An Inventory Model where Customer Demand is Dependent on a Stochastic Price Process

Caner Canyakmaz[*], Fikri Karaesmen[†], and Süleyman Özekici[‡]

Koç University, Department of Industrial Engineering

December 12, 2019

June 2018

Abstract

We investigate the optimal inventory operations of a firm selling an item whose price is driven by an exogenous stochastic price process which consequently affects customer arrivals. This case is typical for retailers that operate in different currencies, or trade products consisting of commodities or components whose prices are subject to market fluctuations. We assume that there is a stochastic input price process for the inventory item which determines purchase and selling prices according to a general selling price function. Customers arrive according to a doubly-stochastic Poisson process which is modulated by the stochastic input price process. We analyze optimal ordering decisions for both backorder and lost-sale cases. We show that under certain conditions a price-dependent base stock policy is optimal. Our analysis is then extended to a price-modulated compound Poisson demand case, and the case with fixed ordering cost where a price-dependent (s, S) policy is optimal. We present a numerical study on the sensitivity of the optimal profits to various parameters of the operational setting and stochastic price process such

[*]ccanyakmaz@ku.edu.tr
[†]fkaraesmen@ku.edu.tr
[‡]sozekici@ku.edu.tr

as price volatility, customer sensitivity to price changes etc. We then make a comparison with a corresponding discrete-time benchmark model that ignores within-period price fluctuations and present the optimality gap when using the benchmark model as an approximation.

Keywords: inventory management, price fluctuations, random selling price, doubly-stochastic Poisson process, modulated demand process

1 Introduction

Price uncertainties are among the most critical challenges that retailers and manufacturers have to face. For instance, companies whose operations require procuring from commodity markets or procuring components from competitive markets are exposed to price fluctuations which experience sharp movements frequently. Unstable economies, supply disruptions due to uncontrolled factors such as earthquakes, strikes, fluctuating exchange rates are all contributing factors to volatile commodity prices or input materials. Considering the fact that in some sectors a significant portion of manufacturers' expenses are due to raw material costs, it is vital for firms to take a variety of risk management measures against undesired price movements. Successful inventory management is an effective approach to mitigate risks due to input or selling price fluctuations. Besides its importance in managing the usual trade-off between holding, shortage and purchase costs, it can create additional value in fluctuating price environments by adjusting the order sizes in response to the price. For instance a manufacturer, in anticipation of raw material price increases, can invest in inventory to avoid high purchase costs and thus benefit from high selling prices in future.

Classical inventory models usually take purchase and selling prices as constants. However, it is clear that for some industries, input prices are volatile. **?** provide a number of real life applications where this is indeed the case. Cash flows of the chemicals manufacturer BASF, for instance, heavily depends on the price of crude oil. Food manufacturer Nestle, suffers from highly volatile coffee prices which are also market driven. Another example is an industrial company purchasing copper (whose price is extremely volatile) in global markets according to a long-term production plan for an end product where copper is one of the main raw materials. It is also common that selling prices in some industries are difficult to predict as well. For instance, a wholesaler that sells in a different currency will bear an exchange rate risk in selling prices. Some industries such as apparel and high technology face the problem of variable selling price due to rapid product substitution and short life cycles. Jewelry retailers which buy and sell products that are made of precious metals or stones such as gold, silver and diamond may reflect the fluctuation in input prices to customers and may charge different prices to customers arriving at different times. Companies may also choose to pass the cost shock to selling prices or absorb the shock itself. This degree of cost pass-through depends on the industry and the level of competition and relative power of the company (**?**). There is, for instance, empirical evidence that cost pass-through from manufacturer to retailer may be substantial in the U.S. coffee industry. **?** state that a 10-percent change in the market price of green coffee beans results in a 3-percent change in both manufacturer and retail prices. It is therefore clear that the price fluctuation and future expectations have a non-trivial influence on how much to buy in each period.

For inputs that are traded in commodity markets, there is a rich literature on modeling commodity price processes (see, for example, **?**, **?** and references therein). Sophisticated models are proposed to take into account both long-term and short-term price volatilities. For certain semi-finished goods and components, manufacturers use their individual forecasting capabilities to model the price process. Despite this focus on understanding input price fluctuations, most inventory models ignore the full effects of such input price volatilities.

In this paper, we focus on an inventory model that can potentially integrate sophisticated input price processes with price-dependent demand. In particular, we propose and investigate a multi-period, single-item, periodic-review inventory control model where we explicitly model a continuous-time stochastic input price process which determines both purchase and selling prices and consequently influences the customer demand. In this environment, the arrival times of customers and the values of random prices are important as they determine the total revenue from sales. Our main contributions to the literature are as follows. First, we capture the effects of continuous stochastic input and selling price fluctuations and their effects on the continuous demand process in a tractable model. The resulting model has both continuous-time and discrete-time components and non-trivial within-period dynamics between fixed time points. Second, we characterize the optimal ordering policy using dynamic programming. Further, our characterization leads to numerically implementable solutions for practically relevant price processes. Using these solutions, we also generate insights on the effect of price volatilities on optimal expected profits and inventory decisions. We also compare our continuous-time model to a corresponding discrete-time benchmark that ignores within-period price fluctuations but models between-period fluctuations, and show that capturing the within-period fluctuations may have significant benefits especially if the prices are volatile.

The rest of the paper is organized as follows. In Section 2, we briefly review the relevant literature. In Section 3, we present our main model with some of its extensions and present the main results. In Section 4, we conduct a sensitivity analysis on the optimal values of the models developed before. Finally, in Section 5 we present our concluding remarks and further research ideas. All of the technical details and proofs are relegated to the Appendix.

2 Literature

There is a long standing literature on inventory management models that incorporates price uncertainties besides demand randomness. Our work mostly relates to the models involving: i) Markov modulated demand, ii) volatile purchase prices, iii) spot market operations, and iv) changing selling prices.

Markov modulated models mostly assume that demand distributions change over time, usually dependent on an external process with discrete state changes. **?** consider an inventory control problem in continuous-time where states evolve as a Markov process and demand arrival rates are state-dependent. **?** investigate a model where both purchasing costs and demand are stochastic and state-dependent. The authors show that a state-dependent (s, S) policy is optimal in the presence of fixed ordering costs. **?** show the optimality of (s, S)-type policies in the presence of lost-sales when demand is Markov modulated. **?** develop a periodic-review inventory model where both supply and the demand processes are modulated by a Markov chain that models the state of the environment. **?** incorporate finite production capacity in a model where supply and demand are Markov modulated. **?**) investigate both pricing and production policies of a firm that faces capacitated supply and fluctuating demand where demand depends on both price and random environment. In our model, an external but continuous price process modulates the demand arrivals.

A number of papers explore the effect of volatile purchase prices on inventory control problems. **?** extends the classical inventory model of **?** by incorporating the random purchase price which is governed by a Markov process. He proves that a price-dependent (s, S) policy is optimal. **?** considers a single-item deterministic demand inventory system. He assumes that at the beginning of each cycle, the ordering price is a random variable with a known distribution function. He derives a policy where it is optimal to order for the next n periods if the price falls into a certain interval. **?** reviews and discusses tools for raw material purchasing problems with fluctuating prices as he highlights the fact that purchasing is a vital part of inventory control. **?** studies a periodic-review inventory problem where unit purchase cost at each stage takes values from a discrete set according to a Markovian transition matrix. **?** consider a multi-period pricing and inventory management problem and show that state-dependent base stock policies are optimal for risk sensitive decision makers which have exponential utility functions. Their results extend to the case where demand and cost parameters are Markov-modulated. **?** study the impact of purchase price volatility in a multi-period stochastic inventory system where input prices at each period are random and independent of the demand distribution. They establish that higher price variability results in lower costs.

