It is known that base-stock policies are not optimal and only a partial characterization of the optimal inventory control policy is available (see Veatch and Wein, 1994 or Karaesmen and Dallery, 2000). Jemaï (2003) presents a numerical investigation of the Nash game when both parties use base stock policies.

3 Modeling Assumptions and Notations

We consider a two-stage supply chain consisting of a manufacturing stage (Stage 1) and a retail stage (Stage 2) which satisfies end-customer demand. Both stages have their own separate inventories. In addition, the manufacturing stage has a limited capacity of production. The retail stage is replenished from the manufacturer's inventory. End-customer demand arrives in single units according to a Poisson process with rate λ. The manufacturer processes items one-by-one using a single-resource. Item processing times have an exponential distribution with rate μ. Let $\rho = \lambda/\mu$ be the utilization rate of the manufacturer.

All customer demand that cannot be satisfied from inventory can be backordered. Both the retailer and the manufacturer manage their own inventories according to base-stock policies (see Buzacott et Shanthikumar, 1993 for a formal definition of a multi-stage base stock policy). Under these assumptions, each demand generates a replenishment order for the retailer, and simultaneously a manufacturing order for the manufacturer. Moreover as in Cachon (1999), the replenishment lead time from the manufacturer's inventory to the retailer's inventory is assumed negligible.

Items incur holding costs of h_s (per item per unit time) in the manufacturer's inventory and h_r (per item per unit time) in the retailer's inventory (where we assume $h_r \geq h_s$). Backorders generate a penalty of b (per item per unit time) for the system. As in Cachon and Zipkin (1999) and Caldentey and Wein (2003) we assume that the system backorder cost is shared between the supplier and the retailer as part of an exogenous arrangement. According to this arrangement, the retailer and the supplier are charged respectively b_r and b_s (where $b_r + b_s = b$) for each system backorder per unit time.

Let I_s and I_r denote the random variables corresponding to the stationary inventory levels respectively at the manufacturer, and the retailer. B denotes the (stationary) number of backorders (unsatisfied customer orders).

The average cost per unit time for the manufacturer is given by:

$$C_s = h_s E[I_s] + b_s E[B]$$

and for the retailer by:

$$C_r = h_r E[I_r] + b_r E[B].$$

4 Centralized and Decentralized Inventory Control

4.1 The Centralized System

Our performance benchmark for the decentralized supply chain is the integrated/centralized system where both the supplier and the retailer belong to the same firm and their inventories are managed by a centralized planning mechanism.

Since replenishments from the supplier to the retailer are assumed to be instantaneous, customer demand can be satisfied from the retailer's inventory or the supplier's inventory. The centralized planner is therefore indifferent in the positioning of inventory from the customer demand satisfaction point of view but because it is less costly to keep inventory at the supplier, it is clear that it is optimal to keep all inventory at the supplier (except in the case of $h_r = h_s$. In that particular case, any partition of the same total inventory would generate the same inventory cost). In order to optimize the centralized system we can then assume that all inventory is held at the supplier and the retailer is only a transition point. The resulting model is a single-stage make-to-stock queue (see Buzacott et Shanthikumar, 1993). Let the supplier's (or the system's) base stock level be S (where S is a non-negative integer).

Following Buzacott and Shanthikumar, let N be the stationary number of manufacturing orders waiting at the supplier's facility. Under base stock policies, N has the same distribution as the stationary queue length of the M/M/1 queue. This leads to the following expressions for the expected inventory level and the expected number of orders as a function of the base-stock level S:

$$E[I] = S - \frac{\rho(1-\rho^S)}{1-\rho} \tag{1}$$

and

$$E[B] = \frac{\rho^{S+1}}{1-\rho}$$

The total average inventory-related cost of the centralized system can then be expressed as : $C(S) = h_s E[I] + bE[B]$

As for the optimal base stock level, let $F_N()$ be the cumulative distribution function of N, it is well known that: $S^* = \min\left\{S : F_N(S) \geq \frac{b}{h_s+b}\right\}$ where $F_N()$ is the cumulative probability distribution of N. This leads to the following expression under exponential assumptions:

$$S^* = \left\lfloor \frac{\ln\left(\frac{h_s}{h_s+b}\right)}{\ln \rho} \right\rfloor \tag{2}$$

where $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.

4.2 The Decentralized System

In the decentralized system, the supplier and the retailer select their individual base stock levels S_s and S_r in order to minimize their own cost functions. To analyze this system, we first obtain the average costs for the two players for any given pair of base stock levels (S_s,S_r). We then investigate the best response of each player to the choice of a given base stock level by the other player. Finally, we investigate the equilibrium strategies.

Lemma 1 *The expected inventory level, $E[I_s]$, of the supplier, the expected inventory level, $E[I_r]$, of the retailer, and $E[B(S_s, S_r)]$ be the average backorder level in the decentralized system are functions of the base stock levels S_s and S_r which can be expressed as:*

$$E[I_s(S_s, S_r)] = S_s - \frac{\rho(1-\rho^{Ss})}{1-\rho} \tag{3}$$

$$E[I_r(S_s, S_r)] = S_r - \frac{\rho^{Ss+1}(1-\rho^{Sr})}{1-\rho} \tag{4}$$

and

$$E[B(S_s, S_r)] = \frac{\rho^{S_s+S_r+1}}{1-\rho} \tag{5}$$

Proof: All proofs can be found in the Appendix. □

Let us also note that $E[I_s(S_s, S_r)]$, $E[I_r(S_s, S_r)]$, and $E[B(S_s, S_r)]$ are all (integer) convex functions in S_s and in S_r.

Next, we investigate the cost structure of the two parties. Using Lemma 1, we can express the supplier's average cost as:

$$C_s\left(S_s, S_r\right) = h_s\left(S_s - \frac{\rho-\rho^{S_s+1}}{1-\rho}\right) + b_s\frac{\rho^{S_s+S_r+1}}{1-\rho} \tag{6}$$

and the retailer's average cost as:

$$C_r(S_s, S_r) = h_r\left(S_r - \frac{\rho^{S_s+1}(1-\rho^{S_r})}{1-\rho}\right) + b_r\frac{\rho^{S_s+S_r+1}}{1-\rho} \tag{7}$$

We now turn to the best responses of each player to the selection of any base stock level by the other player.