Above models assume that demand is a random variable that depends on the price of the product. In our paper, we explicitly model customer arrivals as a stochastic process and track price values to calculate selling prices. In this regard, **?** is closest to our work where the authors investigate a Poisson demand system where the purchase price is a Markov process. They consider both geometric Brownian motion and Ornstein-Uhlenbeck processes for the price and characterize optimal base-stock levels as a series of thresholds and provide an algorithm to calculate them. Following their work, **?** present simple heuristics to calculate those threshold levels efficiently. **?** also incorporates a Poisson arrival process and investigate a continuous-review inventory model where purchase price is a discrete-state Markov process. They show the conditions under which optimal order-up-to levels are monotone decreasing functions of current purchase price. Our paper differs from the above papers because in addition to input price fluctuations, we also model selling price fluctuations and their impact on the demand process. Moreover, in our case, the demand and the revenue within a period depend on the continuous price process which connects the optimal ordering policy to the properties of the price process.

A number of papers consider inventory settings in the presence of inflation or a deterministic continuous price decrease in the context of the Economic Order Quantity (EOQ) model (**?**, **?** and **?**, and **?**). **?** specifically deal with the effect of changing selling price. Referring to the price history of Nokia's two mobile phone models, they consider a single period stochastic demand inventory model with random lead time and continuously decreasing selling price. In contrast to the above papers, we model the case of random demand modulated by a randomly fluctuating price process.

Another related stream of literature considers spot market operations (buying and selling) where prices are constantly changing. **?** consider a multi-period stochastic inventory model where a firm may purchase from both spot and future markets. **?** review models that incorporate spot market procurements with volatile prices in several supply chain operations. **?** consider a multi-period problem of a supplier that has access to a volatile spot market and more stable long-term contractual customers to sell items. In each period, the supplier decides on a production quantity and how much to liquidate in the spot market. **?**, on the other hand, consider a firm that can purchase at any time from a spot market and faces a random demand at a later random time. The firm meets demand as much as possible and salvages the leftovers, if any. They prove that optimal policy is a price-dependent two-threshold policy. **?** compare single-sourcing and dual-sourcing options in a capacity reservation problem with spot market access. **?** studies the warehouse problem of a merchant that is involved in commodity-trading activities. He assumes that in each period, the spot price of the commodity evolves as a Markov process. The key difference of our paper is that we explicitly model the customer demand process in detail and assume that the firm does not have access to an ample spot market. Our model in that sense is more appropriate for retailers who may use commodities or components to manufacture specialized products which are not easily traded in spot markets.

A majority of the models presented in this paper explicitly takes into account customer arrival times and corresponding random selling prices at those times. A similar approach appears in **?** who considers a single-period problem where demand is modeled as a compound renewal process. He assumes that selling price is constant and customers that arrive according to a renewal process demand a random amount of the product. There is no fixed selling period and items are sold until all inventory is depleted. Our model construction is different in the sense that we assume a finite selling season and the selling price is a stochastic process that also modulates the customer arrival process. We also consider multi-period cases and other generalizations.

Overall, the model presented in this paper differs from existing models in the sense that it incorporates the effect of random selling prices by explicitly modeling a stochastic price process, thereby directly modeling the effect of price fluctuations within a planning period. It also relates random purchase and selling prices through a general selling price function unlike classical models that take selling price as constants or as a random variable unrelated to the purchase costs. In addition, instead of having random demands that are realized at the end of sales (or planning) periods, we model the individual customer demand also as a stochastic process that depends on the prevailing stochastic prices. Finally, to our knowledge, there are only few results for the challenging lost sales problem when demand is price dependent and customer arrivals are modeled explicitly to account for random selling prices. We obtain a characterization of the optimal policy under certain restrictions on the price process. The next section presents the details of our model.

3 Inventory Models with Price Fluctuations

In this section, we analyze multi-period and periodic-review inventory models with randomly fluctuating prices. We assume that there are M periods whose lengths are equal to T units of time (without loss of generality) where the firm places an order at the beginning of each sales period. For the models in consideration, we assume that there is a stochastic price process which models the input price of the inventory item. We denote this process with $P = \{P_t; t \geq 0\}$ and assume that it is a continuous-time and time-homogeneous Markov process with state space $\mathbb{R}_+ = [0, \infty)$. Values of the input price process at the beginning of each period (i.e., $P_0, P_T, P_{2T}, ..$) denote the random purchase prices for the firm. We also assume that items are sold to arriving customers according to a nonnegative and deterministic selling price function $f > 0$, where a customer who arrives at time t is charged $f(P_t)$. This general selling price function f may include any potential multiplicative and/or incremental markups that the firm may set. Note that unless f is a constant function, the firm passes any fluctuations in input price of the item to its customers. A constant selling price function is the standard assumption in most basic inventory management models. For now, we do not assume any particular form for the selling price function f. However, in the numerical illustrations presented in Section 4, we assume a multiplicative selling price function $f(p) = \alpha p$ where $\alpha \geq 1$ is firm's proportional markup.

Unlike most of the inventory models that model the random customer demand as a random variable to be realized at the end of each review period, we assume that there is a customer arrival process and it is modulated by the underlying price process. More specifically, we assume that the unit customer demand follows a modulated Poisson process where stochastic arrival rate at time t is $\Lambda_t = \lambda(P_t)$ and $\lambda(.)$ is a deterministic, nonnegative function of random prices. Customers arrive according to this process and at each arrival they demand one unit of the item. This is relaxed in

Section 3.3 as we investigate the compound Poisson case where each customer demands a random amount of the inventory item. We do not necessarily assume that $\lambda(.)$ is a decreasing function of price. Since the firm may potentially deal with materials or commodities whose prices are constantly changing and may be freely traded in markets, the demand for these types of products may not necessarily decrease as price increases. Customers may also be willing to buy in anticipation of future price increases even if prices have already increased. In our setting, since the arrival rate is a function of the stochastic price, we have a stochastic arrival rate process $\Lambda = \{\Lambda_t; t \geq 0\}$ which modulates the customer arrival process. These types of models are referred as doubly-stochastic Poisson processes introduced by **?**, or shortly, Cox processes. If Λ is a deterministic function rather than a stochastic process, one has a nonstationary Poisson process. Since we assume that P is a Markov process, Λ is also Markovian.

We denote the customer demand/purchase process by $N = \{N_t; t \geq 0\}$ where N_t denotes the number of sales by time t and $N_0 = 0$. The arrival times of the customers form a random sequence $S = \{T_n; n \geq 1\}$ where S and N are related as $\{T_n \leq t\} = \{N_t \geq n\}$. Here we remark that equal period length assumption is not a necessity and the models presented in this paper can easily be extended to cover variable period lengths. The structure of the optimal policies remains unchanged, yet the effect of variable period lengths will be reflected in optimal ordering levels. Moreover, we assume that customers do not act strategically in this model.

Figure 1: A realization of the inventory system with sample processes.

An overview of the inventory system with unit demand can be observed in Figure 1 where a sample path is given for both the input price process P and demand process N. Note that, for this particular example, we use a decreasing rate function, and it can be observed that at times where prices are relatively high, inter-arrival times of customers are also high, i.e., they arrive at a lower rate. A potential approximate model would possibly disregard random fluctuations in prices between ordering periods, and estimate the demand distribution based on a deterministic price. For instance, the given sample price path in Figure 1 is drawn from a martingale price process with an initial price of 100 (which also equals to the expected price at any time). If we disregard price fluctuations and calculate expected profits based on the initial price 100, we may be operating with a suboptimal policy. The focus and the challenge of this paper is in modeling the effect of these continuous time price fluctuations within ordering cycles in inventory models. The results of the numerical examples in Section 4 indicate that possible discrete approximations may be off by a significant margin, especially when prices are more volatile and ordering cycles are long. This also implies that discrete approximations are likely to suggest suboptimal ordering policies when inventories are considered. We explore these issues by establishing the theory for the continuous time price model and studying the structure of optimal policies under various assumptions.

In this setup, total revenues from sales in each period are calculated by summing individual revenues generated from arriving customers. The objective of the decision maker is to find an ordering policy that maximizes the expected total discounted profits, by observing current price and inventory levels at the beginning of each period. We assume zero lead time which implies that the entire order is received immediately upon ordering. We mainly investigate two distinct cases where, at first, unsatisfied demand is backordered. Second, we discuss the lost-sale case. We specify the basics of these models in Section 3.1 and Section 3.2, respectively.

Most papers in the literature (see, for instance, **?**, **?**, **?** or **?**) make the assumption that the holding and backorder cost rates are price-independent and linear functions of the inventory or backorder level. This may be a reasonable assumption in a price-fluctuating environment if the firm cannot pass cost increases to its customers (there may be a fixed-price substitute product in the market) or the commodity price is negligible compared to the overall cost of the final product (**?**). We will employ a more general cost rate function and will assume that, for each period, unit inventory holding and unit backorder (or lost-sale) cost rates are general nonnegative functions of the initial price for that period, and use the notation $h(p)$ and $b(p)$, respectively. The reasoning for the holding cost to depend on the initial price is clear as it consists of physical storage costs as well as the opportunity costs for the inventory investments and opportunity cost is driven by the purchase prices. However, to account for the items that are carried over in inventory for several periods, one may need to keep track of all historical price and inventory states, which leads to a highly intractable model. Similarly, it is possible that backorder costs also depend both on the selling price of the item and the length of the backorder period for any customer, which is highly complex to incorporate in a simple model. However, since the distribution of both selling prices and customer demand processes during a period are determined by the observed price at the beginning of each period, one can approximate the backorder and holding costs as functions of observed prices at the beginning of the periods. Note that typical constant unit holding and backorder costs are just special cases where $h(p) = h$ and $b(p) = b$ for some constants h and b. Finally, we also remark that the structure of optimal inventory policies still remain valid under modified assumptions if we have a price-dependent total holding and backorder cost function that is not linear but convex (as in **?**).

In the following sections we use x and p to denote generic inventory level and price states and y to denote the inventory level after ordering. r denotes the interest rate per unit time which is used to discount all monetary values and, for notational convenience, we use $\gamma = e^{-rT}$ to denote the periodic discount factor (for a period length of T). The next section gives the specifics of the backorder model.

3.1 Backorder Model

In this section, we allow the backordering of customer demands in case of inventory shortage. Our assumption in this case is that in the case of backorder, the selling prices are set and the revenue is received at the time of customer arrival rather than at the time of actual product delivery. In other words, any unsatisfied customer is charged at the time of arrival. We assume that the backordered demand is satisfied at the beginning of the next period. At every period, the decision maker observes the current price p and inventory level x to make an ordering decision which maximizes his expected immediate and subsequent discounted profits. Since unit demands arrive according to a doubly-stochastic Poisson process which is modulated by a Markovian price process and length of intervals are the same, probability distribution of total sales in any interval only depends on the initial price at the beginning of that period. In other words, they are conditionally independent. More specifically, given the initial price P_0, the probability distribution of total demand until time t is a Poisson random variable with random mean measure, i.e.,

$$P\{N_t = k \mid P_0\} = E\left[\frac{e^{-M_t} M_t^k}{k!} \mid P_0\right]$$

where $M_t = \int_0^t \lambda(P_s)\, ds$ is the expected number of arrivals until time t given random prices. Note that in case each customer demands one unit of the item, the expected total demand during time interval $[0,t]$ is given by

$$E\left[N_t|P_0\right] = E\left[M_t|P_0\right] = \int_0^t E\left[\lambda\left(P_s\right)|P_0\right] ds.$$

We also do not put any restriction on the price process except for the Markov property. There is no bound on the state space of the price process, hence its state space is $\mathbb{R}_+$. Since, for now, we assume that each demand is of size 1, the state space for inventory level in backorder case is $\mathbb{Z}$, i.e., set of integers.

For both backorder and lost-sales models, the total revenue in each period is calculated by summing sales revenue from each customer. However, based on the particular backorder setting we investigate, the item is sold to each arriving customer regardless of stock availability and each backordered customer yields a backorder and repurchase cost. Let us define the total discounted revenue collected in time interval $[0,t]$ for any $t > 0$ as

$$R_t = \sum_{n=1}^{N_t} e^{-rT_n} f\left(P_{T_n}\right). \tag{1}$$

Note that $f\left(P_{T_n}\right)$ is the selling price for the nth customer who arrived since the beginning of the period and e^{-rT_n} is to discount the unit revenue to the beginning of the period. Summation is performed until the arrival of last customer, i.e., N_tth customer where N_t is the total number of customers who arrived by time t. Let us also define expected total discounted revenue during $[0,t]$ as a function of initial price by $r_t(p) = E\left[R_t \mid P_0 = p\right]$. Note that the expectation in (1) is taken with respect to the random components; number of arrivals N_t, customer arrival times (the former can be obtained from the latter) and random prices during $[0,t]$, i.e., $\{P_s; s \in [0,t]\}$. The expected discounted revenue function can be computed as:

$$r_t(p) = E\left[R_t \mid P_0 = p\right] = \int_0^t e^{-rs} E\left[f\left(P_s\right)\lambda\left(P_s\right)|P_0 = p\right] ds. \tag{2}$$

The complete derivation of (2) is given in the Appendix. Note that if λ is constant, i.e., customers arrive according to a regular Poisson process with constant rate, then $r_t(p) = \lambda t \overline{f}_t(p)$ where $\overline{f}_t(p) = \int_0^t e^{-rs} E\left[f\left(P_s\right)|P_0 = p\right] ds$ is the average discounted selling price and λt is the expected number of customers that arrived by time t.

The dynamics of the model are as follows. At the beginning of any period, if the current inventory level and purchase price are x and p respectively, and order-up-to level decision is $y \geq x$, the immediate expected discounted profit for the period of fixed duration T is

$$g\left(y;x,p\right)=-p\left(y-x\right)+r\left(p\right)-c\left(y;p\right) \tag{3}$$

where we set $r(p)=r_T\left(p\right)$ and

$$c\left(y;p\right)=E\left[b\left(p\right)\left(N_T-y\right)^{+}+h\left(p\right)\left(y-N_T\right)^{+}\mid P_0=p\right]. \tag{4}$$

with $x^{+}=\max(0,x)$.

The first term in (3) is the total purchase cost for $y-x$ units ordered at the initial price p. Second term is the total revenue collected until time T and the last term is the one-period backorder and inventory holding cost function given in (4) in which a cost of $b\left(p\right)\geq 0$ is charged for each unit backordered and a cost of $h\left(p\right)\geq 0$ is charged for every remaining unit. Note that N_T denotes the number of arrivals during the period and one-period expected profit is independent of the period. This is due to the fact that conditional random prices $P_{kT+T_n}|P_{kT}\overset{d}{=}P_{T_n}|P_0$ have the same distribution for any period k since the price process is assumed to be Markovian and time-homogeneous. This in turn implies that N is also Markovian and its distribution depends only on the observed price p.

In the last period, without loss of generality, we assume that all remaining items are valueless, i.e., there is no salvage value. However, if there are backordered customers, it is required to fulfill their order at the prevailing purchase price. Please note that the results are also valid if the retailer has the opportunity to salvage the remaining items in the last period.