The best responses $S_r^*(S_s)$ and $S_s^*(S_r)$ are defined as follows:

$$S_r^*(S_s) = \arg\min_x C_r(S_s, x)$$

and

$$S_s^*(S_r) = \arg\min_x C_s(x, S_r).$$

where x is a non-negative integer.

Proposition 1 *The best response of the supplier to the choice of a base stock level of S_r by the retailer, $S_s^*(S_r)$ and the best response of the retailer to the choice of a base stock level of S_s by the supplier, $S_r^*(S_s)$, are given by:*

$$S_s^*(S_r) = \left\lfloor \frac{\ln\left(\frac{h_s}{h_s + b_s \rho^{S_r}}\right)}{\ln \rho} \right\rfloor . \tag{8}$$

and

$$S_r^*(S_s) = \left\lfloor \frac{\ln\left(\frac{h_r}{h_r + b_r}\right)}{\ln \rho} - S_s \right\rfloor^+ \tag{9}$$

where the operator $(x)^+$ is defined as: max(x,0).

An immediate consequence of Proposition 1 is that both $S_r^*(S_s)$ and $S_s^*(S_r)$ are less than or equal to the base stock level of the centralized system S^*. This enables us to consider only a finite set of strategies (bounded by S^*) for both players: $\Omega_S=\{0,..,S^*\}$ and $\Omega_R=\{0,..,S^*\}$.

The best response functions only depend on the parameters of the problem but the equilibrium pair of strategies depend on the assumptions on the game. Next, we analyze the equilibrium behaviour of the inventory strategies under different assumptions.

4.3 The Nash game and the Nash equilibrium

In the Nash game the supplier and the retailer simultaneously choose their base stock levels with the objective of minimizing their respective costs per unit time. We assume that this is a one-shot game: the players select the base stock levels at time zero and maintain them over an infinite horizon. The strategies of the players are their base stock levels S_s and S_r. As discussed in the previous section, it is sufficient to consider finite sets of strategies for both players: $\Omega_S=\{0,..,S^*\}$ and $\Omega_R=\{0,..,S^*\}$.

A pair of base stock levels (S_s^*, S_r^*) is a Nash equilibrium if neither the supplier nor the retailer can gain from a unilateral deviation from these base stock levels (i.e. $S_s^* = S_s^*(S_r^*)$ and $S_r^* = S_r^*(S_s^*)$).

There are two important issues regarding the determination of the Nash equilibrium: existence and uniqueness. The existence issue is addressed in the appendix, where we show that a Nash equilibrium always exists in this game. On the other hand, it turns out that the uniqueness of the equilibrium is not always guaranteed. This is not surprising since the strategy space is discrete. In order to have a concrete example of multiple Nash equilibria, consider the following set of parameters: $\rho = 0.9$, $h = 1$,$b = 10$ and $b_r = 6$. There are two Nash equilibria in this case: (7,11) and (8,10). The respective costs are: for the first equilibrium $C_s(7, 11) = 7.708$, $C_r(7, 11) = 16.151$, and for the second equilibrium $C_s(8, 10) = 8.278$, $C_r(8, 10) = 15.581$. The expected total cost generated by the decentralized system is then 23.959 regardless of the equilibrium selected (since the total inventory level is identical in both cases). Let us also note for completeness that, the corresponding centralized system has an optimal base stock level of 22, and generates an expected cost of 22.749. Fortunately, as also reported in Cachon (1999), the occurrence of multiple Nash equilibria is rare and when multiple equilibria exist the difference in respective payoffs between different equilibria seem to be relatively small as in the previous example. Since the qualitative results seem to be robust, we avoid the rather complicated debate on which of the equilibria may be realized. For reporting purposes however, we report the equilibrium that minimizes the total supply chain cost.

Next, we look into the structure of the Nash equilibria. The structure of the Nash equilibrium strategy depends on the ratios b_r/h_r and b_s/h_s as outlined in the following proposition:

Proposition 2 *The Nash equilibrium pair of strategies (S_s^* , S_r^*) has the following properties:*
i. if $b_r/h_r \leq b_s/h_s$ then:

$$S_r^* = 0, \;\; S_s^* = \left\lfloor \frac{\ln\left(\frac{h_s}{h_s+b_s}\right)}{\ln \rho} \right\rfloor \quad ,$$

ii. if $b_r/h_r > b_s/h_s$ then S_s^* and S_r^* are such that:

$$S_r^* + S_s^* = \left\lfloor \frac{\ln\left(\frac{h_r}{h_r+b_r}\right)}{\ln \rho} \right\rfloor$$

.

Remark: An explicit formula for the second case of the lemma ($b_r/h_r > b_s/h_s$) can be obtained if the condition that the base stock levels have to be integers are relaxed. The resulting continuous approximation is:

$$S_s^* = \frac{\ln\left(1 - \frac{h_r b_s}{h_s(h_r+b_r)}\right)}{\ln \rho}, \quad S_r^* = \frac{\ln\left(\frac{h_s h_r}{h_s(h_r+b_r)-h_r b_s}\right)}{\ln \rho}.$$

Let us discuss some of the qualitative consequences of Proposition 2. When $b_r/h_r < b_s/h_s$, the supplier is the more concerned party in terms of backorder to holding cost ratios. The proposition then establishes that under this condition, the supplier holds all the inventory while the retailer operates in a replenish-to-order mode. In addition, the equilibrium base stock level only depends on b_s/h_s (and ρ) but not on b_r/h_r. When $b_r/h_r > b_s/h_s$, in general both parties end up holding some inventory in the equilibrium but the total supply chain inventory is determined only by the ratio b_r/h_r. Finally, let us note that the total equilibrium base stock level can be expressed as the maximum of two quantities as follows:

$$S_r^* + \; S_s^* = \max\left(\left\lfloor \frac{\ln\left(\frac{h_r}{h_r+b_r}\right)}{\ln \rho} \right\rfloor, \left\lfloor \frac{\ln\left(\frac{h_s}{h_s+b_s}\right)}{\ln \rho} \right\rfloor \right). \tag{10}$$

From equation (10), it immediately follows that, at equilibrium, the total base stock level in the decentralized supply chain is always less than or equal to optimal base stock level of the corresponding centralized system. In fact, if we disregard the discrete nature of the base stock levels momentarily, it can be seen that when $h_s < h_r$ the equality only takes place if the supplier pays the full backorder cost of the system ($b_s = b$). For $h_s = h_r = h$, the equality of the base stock levels can only take place when either one of the parties pays the full backorder cost (i.e. $b_s = b$ or $b_r = b$). When both parties take some but not all of the responsibility of backorder costs ($0 < b_s,\ b_r < b$), the supply chain is understocked with respect to its optimal level. These conclusions extend to the total supply chain costs. The decentralized system incurs a higher total cost per unit time than the centralized system unless $b_s = b$ (respectively $b_s = b$ or $b_r = b$ when $h_s = h_r = h$). To summarize these qualitative conclusions, in the Nash equilibrium, the decentralized system is not coordinated except under special cases (one party taking full responsibility of the backorder costs or due to the discrete nature of the optimal base stock levels especially if these levels are low). Finally, the lack of coordination is always due to understocking with respect to the optimal stocking levels of the centralized system.

4.4 The Performance of the Decentralized System

The preceding subsection established that the decentralized system is not coordinated except in particular circumstances. In this section, we investigate the loss of efficiency in the decentralized system with respect to the corresponding centralized system. Let us define the competition penalty as follows:

$$CP = \frac{(C_s(S_s^*, S_r^*) + C_r(S_s^*, S_r^*)) - C(S^*)}{C(S^*)}$$

Below, we denote by $CP(b_r)$ the competition penalty as a function of the retailer's backorder cost b_r, when all other parameters are held fixed.

Proposition 3 *If $h_s = h_r = h$ then:*
i. CP(b_r) is symmetrical with respect to $b_r = b_s = b/2$ in the interval $[0, b]$
ii. CP(b_r) is non-decreasing in the interval $[0, b/2]$ (and non-increasing in the interval $[b/2, b]$).

Proposition 3 establishes that the highest CP occurs when each party is partially (50%) responsible from the backorder costs. This is in accord with the results in Cachon for the lost sales system but in contrast with the results in Cachon and Zipkin (1999) and Caldentey and Wein (2003) where the worst CP occurs at extreme backorder cost allocations. The explanation for this discrepancy is as follows: in our setting the positioning of the inventory is irrelevant as long as the total inventory in the supply chain is sufficient. This makes extreme backorder cost allocations efficient in our setting because the more responsible party can always compensate for the other. However, in an equal backorder cost allocation, both parties tend to understock with respect to ideal levels which turns out to be the worst situation for the supply chain.

Figures 1 and 2 depict the result of Proposition 3 on numerical examples. In particular, Figure 1 presents the competition penalty curves for three different systems differing in their total backorder costs (b = 10, 100 or 1000). While the total supply chain cost is increasing b, the competition penalty seems to be decreasing. Increasing backorder cost to holding cost ratios makes both parties more responsible about total inventory levels in the supply chain. This is also in accord with a corresponding observation in Cachon (1999) where increasing profit margin to holding cost ratios decrease inefficiency. It should be noted that, in our setting, only major increases in b/h have a notable effect on the CP since inventories are roughly proportional to $\log(b/h)$

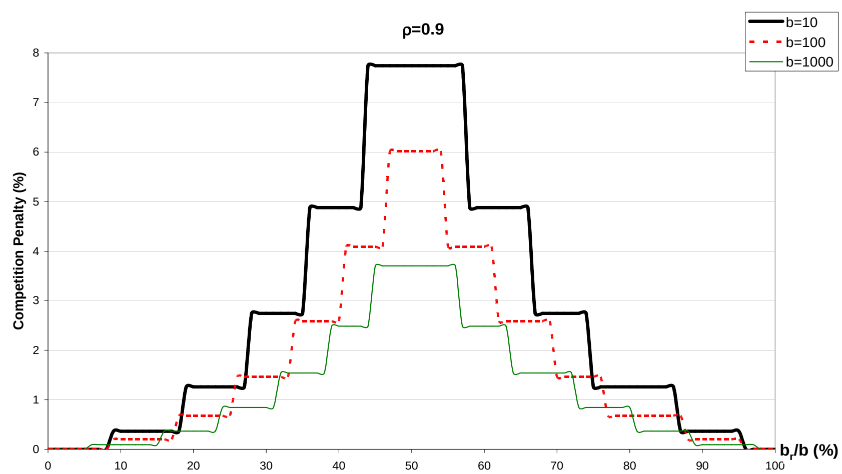

Figure 1: The competition penalty as a function of b_r/b for different values of b ($h_s = h_r = 1$)

Figure 2 presents the competition penalty curves for three different systems differing in utilization levels (ρ = 0.8, 0.9 or 0.98). Although the total supply chain cost is increasing in ρ (is roughly proportional to $-1/\ln\rho$), the competition penalty does not seem to be strongly affected by ρ. Regardless of the utilization rate, the parties respond in similar ways in terms of relative inventories in a decentralized setting. This is in contrast with the observations in Cachon (1999) where the utilization rate has a significant impact. Proposition 2 sheds some light onto this interesting behaviour. A close examination of the equilibrium structure reveals that the total base stock level in the supply chain is always proportional $-1/\ln\rho$. This implies that the ratio of the total inventory in the decentralized system to the total inventory in the centralized system (i.e. $(S_s^* + S_r^*)/S^*$) does not depend on ρ.