We assume that the decision maker is risk-neutral and aims to maximize the expected total discounted profits. We use dynamic programming to solve this problem to optimality. We define the value function $V_k(x,p)$ as the maximum total expected discounted profit for periods from k to M if the initial inventory is x and price is p. We also define the expected discounted one-period profit function (assuming no initial inventory, i.e., $x=0$) given observed price p as

$$g(y;p)=-py+r\left(p\right)-c(y;p). \tag{5}$$

Then, the dynamic programming equation (DPE) is

$$V_k(x,p) = \max_{y \geq x} \{g(y;p) + \gamma \Psi_k(y;p)\} + px = \max_{y \geq x} G_k(y;p) + px \tag{6}$$

where the expected discounted total future profits for the remaining periods is

$$\Psi_k(y;p) = E\left[V_{k+1}(y - N_T, P_T)|P_0 = p\right] \tag{7}$$

and

$$G_k(y;p) = g(y;p) + \gamma \Psi_k(y;p). \tag{8}$$

Note that $\Psi_k(y;p)$ is the expected discounted total future profits for the remaining periods. Since we allow backorders, the inventory level for the next period upon ordering y units is $y - N_T$, which in fact can be negative. Note that for any period k, the objective is to maximize $G_k(y;p)$, which is the sum of the expected one-period profit and expected discounted future profits resulting from the ordering decision. Also, since we assume that the seller serves all arriving customers, the revenue term is independent of decision variable y. Last, for each (x,p) pair, the terminal value function is $V_{M+1}(x,p) = -px^-$ where $x^- = (-x)^+$.

Now we present the structural properties of function $G_k(y;p)$ and the form of the optimal policy. In the following discussion, $\triangle f(x) = f(x+1) - f(x)$ represents the forward difference of a discrete function f.

Theorem 1 *For $0 \leq k \leq M$, $V_k(x,p)$ is concave in x and $G_k(y;p)$ is concave in y for every p and a price-dependent-base-stock policy is optimal, i.e., there exists a base-stock level $S_k(p)$ for each period k such that if the inventory level is less than the base-stock level, it is optimal to raise the inventory up to $S_k(p)$; otherwise, it is optimal to order nothing. Moreover, optimal base-stock level for period k is given by*

$$S_k(p) = \inf\left\{ y \geq 0 : P\left\{N_T \leq y \mid P_0 = p\right\} \geq \frac{-p + b(p) + \gamma \triangle \Psi_k(y;p)}{b(p) + h(p)} \right\}. \tag{9}$$

We establish that a price-dependent-base-stock-type policy is optimal. This is consistent with similar models with discrete dynamics such as **?**. Next, we present a more explicit single-period solution. First, we define the expected discounted price at time t as a function of initial price p as $z_t(p) = e^{-rt}E[P_t \mid P_0 = p]$ and set $z(p) = z_T(p)$.

Corollary 1 *The optimal base-stock level at period M is given by*

$$S_M(p) = \inf\left\{y \geq 0 : E\left[(b(p) + h(p) + \gamma P_T) 1_{\{N_T \leq y\}} \mid P_0 = p\right] \geq -p + b(p) + z(p) \right\}. \quad (10)$$

Observe that for a given value of P_T, $-p + b(p) + \gamma P_T$ is the unit underage cost whereas $p + h(p)$ is the unit overage cost. This implies that the characterization in (10) reduces to well-known newsvendor solution if P_T is independent of N_T, or if stochastic input price process P is constant (hence N and P are independent). Note also that if $-p + b(p) + z(p) \leq 0$, the optimal base-stock level will be $S_M(p) = 0$. Although it does not affect the concavity of the expected profit function, it is reasonable to assume that $-p + b(p) + z(p) \geq 0$. The expression $-p + b(p) + z(p)$ can economically be interpreted as the expected cost of ordering one less unit. If it is negative, it is optimal not to order.

Now, we analyze the behavior of the optimal base-stock level of the last period as we have an explicit formula for $S_M(p)$. It is clear that the stochastic behavior of the input price process conditional on the observed price and behavior of the arrival rate function $\lambda(.)$ play a key role. We make the following three assumptions in which the first two are very plausible for a real-life inventory system and the third can be justified in the context of the specific model setup.

Assumption 1 *P_t stochastically increases in the initial price $P_0 = p$.*

Assumption 2 *$\lambda(.)$ is a decreasing function.*

Assumption 3 *$-p + b(p) + z(p)$ is decreasing in p.*

Assumption 1 requires that the future prices are stochastically higher if the initial price is higher, which is highly intuitive and satisfied by the most practical stochastic price processes. For instance, let us assume that price follows a geometric Brownian motion (gBm) process $P_t = P_0 e^{vt+\sigma W_t}$ with drift v and volatility σ where W_t is a Wiener process with $E[W_t] = 0$ and $Var(W_t) = t$. Then, Assumption 1 trivially holds.

Assumption 2, on the other hand, requires that the deterministic rate function $\lambda(.)$ is a decreasing function of price. Although there may be cases that violate this assumption in volatile markets as explained before, it is a very common assumption in the literature that the customer demand decreases as the price increases. We only need this assumption to show the monotonicity of $S_M(p)$. Price-dependent base-stock policy is an optimal ordering policy regardless of the structure of $\lambda(.)$.

Assumption 3 requires that sum of the expected discounted price increase until time T and the unit backorder cost is decreasing in the initial price. Note that we can interpret both $b(p)$ and $z(p)$ $-p$ as the loss from ordering one less unit. The latter is due to the difference between two successive ordering prices (discounted) while the former is by the definition of backorder cost. Therefore, Assumption 3 essentially implies that total loss from ordering one less unit should be lower for higher initial prices. If $-p + b(p) + z(p)$ does not decrease in initial price p, one can find cases where optimal base-stock level does not decrease as initial price increases.

Theorem 2 *If Assumptions 1, 2 and 3 hold, then $S_M(p)$ is decreasing in initial price p.*

In the proof of Theorem 2, we used a direct approach. It can also be proved by showing that $\triangle G_M(y,p)$ is decreasing in p, i.e., $G_M(y,p)$ is submodular under Assumptions 1, 2 and 3. However, in either approach, a conclusion can not be drawn for intermediate periods $k < M$ as observed price affects several factors including demand distribution and sales revenues. Hence, extracting the effect of initial observed price is difficult for this general model. In fact, this is also consistent with general price-dependent base-stock models where there is no within-period price fluctuations (e.g., **?**).

Here we note that a positive lead time can be also be incorporated in the backorder model and the results mostly extend. First, regardless of a positive lead time, the total expected revenue during a period is the same since each customer is backordered at the prevailing sales price at arrival times. However, assuming that the lead time is less than the length of a sales period, there is a possibility that there are backordered customers during the lead time. Then the total backorder cost (hence the total inventory-related costs defined in (4)) is a function of both the observed inventory level and the ordering decision. More specifically, let $0 < L < T$ denote the lead time. Then, we can write the total expected backorder and holding cost function as

$$c(y;x,p) = E\left[b(p)(N_L - x)^+ + b(p)(N_{[L,T]} - y)^+ + h(p)(y - N_T)^+ \mid P_0 = p\right]$$

where the first term is the total backorder cost during the lead time and the second term is during the remaining time until next period, i.e., in $[L, T]$. Handling this extra term in calculating the value functions does not create any problem since this term is also concave and the optimality of price-dependent base-stock policies remains valid. However, the optimal base-stock levels may change during intermediate periods except for the last-period where it remains unchanged.

3.2 Lost-Sales Model

In this section, we explore the lost sales case where we assume that any arriving customer that can not find an available item is lost. This case is more challenging than the backorder case because the expected revenue now depends on the ordering policy. To our knowledge, few results on the structure of the optimal policy exist for the lost sales case with price-dependent demand even for simpler models.

In analogy with the backorder model, let us write the total expected revenue during a period as a function of initial price p and order-up-to decision y as

$$r(y;p) = E\left[\sum_{n=1}^{N_T \wedge y} e^{-rT_n} \alpha P_{T_n} \mid P_0 = p\right] = \sum_{n=1}^{y} E\left[e^{-rT_n} f(P_{T_n}) 1_{\{T_n \le T\}} \mid P_0 = p\right] \tag{11}$$

where $a \wedge b = \min(a, b)$. Note that only the revenue term is different than the previous model by which we are now collecting the revenues until the firm runs out of inventory, i.e., until the arrival of $(N_T \wedge y)$th customer. The total expected discounted one-period profit can be written similarly as

$$g(y;p) = -py + r(y;p) - c(y;p) \tag{12}$$

where $c(y;p)$ is given in (4). We write the dynamic programming equation for period k as in (6) where $G_k(y;p)$ is given in (8) and with a slight change in the future expected profit which is given as

$$\Psi_k(y;p) = E\left[V_{k+1}((y - N_T)^+, P_T) | P_0 = p\right].$$

Since there is no backordering, the inventory level can not be negative in the next period. It should be zero if the demand turns out to be more than the total inventory in the current period. As in the backorder case, we assume that the salvage price is zero. Therefore, the terminal value function for the lost-sale model is

$$V_{M+1}(x,p) = 0. \tag{13}$$

Next, we present the structural properties of $G_k(y;p)$ and the form of the optimal policy. Note that we can use the transformations $(y - N_T)^+ = y - \sum_{n=1}^{y} 1_{\{T_n \le T\}}$ and $(N_T - y)^+ = N_T - \sum_{n=1}^{y} 1_{\{T_n \le T\}}$. Moreover, $y = \sum_{n=1}^{y} 1$, trivially. Then, (12) can be written as

$$g(y;p) = \sum_{n=1}^{y} E\left[1_{\{T_n \le T\}}\left(e^{-rT_n} f(P_{T_n}) + b(p) + h(p)\right) - p - h(p) \mid P_0 = p\right] - b\left(p\right) E\left[N_T \mid P_0 = p\right]. \tag{14}$$

One-period expected discounted profit function $g(y;p)$ consists of a finite sum where the upper limit of the summation is the decision variable y. and a constant. Clearly, the behavior of this function is directly determined by the behavior of the term inside the summation.

It turns out that expected total profit function in lost-sale case is not necessarily unimodal in order quantity as we illustrate later. The following assumption on the price process ensures that the objective function is concave which is presented in the subsequent theorem.

Assumption 4 $E\left[1_{\{T_n \le T\}}\left(e^{-rT_n} f(P_{T_n}) + b\left(p\right) + h\left(p\right)\right) \mid P_0 = p\right]$ *is decreasing in* n.

Theorem 3 *Under Assumption* (4) *,* $G_k(y;p)$ *is concave in* y *and* $V_k(x;p)$ *is concave in* x *for every* p *and a base-stock policy is optimal, i.e., there exists a base-stock level* $S_k(p)$ *for each period* k *such that if the inventory level is less than the base-stock level, it is optimal to raise the inventory up to* $S_k(p)$*; otherwise, it is optimal to order nothing. Moreover, optimal base-stock level for period* k *is given by*

$$S_k(p) = \inf\left\{ y \ge 0 : P\left\{N_T \le y \mid P_0 = p\right\} \ge \frac{-p + b(p) + E\left[1_{\left\{T_{y+1} \le T\right\}} e^{-rT_{y+1}} f(P_{T_{y+1}}) \mid P_0 = p\right] + \gamma \triangle \Psi_k(y;p)}{b(p) + h(p)} \right\}. \tag{15}$$

For the single-period problem, the optimal order quantity is

$$S_M(p) = \inf\left\{ y \ge 0 : P\left\{N_T \le y \mid P_0 = p\right\} \ge \frac{-p + b(p) + E\left[1_{\left\{T_{y+1} \le T\right\}} e^{-rT_{y+1}} f\left(P_{T_{y+1}}\right) \mid P_0 = p\right]}{b(p) + h(p)} \right\}. \tag{16}$$

Note that Assumption 4 implies that the marginal expected profit is decreasing in n, which is required for $g(y,p)$ to be concave. Although this condition is rather complicated, we next show that a sufficient condition is the expected discounted price $z_t(p)$ being decreasing in time. We give the motivation in the following result.

Proposition 1 *If the expected discounted price $z_t(p)$ given initial price p is decreasing in t, then $E\left[1_{\{T_n \le T\}}\left(e^{-rT_n} f\left(P_{T_n}\right) + b\left(p\right) + h\left(p\right)\right) \mid P_0 = p\right]$ is decreasing in n.*

Proposition 1 is usually easy to verify for most price processes. For instance, for the geometric Brownian motion process given earlier, the expected discounted price at time t is $z_t(p) = pe^{\left(\mu+\frac{1}{2}\sigma^2-r\right)t}$. Observe that if $\mu + \frac{1}{2}\sigma^2 - r \le 0$, then expected discounted price is nonincreasing in time and Assumption 4 is satisfied.

Unfortunately, the situation may be more complicated when Assumption 4 does not hold and its violation may lead to non-base-stock situations even in very simple cases. For instance, consider the following case where $f(p) = 2p$, $b = h = 0$ and $\lambda(t) = 40$, i.e., customer arrivals are Poisson independent of the price process. Also assume that the price process is deterministic but is a function of time given in Figure 2.

Figure 2: A non-base-stock system.

This simple price process yields a non-base-stock system as observed in Figure 2 which plots the total expected profits as a function of order quantity. We observe two critical points, which are local maxima $y^{(1)} = 12$ and $y^{(2)} = 37$. The optimal policy in this case is to order $Q = 12 - x$ units when $0 \le x < 12$, to order nothing when $12 \le x \le 20$, to order $Q = 37 - x$ units when $21 \le x < 37$ and to order nothing when $x \ge 37$.

We also note that the treatment of a positive lead time in this particular lost-sales case appears to be more challenging than the backorder model. Handling the sums that take into account both the total revenue during the lead time and after the lead time appears to be intractable.

Backorder and lost-sale models can be considered as two extreme cases for this inventory system. The reason is that in the backorder case, all unsatisfied customers are assumed to accept backordering with probability 1 and to pay the prevailing selling price. In the lost-sale case, on the other hand, each unsatisfied customer is assumed to be lost completely. Since arrival times and selling prices are explicitly used in the revenue calculation, one can extend these two extreme cases to other models with partial backorders and/or with backordered customers paying at the time of next replenishment. These extensions are mostly combinations of these models. In the former case, for example, one can write the total revenue as a summation similar to (2), but only the terms with unsatisfied demand should be multiplied with a backorder probability.

To be more specific, assume that each customer, independently from each other, agrees to be backordered with probability α in case of shortage. Then, one can rewrite the expected total revenue (2) as

$$r(y;p) = E\left[\sum_{n=1}^{N_T \wedge y} e^{-rT_n} f(P_{T_n}) + \alpha \sum_{n=N_T \wedge y+1}^{N_T} e^{-rT_n} f(P_{T_n}) \mid P_0 = p\right]. \tag{17}$$

The first term is the total revenue collected from customers who are satisfied with the available stock. The second term, on the other hand, is the total revenue collected from only backordered customers. Via a simple manipulation, one can also rewrite (17) as the convex combination of expected revenues from the two extreme scenarios, backorder and lost-sales cases, such that

$$r(y;p) = \alpha E\left[\sum_{n=1}^{N_T} e^{-rT_n} f(P_{T_n}) \mid P_0 = p\right] + (1-\alpha)\, E\left[\sum_{n=1}^{N_T \wedge y} e^{-rT_n} f(P_{T_n}) \mid P_0 = p\right]. \tag{18}$$

Assuming that unit backorder and lost-sale costs are equal, the total expected inventory-related costs can be written exactly as in (4). Then, the analysis is very similar to the lost-sale case as the first term in (18) is a state-dependent constant while the second term is only a scaled version of (11).

In the latter case where backordered customers agree to pay the price at the end of the period, we can write the total expected discounted revenue as

$$r(y;p) = E\left[\sum_{n=1}^{N_T \wedge y} e^{-rT_n} f(P_{T_n}) + (y - N_T)^+ \, e^{-rT} f(P_T) \mid P_0 = p\right]$$

which can also be treated very similarly to the lost-sale model. Note that total expected inventory holding and lost-sale cost also does not change.