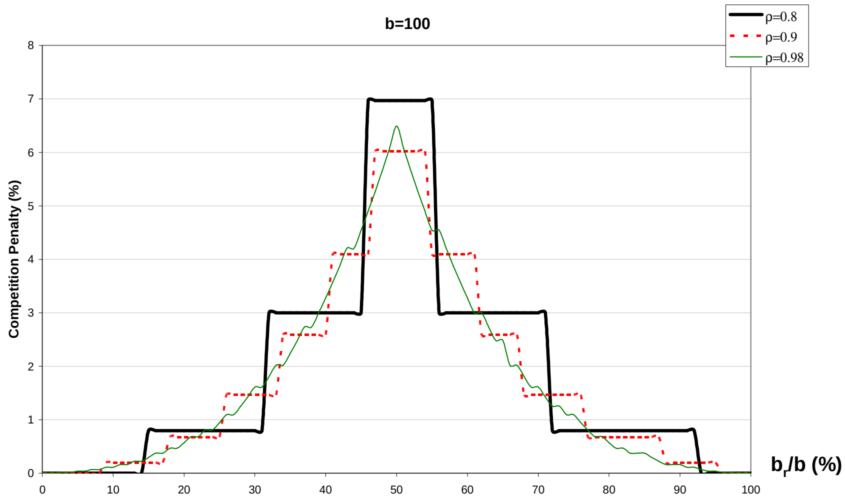

Figure 2: The competition penalty as a function of b_r/b for different utilization rates($h_s = h_r = 1$)

Figures 1 and 2 also indicate that the competition penalty is relatively modest when $h_s = h_r$ (less than 10% in the worst situation in these figures). This seems to be the general pattern in other examples as well. There are nevertheless some special extreme cases in which the competition penalty can be more significant. For instance, let $\lambda = 0.1$, $\mu = 1$, $b = 20 - \epsilon$ ($\epsilon > 0$), $h_s = h_r = 1$ and $b_r = b_s = (1/2)b$. In this particular case, it can be observed that the competition penalty approaches 100% as ϵ goes to zero. The reason for this is as follows: the optimal centralized base stock level is one unit whereas the decentralized supply chain has an optimal base stock level of zero. This small difference in the base stock levels reflects into a significant relative difference for optimal costs. Having mentioned this special case, we maintain our focus on systems with relatively high base stock levels in the rest of the analysis.

Next, we investigate systems with $h_r > h_s$. In this case, the structure of the competition penalty (as a function of b_r/b) is somewhat different. In fact, the global symmetry of the CP curve is lost but there is still some local symmetry around $b_r = (h_r/h_s)b_s$ where the penalty reaches its maximum. To the left of this point, the curve is similar to the ones in Figures 1 and 2 with zero penalty at $b_r = 0$ and then increasing until $b_r = (h_r/h_s)b_s$. To the right of this point, the curve is decreasing as can be seen in Figures 3 and 4.

Other than the structure of the curves, Figures 3 and 4 confirm some of the general results observed in Figures 1 and 2. For example, the competition penalty is only slightly affected by the utilization rate as seen in both figures. The effect of total backorder costs (b) when we compare the two figures (3 and 4) is now more complicated. To the left of the point $b_r = (h_r/h_s)b_s$, the retailer does not hold any inventory. It turns out that in this region the CP is decreasing in

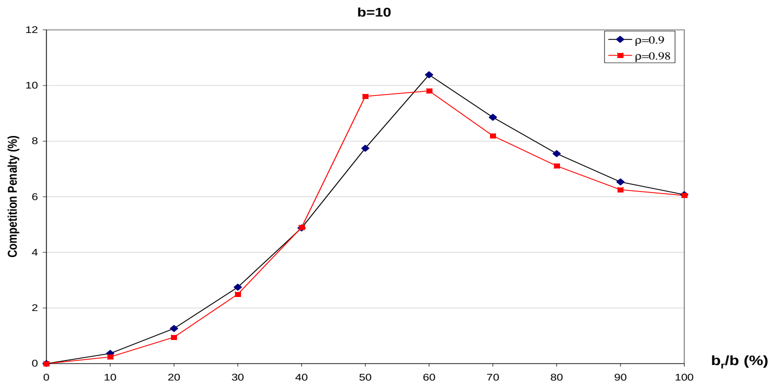

Figure 3: The competition penalty as a function of b_r/b for $b = 10$ at different utilization rates ($h_s = 1$ $h_r = 1.1$)

b (when b_r/b is constant). The maximum CP occurs at $b_r = (h_r/h_s)b_s$ and does not seem to depend on b. On the other hand, to the right of $b_r = (h_r/h_s)b_s$ (where both parties hold some inventory in general), the CP is increasing in b (when b_r/b is constant).

5 Coordination of the Supply Chain

It is clear that, in general, the decentralized operation is less efficient in terms of the total supply chain profit. In this section we investigate contracts which can be used to improve the decentralized operation. We focus, once again, on the Nash game where the supplier and the retailer simultaneously select their base stock levels. As in Cachon and Zipkin (1999), Cachon (1999) and Caldentey and Wein (2003) we focus on contracts with linear transfer payments. An important starting point comes from the earlier established fact that the decentralized system always holds less total inventory than the centralized system. Any coordinating contract must then increase the total amount of inventory in the system. Below we describe a general holding cost subsidy contract and investigate its properties. We also briefly discuss some interesting and useful special cases of the holding cost subsidy contract. A more extensive treatment of these issues can be found in Jemaï and Karaesmen (2004).

Let us investigate a two-parameter contract that we refer to as "the generalized α" (α_G) contract which allows both way holding cost subsidies between the supplier and the retailer. The contract has two parameters: α_S and α_R. For simplicity, we only present the identical holding costs case ($h_s = h_r = h$) but the extension to the general case is straightforward. The

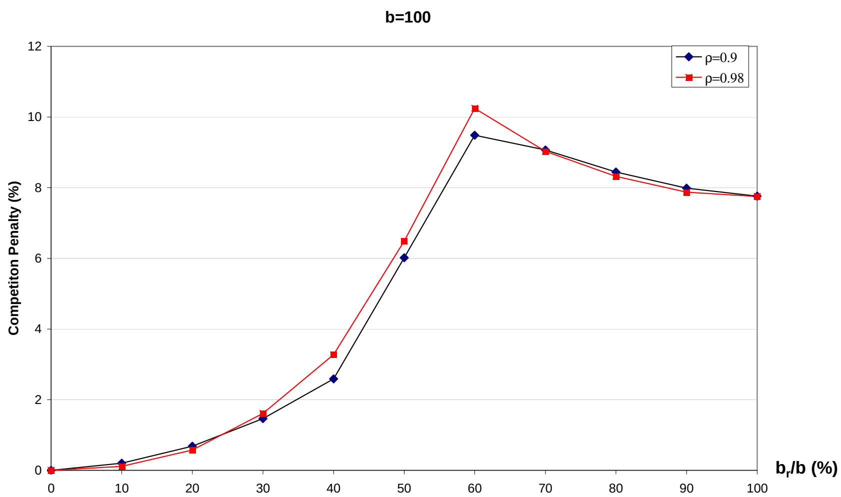

Figure 4: The competition penalty as a function of b_r/b for $b = 100$ at different utilization rates($h_s = 1$ $h_r = 1.1$)

corresponding cost functions are:

$$C_s(S_s, S_r) = h(1 - \alpha_R)E[I_s(S_s, S_r)] + \alpha_S\, h\, E[I_r(S_s, S_r)] + b_s E[B(S_s + S_r)]$$

and

$$C_r(S_s, S_r) = h(1 - \alpha_S)E[I_r(S_s, S_r)] + \alpha_R h_s E[I_s(S_s, S_r)] + b_r E[B(S_s + S_r)]$$

The analysis of this contract is similar to the analysis of the decentralized case without a contract. The response functions can be written as:

$$S_s^*(S_r) = \left\lfloor \frac{\ln \frac{(1-\alpha_R)h}{(1-\alpha_R)h - \alpha_S h + (b_s + \alpha_S\, h)\, \rho^{S_r}}}{\ln \rho} \right\rfloor \tag{11}$$

and

$$S_r^*(S_s) = \left\lfloor \frac{\ln \frac{(1-\alpha_S)h}{(1-\alpha_S)h + b_r}}{\ln \rho} - S_s \right\rfloor^+ \tag{12}$$

The next question is whether such a contract can coordinate the supply chain and if so for which values of the contract parameters. To investigate this issue let us start by two special cases that are of interest on their own. First let us assume that $\alpha_R = 0$. This is the case where only the supplier subsidizes the retailer's inventory costs. As pointed out by Cachon (1999), this is the analogue of a buyback contract that is known to coordinate the supply chain in a single-period newsvendor setting (see Pasternack 1985).

Then following a similar approach as in Proposition 2, when $\alpha_R = 0$ the Nash equilibrium strategies are obtained as:

If $b_r < (1-\alpha_S)b_s$ then $S_r^* = 0$ and

$$S_s^* = \left\lfloor \frac{\ln\left(\frac{h}{h+b_s}\right)}{\ln \rho} \right\rfloor, \tag{13}$$

and if $b_r \geq (1-\alpha_S)b_s$ then S_s^* and S_r^* are related by:

$$S_r^* + S_s^* = \left\lfloor \frac{\ln\left(\frac{(1-\alpha_S)h}{(1-\alpha_S)h+b_r}\right)}{\ln \rho} \right\rfloor. \tag{14}$$

The next proposition establishes a condition for the parameter α_S which makes the contract coordinating.

Proposition 4 *Under the α_G-contract with $\alpha_R = 0$, when $\alpha_S = 1-(b_r/b)$ then the corresponding Nash equilibrium (S_s^*, S_r^*) is such that $S_s^* + S_r^* = S^*$.*

Pursuing the findings of Proposition 4, a closer examination of equations (11) and (12) establishes that $(0, S^*)$ is always a Nash equilibrium for the coordinating α_G contract with $\alpha_R = 0$. This solution, however, may not be unique. By Proposition 4, this contract always coordinates the supply chain similar to the analogous buyback contract in the newsvendor setting. Next, we investigate the flexibility of this contract in the allocation of the benefits of coordination. For any given b_r, there is a unique value of α_S that coordinates the supply chain allowing only a unique partition of the savings due to coordination. This parallels again the results for the newsvendor model under the buyback contract: for a fixed wholesale price, there is a unique coordinating buyback price which results in a unique split of the coordination benefits. On the other hand, if the wholesale price is varied along with the unit buyback price, one obtains infinitely many coordinating buyback contracts with different splits of the coordination benefits. This suggests an alternative interpretation of the contract. Let b_r be an additional contract parameter. This, of course, requires renegotiating the initial backorder cost allocation at the same time with the holding cost subsidy rate α_S. In this case, for any α_S, there always exists a b_r which leads to coordination, and different coordinating (α_S, b_r) pairs allow different allocations of the supply chain profit. This contract is very similar to a buyback contract with the flexibility of negotiating the unit wholesale price. The corresponding (α_S, b_r) contract is always coordinating and flexible in terms of allocation of savings. Moreover, just like a coordinating buyback contract, it has the attractive feature that it does not depend on the demand rate λ.

The α_G contract can also coordinate the supply chain with $\alpha_S = 0$ if $\alpha_R = b_r/b$. This version of the contract also suffers from the same drawbacks in terms of the allocation of the supply chain profits.

Finally, the α_G contract coordinates the supply chain for a variety of other parameter settings requiring two-way monetary transfers. It turns out that the Nash equilibrium always has the structure: $(S_s^*, S^* - S_s^*)$ for $\alpha_S = b_s/b$ and $\alpha_R \leq b_r/b$ or for $\alpha_R = b_r/b$ and $\alpha_S \leq b_s/b$. For instance, if $\alpha_S = b_s/b$, any $\alpha_R \leq b_r/b$ leads to coordination. Of course unless $\alpha_R = 0$, the result is a more complicated contract that requires two-way monetary transfers. On the other hand, this enables a more diverse allocation of the coordination savings through the choice of the right parameters. Unfortunately, a completely arbitrary allocation of the savings cannot be achieved unless b_r is used as a third contract parameter.