3.3 Compound-Poisson Demand Model

In previous sections we assumed that each arriving customer demands a unit of the product. We now extend this model to a case where each arriving customer requires random amounts of the product independent of the arrival process. Individual demand of arriving customers forms an independent and identically distributed random sequence $\{D_n; n \geq 1\}$ which are drawn from a continuous distribution having a cumulative distribution function F. Similar to models analyzed previously, customer arrival process is a doubly-stochastic Poisson process modulated by the input price process P. We again assume that the decision maker sets the order-up-to levels at each T units of time and as the customers arrive, demanded amounts of the item are sold at the corresponding selling price at that time. For this particular model, we also assume that the customers will always require a positive amount, i.e., there is no possibility that they will demand nothing.

The analysis of the backorder case in a compound Poisson model is very similar to the backorder models with unit demand since revenue terms are still independent of the ordering decision. Therefore, we start our analysis with more challenging lost-sales case. For this particular model, we assume that at any period, if the last arriving customer's demand exceeds on-hand inventory, his demand is partially satisfied and stock is emptied. The remaining part of this sale along with future sales in that period are assumed to be lost forever. To this end, we define the following new notation. We let $\overline{D}_n = \sum_{k=1}^{n} D_k$ to denote the cumulative demand including the nth customer. We additionally define $N(y) = \inf\left\{n \geq 1; \overline{D}_n \geq y\right\}$ to denote the sequence order of the last customer who makes a purchase (full or partial) for y units to be depleted. Observe that this quantity is independent of the period length T. It basically indicates how many customers should arrive for the current inventory to be sold. With these new and old notations, we can write the total expected discounted profit in the kth period given initial price p as in (8) where the one-period expected profit now becomes

$$g\left(y;p\right) = -py + r(y;p) - E\left[b\left(p\right)\left(\overline{D}_{N_T} - y\right)^{+} + h\left(p\right)\left(y - \overline{D}_{N_T}\right)^{+} \mid P_0 = p\right] \tag{19}$$

and the new revenue term is

$$r(y;p) = E\left[\sum_{n=1}^{N(y)-1} e^{-rT_n} D_n f(P_{T_n}) 1_{\{T_n \le T\}} + e^{-rT_{N(y)}} \left(y - \overline{D}_{N(y)-1}\right) f(P_{T_{N(y)}}) 1_{\{T_{N(y)} \le T\}}\right]. \quad (20)$$

The first summation inside of the expectation in (20) is the total discounted revenue that is collected from fully satisfied customers who arrived during the period whereas the subsequent term is the revenue collected from the possibly-last customer who is partially satisfied. The other terms are very similar to the previous profit functions. The only distinction is that instead of N_T which we used to denote total amount of demand in a period in the unit demand case, we now write $\overline{D}_{N_T}$ with the same meaning. As in the previous models, we denote expected future profits as

$$\Psi_k(y;p) = E\left[V_{k+1}(\left(y - \overline{D}_{N_T}\right)^+, P_T) | P_0 = p\right]$$

and the value function for period k and the boundary condition as (6) and (13) , respectively.

We first analyze the structural properties of $G_k\left(y;p\right)$ as it is the function that the firm aims to maximize at each period k. However, it is difficult to perform a probabilistic analysis as in the unit demand case since we now have additional random variables such as $N(y)$ and D_n. In addition, we do not have a discrete problem anymore. We perform a sample path analysis on $G_k\left(y;p\right)$. For now, we think of $N(y)$, T_n and D_n as the realizations of these random variables and observe that the firm will only be able to sell an additional infinitesimal amount dy when $T_{N(y)} \le T$ since otherwise the last customer will arrive after this period although we might have a positive amount of inventory. This is due to the definition of $N(y)$. If $T_{N(y)} \le T$, the firm sells dy units with a total revenue of $dy e^{-rT_{N(y)}} P_{T_{N(y)}}$. Therefore, we can write the marginal expected revenue as

$$r'\left(y;p\right) = \lim_{dy \downarrow 0} \frac{r\left(y + dy; p\right) - r(y;p)}{dy} = E\left[e^{-rT_{N(y)}} f(P_{T_{N(y)}}) 1_{\{T_{N(y)} \le T\}}\right]. \quad (21)$$

Note that in this analysis, the possibility that $\overline{D}_{N(y)}$ is exactly y is ruled out. However, this is not an issue since $P\left\{\overline{D}_{N(y)} = y\right\} = 0$ as D_n's are assumed to be continuous random variables and, by definition of $N(y)$, the last customer is always partially satisfied. We remark that in the unit demand lost-sale case, profit-to-go function for any period is concave under Assumption (4). For the compound Poisson demand case, we also need a condition to ensure concavity. To state our theorem, let us assume that the condition in Proposition 1 holds, i.e., the expected discounted price $z_t(p)$ is decreasing in t.

Theorem 4 *If the expected discounted price $z_t(p)$ given initial price p is decreasing in t, then $G_k(y;p)$ is concave in y and $V_k(x;p)$ is concave in x and a base-stock policy is optimal, i.e., there exists a base-stock level $S_k(p)$ for period k such that if the inventory level is less than the base-stock level, it is optimal to raise the inventory up to $S_k(p)$; otherwise, it is optimal to order nothing. Moreover, optimal base-stock level for period k is*

$$S_k(p) = \inf\left\{ y : P\left\{\overline{D}_{N_T} \leq y \mid P_0 = p\right\} \geq \frac{-p+b(p)+E\left[1_{\left\{T_{N(y)}\leq T\right\}} f(P_{T_{N(y)}}) e^{-rT_{N(y)}}\right]+\gamma\Psi_k'(y;p)}{b(p)+h(p)} \right\}. \quad (22)$$

In the case of complete backordering, the extension to compound Poisson demand is rather straightforward. As before, the revenue terms do not depend on the order-up-to decision y. Therefore, the analysis for this extension will be exactly the same as in Section 3.1 when we replace N_T with $\overline{D}_{N_T}$ and take expectations accordingly. In particular, expected revenue function becomes

$$r(p) = E\left[\sum_{n=1}^{N_T} e^{-rT_n} D_n f(P_{T_n})\right] = E[D] \int_0^t e^{-rs} E\left[f(P_s)\lambda(P_s) | P_0 = p\right] ds$$

and expected total backorder and holding cost given in (4) becomes

$$c(y;p) = E\left[b(p)(\overline{D}_{N_T} - y)^+ + h(p)(y - \overline{D}_{N_T})^+ \mid P_0 = p\right].$$

These modifications to the unit-demand backorder model does not alter the structure of the optimal policy as concavity of the corresponding value functions are preserved. The only change is the demand measure $\overline{D}_{N_T}$ which only affects the optimal base-stock levels. Similarly, the expected revenue function is scaled by the constant $E[D]$ which denotes the expected individual demand.

3.4 Fixed Ordering Cost Case

In previous sections, only variable unit purchase costs were incorporated in the profit function. However, it is well-known that independent of the order size, a prevalent fixed cost may be incurred for each order. This may be a fixed cost of using a vehicle of transportation for procurement, etc. Previous analysis can be extended to the case where there is a fixed order cost of $K > 0$ for each positive order amount. In this case, the value function for period k becomes

$$V_k(x,p) = G_k^*(x,p) + px \tag{23}$$

where

$$G_k^*(x,p) = \max\left\{G_k(x;p),\ \max_{y\geq x} G_k(y;p) - K\right\}. \tag{24}$$

The first function inside maximum operator refers to not ordering. In the last period, we assume that there is no fixed cost and the terminal value function is again given by $V_{M+1}(x,p) = -px^-$ for the backorder case and $V_{M+1}(x,p) = 0$ for the lost-sale case. Existence of a fixed order cost fundamentally changes the structure of the problem since one does not necessarily have concave profit functions as in the linear order cost case. Therefore, a base-stock policy is not optimal for this case. For our problem, the profit-to-go function that is maximized at each period is $K-$concave.