6 Stackelberg games and Stackelberg equilibria

In this section, we focus on Stackelberg games where one of the player selects (and announces) its base stock level before the other one. This gives an advantage to the first-moving player because its choice takes into account the anticipated best response of the other player. This provides a coherent framework to model a situation where one of the players dominate the inventory positioning decisions in the supply chain.

Let us start by the case where the supplier is the Stackelberg leader: in this case, the supplier selects S_s^* that minimizes $C_s(S_s, S_r)$ with the knowledge that the retailer will select S_r^* that minimizes $C_r(S_s, S_r)$. Then:

$$S_s^* = \arg\min_{S_s}(C_s(S_s, S_r^*(S_s))) \text{ and } S_r^* = S_r^*(S_s^*).$$

Similarly when the retailer is the Stackelberg leader, she will select the optimal base stock level given by:

$$S_r^* = \arg\min_{S_r}(C_r(S_s^*(S_r), S_r)) \text{ and } S_s^* = S_s^*(S_r^*).$$

Analytical expressions for the optimal base stock levels in the Stackelberg games are difficult to obtain. On the other hand, numerical results easily follow from Proposition 1. Below, we present a summary of these results focusing on the contrasts with respect to the Nash game.

The CP curves for Stackelberg equilibria manifest certain similarities with the corresponding CP curves for Nash equilibria. Figure 5 presents the CP as a function of the retailer's backorder cost for cases where the Stackelberg leader is respectively the supplier, and the retailer. The CP for the Nash game can also be observed in the same figure. As seen in these figures, when $b_r < b_s$ the Stackelberg equilibrium solution led by the supplier is less efficient than the Stackelberg solution led by the retailer. The situation reverses when $b_r > b_s$ and the Stackelberg equilibrium solution led by the supplier becomes more efficient. In addition when $h_s = h_r = h$, the two

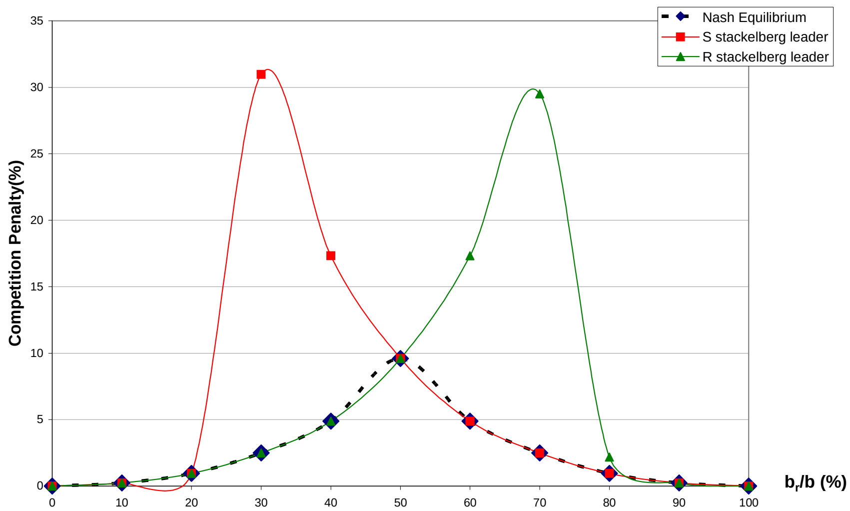

Figure 5: Comparison of the competition penalty for Stackelberg and Nash equilibria ($h_r = h_s = 1,\ \rho = 0.9,\ b = 10$)

different equilibria (the ones led by the supplier and retailer) yield symmetrical competition penalties.

An important difference, with respect to the Nash case, that can be observed in Figures 5 is that the CP can now be extremely significant (exceeding 30%) for certain cases. The most significant efficiency seems to take place as the Stackelberg leader gradually decreases its base stock level to zero while the other player is only moderately (around 30%) responsible for the total backorder cost. On the other hand, the inefficiency is insignificant if either party is almost fully (more than 80%) responsible of the total backorder cost, regardless of the dominant player.

It can be seen in Figure 5 that the Nash equilibrium is always more efficient (in a non-strict sense) than both of the Stackelberg cases. The existence of a dominating player has a non-positive influence on the supply chain performance. More interestingly, the Nash equilibrium always coincides with one of the Stackelberg equilibria (supplier led or retailer led) for any b_r/b (Jemai (2003) gives a proof of this property under the continuous approximation of the base stock levels). This implies that for certain cost structures (when the more responsible party is the Stackelberg leader), the Stackelberg equilibrium coincides with the Nash equilibrium. Inversely, the worst supply chain efficiency is realized when the less responsible party is the Stackelberg leader.

In other examples not reported here, similar results (to the Nash case) on the effects of other parameters (b and ρ) on the CP for Stackelberg games were observed. For instance, the effect of the utilization rate ρ seems to be insignificant. The effect of the backorder cost rate b seems more important, and in particular, the maximum CP is increasing in b but the overall qualitative behavior of the CP appears to be rather robust with respect to the changes in parameters.

7 Conclusion

We investigated the decentralized operation of a capacitated supplier-retailer supply chain using stationary base stock policies for inventory control. We showed that the decentralized system is inefficient because the decentralized supply chain keeps lower base stock levels than what is required for coordination. If there is little holding cost difference between the stages, the relative inefficiency reaches its maximum if each of the two parties is responsible from half of the total backorder cost. The consolation is that the relative inefficiency does not surpass 10% except in extreme cases (such as extremely low base stock levels). Surprisingly, the utilization rate of the supply chain, which has a huge impact on the total supply chain performance, has little influence on the relative inefficiency of decentralized operation. The additional cost due to increased utilization rate seems to be shared fairly and proportionally between the supplier and the retailer. With unequal holding costs, the inefficiency of the supply chain can be much more significant especially if the supplier (who has lower holding costs) is less responsible of the backorder cost than the retailer. Similarly, if one party is more powerful in leading the inventory decisions (i.e. is the Stackelberg leader), the inefficiency of the decentralized operation can be significant.

Our analysis reveals that there are several simple linear contracts that coordinate the supply chain. For instance, a holding cost subsidy by the retailer to the supplier coordinates the supply chain even in the case of unequal holding costs. A common shortcoming of all simple single-parameter contracts is that, unless the backorder cost allocation becomes an additional contract parameter, coordination results in a unique allocation of the savings. Another downside seems to be the issue of voluntary compliance. In general, only one of the two parties is better off with the contract (than without). This implies that additional side payments may be necessary in order to achieve coordination by contracting.

The model considered in this paper assumes negligible transfer times between the supplier and the retailer. While this may restrict the scope of its relevance, it has the virtue that pure inventory ownership (rather than inventory positioning) in the supply chain becomes the focus. Our analysis then sheds light onto inefficiencies of decentralized operation from this point of view. It is interesting to note that in this context simple contracts can coordinate the supply chain.

It would be interesting to investigate the validity of the insights obtained with extensions to non-negligible transfer times and multiple retailers. Although exact analysis for these extensions seems impossible in the case of a capacitated supplier, approximations and simulation could be employed to investigate the effects of stock positioning.

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Appendix: Proofs

Proof of Lemma 1: Under a base stock policy, the inventory level of the supplier in the decentralized system is (stochastically) equal to the inventory level in a corresponding single-stage system with the same base stock level (S_s). Using (1), this leads to expression (3).

Moreover the total (retailer+supplier) inventory level in a decentralized system with base-stock levels S_s and S_r is stochastically equal to the inventory level in a corresponding single-stage system whose base stock level is: $S = S_s + S_r$. Once again, using (1) yields: $E[I_s(S_s, S_r)] + E[I_r(S_s, S_r)] = E[I(S_s + S_r)]$. Expanding the previous expression enables us to reach (4).

Finally, in the decentralized system with instantaneous replenishments, demands that cannot be satisfied from the retailer's inventory can be satisfied from the supplier's inventory with an instantaneous transfer through the retailer. In fact, the decentralized system only backorders an arriving demand when the total inventory in the system runs out. A direct coupling argument then shows that the number of backorders in the decentralized system is stochastically equal to the number of backorders in a corresponding single-stage system whose base stock level is : $S = S_s + S_r$. This leads to the expression in (5). □

Proof of Proposition 1: The supplier's cost function is convex in S_s since the average inventory level and backorder level are both convex in S_s. A direct investigation of the first order condition leads to expression (8) (in fact, the supplier's cost function is very similar to the cost function of the corresponding single stage system with backorder cost b' defined as: $b' = b_s \rho^{Sr}$). Similarly, the retailer's cost function is convex in S_r and is similar to the cost function of the corresponding single-stage system whose base stock level is given by: $S = S_s + S_r$. For the corresponding single-stage system this leads to:

$$S^* = \left\lfloor \frac{\ln\left(\frac{h_r}{h_r+b_r}\right)}{\ln \rho} \right\rfloor^+$$

Hence, for a given supplier base stock level S_s, S_r^* can be expressed as in (9). □

Proof of Proposition 2:

In order to prove part i., let us define the function $\tilde{S}_r^*$ such that :

$$\tilde{S}_r^*(S_s) = \left\lfloor \frac{\ln\left(\frac{h_r}{h_r+b_r}\right)}{\ln \rho} - S_s \right\rfloor$$

Note that $S_r^* = 0$ when $\tilde{S}_r^* < 0$ and $S_r^* = \tilde{S}_r^*$ when $\tilde{S}_r^* \geq 0$. In addition, $\tilde{S}_r^*$ is increasing in b$_r$ and $\tilde{S}_r^* = 0$ when $b_r = b_s h_r / h_s$.

Then for $b_r = b_s h_r / h_s$, $S_r^* = 0$ and

$$S_s^*(S_r) = \left\lfloor \frac{\ln\left(\frac{h_s}{h_s+b_s}\right)}{\ln \rho} \right\rfloor$$

as given by Proposition 1. This proves part i.

In order to prove part ii., let us note that from proposition 1 it can be seen that

$$S_s^*(S_r) \leq \left\lfloor \frac{\ln\left(\frac{h_s}{h_s+b_s}\right)}{\ln \rho} \right\rfloor$$

then

$$\left\lfloor \frac{\ln\left(\frac{h_r}{h_r+b_r}\right)}{\ln \rho} \right\rfloor - S_s^*(S_r) \geq \left\lfloor \frac{\ln\left(\frac{h_r}{h_r+b_r}\right)}{\ln \rho} \right\rfloor - \left\lfloor \frac{\ln\left(\frac{h_s}{h_s+b_s}\right)}{\ln \rho} \right\rfloor$$

which leads to:

$$\tilde{S}_r^* \geq \left\lfloor \frac{\ln\left(\frac{h_r}{h_r+b_r}\right)}{\ln \rho} \right\rfloor - \left\lfloor \frac{\ln\left(\frac{h_s}{h_s+b_s}\right)}{\ln \rho} \right\rfloor.$$

However, for $b_r > b_s h_r / h_s$, we have:

$$\left\lfloor \frac{\ln\left(\frac{h_r}{h_r+b_r}\right)}{\ln \rho} \right\rfloor - \left\lfloor \frac{\ln\left(\frac{h_s}{h_s+b_s}\right)}{\ln \rho} \right\rfloor > 0$$

which implies that $\tilde{S}_r^* > 0$

then

$$S_r^* = \left\lfloor \frac{\ln\left(\frac{h_r}{h_r+b_r}\right)}{\ln \rho} - S_s \right\rfloor$$

and finally

$$S_r^* + S_s^* = \left\lfloor \frac{\ln\left(\frac{h_r}{h_r+b_r}\right)}{\ln \rho} \right\rfloor .$$

which implies the characterization of part ii.

Proof of Proposition 3: Using the definition of the competition penalty, for equal holding costs we can write:

$$CP(b_r) = \frac{C(S_s^* + S_r^*)}{C(S^*)} - 1$$

but $C(S_s^* + S_r^*)$ is given by equation (10) which is symmetrical with respect to $b_r = b_s = b/2$.