Theorem 5 *$G_k(y;p)$ is K-concave for any initial price p and a price-dependent (s,S) policy is optimal, i.e., there exists $s_k(p) \leq S_k(p)$ such that whenever the inventory level x is below $s_k(p)$, it is optimal to order up to $S_k(p)$; otherwise it is optimal not to order. The optimal order-up-to level is given by* (9) *and* (15) *for the backorder and lost-sale cases, respectively and the reorder level is given by*

$$s_k(p) = \inf\{x : G_k(x,p) \geq G_k(S_k(p),p) - K,\ x \geq 0\}. \tag{25}$$

4 Numerical Analysis

In this section, we numerically perform several sensitivity analyses on the impact of price volatility. To set up the numerical analysis, we begin by reviewing some of the well-studied financial price processes that are used to model the movements of financial instruments, commodities, exchange rates, etc. Then we construct and use a suitable price process for our numerical experiments.

4.1 Price Process

One of the most important financial price models is the geometric Brownian motion. In this model, the stock price at time t is given by the stochastic differential equation $dS_t = \mu dt + \sigma dW_t$ where W_t is a Wiener process with drift μ and volatility σ. This model is the basis for Black-Scholes option pricing formulas and due to its simplicity, the calculations with this process are relatively easy and leads to closed form solutions (**?**). Another well-known price model is the Ornstein-Uhlenbeck process where prices follow the stochastic differential equation $dS_t = -\kappa(\mu - S_t)\,dt + \sigma dW_t$. In this model, contrary to geometric Brownian motion process, prices tend to revert to their long term mean μ with a degree of volatility σ and a reversion rate parameter κ. This model is particularly useful to model commodity prices as they are known to exhibit some mean-reversion.

A more specialized model is developed by **?** which uses both of the above models to represent the commodity price movements by taking into account both long and short term behaviors. In the short term, the commodity prices show mean-reversion properties whereas in the long term they revert to an equilibrium. In particular, it is assumed that prices follow

$$P_t = e^{\chi_t + \xi_t} \tag{26}$$

where χ_t is an Ornstein-Uhlenbeck process $d\chi_t = -\kappa\chi_t dt + \sigma_\chi dW_t^{(\chi)}$ which models the short term deviations by reverting towards zero. On the other hand, long-term equilibrium level ξ_t is a Brownian motion process $d\xi_t = \mu_\xi dt + \sigma_\xi dW_t^{(\xi)}$. Moreover, $W_t^{(\chi)}$ and $W_t^{(\xi)}$ are correlated Wiener processes with a correlation coefficient of ρ, i.e., $dW_t^{(\chi)} dW_t^{(\xi)} = \rho dt$ (see **?**).

In our numerical setup, our aim is to test the effect of within-period price volatilities on expected profits and optimal controls. To accomplish this, we use a risk-neutral probability measure that makes the price process given in (26) a martingale. This is particularly interesting since changing the volatility related parameters of a martingale price process does not change its expected values in time, which we desire in order to capture the sole effect of volatility. To find a risk-neutral version of (26), we first apply Itó's formula to P_t to write

$$dP_t = \frac{\partial P_t}{\partial \chi_t} d\chi_t + \frac{\partial P_t}{\partial \xi_t} d\xi_t + \frac{1}{2}\frac{\partial^2 P_t}{\partial \chi_t^2} d\chi_t d\chi_t + \frac{1}{2}\frac{\partial^2 P_t}{\partial \xi_t^2} d\xi_t d\xi_t + \frac{\partial^2 P_t}{\partial \chi_t \partial \xi_t} d\chi_t d\xi_t.$$

Noting that $d\chi_t = -\kappa\chi_t dt + \sigma_\chi dW_t^{(\chi)}$, $d\xi_t = \mu_\xi dt + \sigma_\xi dW_t^{(\xi)}$, $d\chi_t d\chi_t = \sigma_\chi^2 dt$, $d\xi_t d\xi_t = \sigma_\xi^2 dt$, $d\xi_t d\chi_t = \sigma_\chi \sigma_\xi \rho dt$ and all partials are trivially equal to P_t, we obtain

$$dP_t = \left(-\kappa\chi_t + \mu_\xi + \sigma_\chi\sigma_\xi\rho + \frac{\sigma_\chi^2}{2} + \frac{\sigma_\xi^2}{2}\right) P_t dt + \sigma_\chi P_t dW_t^{(\chi)} + \sigma_\xi P_t dW_t^{(\xi)}.$$

We now need to find a probability measure that makes P_t a martingale. To do this, we first construct two independent Brownian motions $W^{(1)}, W^{(2)}$ by setting $W_t^{(\xi)} = W_t^{(1)}$ and $W_t^{(\chi)} = \rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(2)}$. Using these transformations, we can now write

$$dP_t = \left(-\kappa\chi_t + \mu_\xi + \sigma_\chi\sigma_\xi\rho + \frac{\sigma_\chi^2}{2} + \frac{\sigma_\xi^2}{2}\right) P_t dt + \left(\sigma_\xi + \sigma_\chi\rho\right) P_t dW_t^{(1)} + \sigma_\chi\sqrt{1-\rho^2} P_t dW_t^{(2)}.$$

Then, we can use Cameron-Martin-Girsanov theorem for n-factor models which states that there exists a probability measure $\mathcal{Q}$ such that

$$dP_t = \sigma_1 P_t dW_t^{(1)} + \sigma_2 P_t dW_t^{(2)} \tag{27}$$

is a martingale where $\sigma_1 = (\sigma_\xi + \sigma_\chi\rho)$ and $\sigma_2 = \sigma_\chi\sqrt{1-\rho^2}$ and $W_t^{(1)}, W_t^{(2)}$ are two independent Brownian motions with respect to Q (see, for example, **?**). In our model, we will use (27) as our input price process.

Here we remark that the model in (26) can be used with parameters estimated using real data. For instance, in **?**, the authors use oil future contracts to estimate the parameters. In our model, we will use hypothetical parameters for price and demand processes.

4.2 Numerical Setup

It is possible to investigate the impact of salient characteristics of this inventory system, including the backorder/lost-sale or holding costs. However, the more interesting case is to investigate the effect of price related parameters and the form of demand functions on optimal expected profits. We mainly take the lost-sales model and experiment with three different rate functions, namely exponential, normal and piecewise linear. The exponential rate function is assumed to have the form $\lambda_E(p) = \overline{\lambda}_E e^{-\theta\alpha p}$ where θ is a sensitivity parameter for the arriving customers. We remark that this sort of a rate function also applies to the cases where individual customer arrivals form an independent Poisson process with rate $\overline{\lambda}_E$ and arriving customers have i.i.d. reservation prices which are exponentially distributed random variables with parameter θ. Then arriving customers only buy the product if their reservation prices are lower then the prevailing selling price at that time. Similarly, normal rate function is assumed to have the form $\lambda_N(p) = \overline{\lambda}_N (1 - \Phi((\alpha p - \mu_N)/\sigma_N))$ where Φ is the cumulative distribution function of the standard normal random variable and μ_N and σ_N are the mean and standard deviation, respectively. Finally, the piecewise linear rate function is of the form $\lambda_L(p) = (A - B\alpha p)^+$ where A represents the potential arrival rate and B represents the customer sensitivity. We use these three functions to test the effect of different rate functions to price changes on the optimal expected profits.

We employ a simulation approach to estimate the one-period expected profits since a direct analytical approach is challenging in the lost-sales model for this price process. To generate the random price process, we use $n = 100$ equally-spaced discretization of each period to simulate random paths for the Brownian motions $W_t^{(1)}$ and $W_t^{(2)}$. We then use these realizations to generate the input price process using (27) . For each sample path of input prices, we generate a nonhomogeneous Poisson arrival stream using the thinning algorithm (see **?**). Furthermore, we use another discretization for the price state in the dynamic programming recursions to compute the value functions.