In order to prove part ii, we note, using (10), that in the interval $[0, b/2]$ $S_s^* + S_r^*$ is non-increasing in b_r. From proposition 2, $S_s^* + S_r^* \leq S^*$ and the cost function of the centralized system is convex in S and reaches its minimum at S^*. Then $C(S_s^* + S_r^*)$ is increasing in b_r implying that $CP(b_r)$ is non-decreasing. □

Proof of Proposition 4: From equations (13) and (14), it is seen that for any given α_S, there are two different solution structures depending on the comparison between b_r and $(1-\alpha_S)b_s$. For $\alpha_S = 1 - (b_r/b)$, b_r is greater than or equal to $(1-\alpha_S)b_s$, leading to the equilibrium structure in (14). It can then be directly verified that $S_s^* + S_r^* = S^*$ when $\alpha_S = 1 - (b_r/b)$. □

In this part of the appendix, we prove the existence of a pure-strategy Nash equilibrium for the Nash game presented in Section 4.3. The proof will be based on a result by Topkis (1998) which establishes that a pure strategy Nash equilibrium exists in a supermodular game. In the sequel, we prove that our inventory game is indeed supermodular and satisfies the conditions of Topkis (1998).

Definition 1 *The set S is a lattice of $I\!R^m$ if for all $s(x_1, .., x_m \in S$ and $s'(x'_1, .., x'_m \in S$, $s \wedge s' \in S$ and $s \vee s' \in S$ (where $s\wedge s' \equiv (min(x_1, x'_1), \ldots, (min(x_m, x'_m))$ and $s\vee s' \equiv (max(x_1, x'_1), \ldots, (max(x_m, x'_m))$.*

Definition 2 *The function $U_i(S_s, S_r)$ has increasing differences in (S_s, S_r) if for all $(S_s, S'_s) \in SS^2$ and $(S_r, S'_r) \in SR^2$ such that $S_s \geq S'_s$ and $S_r \geq S'_r$*

$$U_i(S_s, S_r) - U_i(S'_s, S_r) \geq U_i(S_s, S'_r) - U_i(S'_s, S'_r).$$

Definition 3 *The function $U_i(S_s, S_r)$ is supermodular in S_s if:*

$$U_i(S_s, S_r) + U_i(S'_s, S_r) \leq U_i(S_s \wedge S'_s, S_r) + U_i(S_s \vee S'_s, S_r) \;\; \forall \;\; (S_s, S'_s) \in SS^2.$$

Let us note that supermodularity is automatically satisfied if SS is one dimensional.

Definition 4 *A game is supermodular if for all players i, S_i is a lattice of $\mathbb{R}^m$, U_i has increasing differences in (S_i, S_{-i}) and U_i is supermodular S_i (where S_{-i} denotes the strategies used by all other players than i).*

Using the above definitions, we will show that our decentralized inventory game is a supermodular game. To this end, let us introduce an order similar to the one used by Milgrom and Roberts (1990) and Cachon (2001).

Let SS and SR be the respective strategy sets of the supplier and the retailer, and let $(S_s, S'_s) \in SS^2$ be two strategies of the supplier. We say S_s is superior to S'_s and write $S_s \underset{o}{\geq} S'_s$ if $S_s \geq S'_s$. Similarly, let $(S_r, S'_r) \in SR^2$ be two strategies of the retailer. We say S_r is superior to S'_r and write $S_r \underset{o}{\geq} S'_r$ if $S_r \geq S'_r$.

Lemma 2 *The Nash inventory game between the supplier and the retailer whose utility functions are given by $U_s(S_s, S_r) = -C_s(S_s, S_r)$ and $U_r(S_s, S_r) = -Cr(S_s, S_r)$ is a supermodular game.*

Proof: We need to check that the inventory game satisfies the conditions of a supermodular game. First, it is easily verified that SS and SR are lattices of $\mathcal{R}$. Second, U_s and U_r have increasing differences in (S_s, S_r) because for all $(S_s, S'_s) \in SS^2$ and $(S_r, S'_r) \in SR^2$ such that $S_s \underset{o}{\geq} S'_s$ and $S_r \underset{o}{\geq} S'_r$ we have:

$$U_s(S_s, S_r) - U_s(S'_s, S_r) \geq U_s(S_s, S'_r) - U_s(S'_s, S'_r)$$

since $S_r \leq S'_r$, and:

$$U_r(S_s, S_r) - U_r(S_s, S'_r) \geq U_r(S'_s, S_r) - U_r(S'_s, S'_r)$$

since $S_s \geq S'_s$ and $\rho^{S'_r} - \rho^{S_r} \leq 0$. Finally $U_s(S_s, S_r)$ (respectively $U_r(S_s, S_r)$) is supermodular in S_s (respectively in S_r) because $S_s \wedge S'_s = \min(S_s, S_r)$ and $S_s \vee S'_s = \max(S_s, S_r)$ (respectively $S_s \wedge S'_s = \min(S_s, S_r)$ and $S_s \vee S'_s = \max(S_s, S_r)$) hence

$$U_i(S_s, S_r) + U_i(S'_s, S_r) = U_i(S_s \wedge S'_s, S_r) + U_i(S_s \vee S'_s, S_r), \quad i = \{S, R\}$$

Supermodularity of the game is a critical requirement for our proof of existence of Nash equilibria. There are, however, some other technical requirements necessary to complete the proof. A complete verification of these requirements can be found in Jemai (2003). Below, we outline the major steps.

Proposition 5 *There exists a Nash equilibrium in the supplier-retailer inventory game.*

Proof: Topkis (1979) established that, if, for all players i, the strategy set S_i is compact, the utility functions U_i are upper semi-continuous in S_i for all S_{-i}, and if the game is supermodular, then there exists a Nash equilibrium. These conditions are verified for our inventory game. In particular, the set of strategies is compact since it is bounded. We have already shown that $SS = \{0, .., S^*\}$ and $SR = \{0, .., S^*\}$ where S^* is the optimal base stock level of the centralized system. SS and SR are finite sets hence all series x_n that converge to x in SS (respectively in SR), hence, the utility functions U_s and U_r are upper semi-continuous. Finally, the game is supermodular by Lemma 2.This establishes that a Nash equilibrium exists in the inventory game. $\square$