Throughout the numerical analysis, we use a multiplicative selling price function $f(p) = \alpha p$. We arbitrarily take initial price $P_0 = 100$, multiplicative markup parameter $\alpha = 4$, fixed holding cost $h = 5$, fixed lost-sale cost $b = 20$ and zero interest rate $r = 0$. For the demand rate functions, we use $A = 380$, $B = 0.8$ for the linear case, $\lambda_E = 160, \theta = 0.0025$ for the exponential case and finally $\lambda_N = 120$ and mean and standard deviation of the normal distribution as $\mu_N = 400$ and $\sigma_N = 100$, respectively.

Our model is rich in terms of parameters and there are many issues that can be explored under this numerical setup. Due to space constraints, our analysis below focuses on the effects of price volatility, a central component of our model, whose effects are arguably more difficult to predict than the effects of some of the other parameters. In particular, we explore the effects of price volatility on expected profits and for different price-demand functions and also investigate the effects of ignoring price volatility by using a simpler model with respect to our approach.

4.3 Impact of Price Volatility on Optimal Expected Profits

We first numerically analyze the sensitivity of the optimal expected profits to the level of price volatility. We solve a four-period model ($M = 4$) where we use financial parameters $\rho = 0.3$, $\sigma_\xi = 0.05$ and change the value of σ_χ from 0 to 0.2. Note that since the price process given in (27) is a martingale, i.e., constant in expectation, altering the values of σ_χ increases only the volatility of within-period prices. The result of this numerical experiment is the following: In Figure 3 we observe that the optimal expected profits decrease as the price volatility increases for each of three rate functions, which suggests that price volatilities are undesirable for the firm. There are also differences in the magnitude of the effect of volatility for these rate functions. This is due to the robustness of these functions to price changes. However, for each of the three rate functions, we observe the negative effect of increased volatility on the optimal expected profits. This observation holds for the vast majority of the cases with plausible parameter values. Only in some extreme cases where, for instance, a more volatile price process leads to much higher arrival rates, the optimal expected profits may increase by increased volatility. In terms of optimal base-stock levels, on the other hand, we do not particularly observe any monotonicity with respect to price volatility.

Figure 3: Effect of price volatility on optimal expected profits.

A similar analysis can also be conducted to observe the effect of correlation parameter ρ. However, it is also observed that as ρ increases, the optimal expected profits decrease. This is again due to the fact that a higher ρ means a more volatile price process. More specifically, the variance of P_t whose differential equation is given in (27) is $Var\left(P_t\right) = P_0^2\left(e^{\sigma_\xi^2+2\sigma_\xi\sigma_\chi\rho+\sigma_\chi^2}-1\right)$ which is an increasing function of ρ.

4.4 Impact of Ignoring Price Volatility on Optimal Expected Profits

In another numerical setup, we compare our proposed model with an approximate discrete model to test the effectiveness of modeling within-period price fluctuations explicitly. In particular, we take the model in **?** as benchmark, where prices are constant within sales periods, however they are still random with the same probability distribution at the end (and beginning) of each period. We again use the price process given in (27) with $\rho = 0.3$ and $\sigma_\xi = 0.05$. For a given volatility level σ_χ, we find the optimal base-stock levels for both models and use them in the proposed model which considers within-period price fluctuations to perform a consistent comparison between the resulting expected profits. We use the piecewise linear rate function $\lambda_L(p) = (A - B\alpha p)^+$ and assume that $B = 0.8$. Other parameters are assumed to be the same as above. The result of this comparison is shown in Figure 4 which plots the percentage deviations from the optimal expected profits with respect to price volatility when the decision maker uses the base-stock levels obtained by solving the benchmark model for three different cases. The three curves in Figure 4 represent the cases where the potential customer arrival rate is $A = 340$, $A = 360$ and $A = 380$. Figure 4 shows that as prices get more volatile, the benefit of using the proposed model increases. We also note that the benefit of using the proposed model greatly increases when the potential arrival rate A decreases since a lower A implies an arrival rate process which is more sensitive to price increases.

Figure 4: Comparison with a benchmark model that ignores within-period price volatilities.

We also observe that as period length T increases, then the gap between the benchmark model and the proposed model increases. This can be observed in Figure 5a which plots the percentage deviation from optimal expected profit with respect to changes in within-period length when benchmark model is used. This is for the single-period model and the period length is increased from $T = 0.6$ to $T = 3$ for three different volatility levels. In another experiment, we see that as the number of periods increase, the percentage deviation from optimal expected profits decreases. This is also intuitive as the decision maker has additional opportunities to react to price changes as the number of ordering periods increase. The effect of the number of periods can be observed in Figure 5b which plots the percentage deviation from optimal expected profit when the benchmark model is used as an approximation with respect to number of periods.

We conclude from these experiments that: i) the effect of price volatility on the expected profits is significant so modeling the volatility correctly is important and ii) modeling the within-period price volatility is also important and approximations that ignore the within-period dynamics may underperform. More specifically, the deviation from the approximate model increases in the level of volatility, sensitivity of demand function to price changes and the length of the ordering cycle. On the other hand, the deviation decreases in the number of ordeirng periods.

(a) Effect of period length on the gap with the benchmark model.

(b) Effect of number of periods on the gap with the benchmark model.

Figure 5: Optimality gap with the benchmark model when problem parameters are changed

5 Concluding Remarks

In this paper, we analyzed an inventory management problem where purchase and selling prices are described by a continuous-time stochastic price process which also influences the customer demand. In contrast with most of the existing literature, within each period demand arrives continuously and is influenced by the continuous price process. In this setting, sales revenues depend on individual arrival times of demands and not simply on total accumulated demand. This is an appropriate model for consumer demand that is strongly affected by fluctuating prices that are transferred to the customer. We show that for the backorder case, price-dependent base-stock policies are optimal under standard assumptions. Moreover, this also extends to the more challenging lost-sales case under an additional plausible condition.

The managerial insights this paper presents follow from the capabilities of the model in capturing within-period price fluctuations. To summarize:

- For non-increasing price processes, price-dependent base stock policies are optimal but non-decreasing price processes may lead to non-trivial purchasing policies in the lost sales case.
- Price volatility is, in general, undesirable for the firm since the optimal expected profits decrease as variability of prices increase.
- Modeling within-period price fluctuations proves to be critical as using operational policies that ignore these fluctuations may lead to significant deviations from optimal expected profits. In addition, the suboptimality gap is greater if review cycles are longer and variability of the input price process are higher.

A few other research directions can be considered for the class of models analyzed in this paper. First, all models presented in this paper can be extended to the infinite time horizon case. Second, our assumption of a risk-neutral decision maker can be relaxed. In the backorder model, for instance, the retailer takes the risk of repurchasing the backordered items later at a higher price. By introducing an appropriate risk-measure, risk-sensitive inventory management in the presence of fluctuating costs and revenues can be examined. Similarly, financial hedging opportunities may be considered. It seems interesting to investigate the portfolio decisions of a risk-sensitive decision maker in the presence of financial securities whose prices are correlated with the input price process. Second, it is worth examining the problem of pricing in the presence of price fluctuations. For instance, in the special case where the selling price function is of multiplicative markup form, one may be interested in finding the jointly optimal markup and ordering policy. This joint optimization is not trivial for our problem and multi-period characterizations appear extremely challenging.

Acknowledgement. This research is supported by the Turkish Scientific and Technological Research Council through grant 110M620.

Appendix

Derivation of Expected Total Revenue given in (2)

Let $\mathcal{P} = \{P_s : s \in [0,t]\}$. Then,

$$
\begin{aligned}
r_t(p) &= E\left[\sum_{n=1}^{N_t} e^{-rT_n} f(P_{T_n})\right] = E\left[E\left[\sum_{n=1}^{N_t} e^{-rT_n} f(P_{T_n}) \mid \mathcal{P}\right]\right] \\
&= E\left[\sum_{k=0}^{\infty} P\{N_t = k \mid \mathcal{P}\} E\left[\sum_{n=1}^{k} e^{-rT_n} f(P_{T_n}) \mid \mathcal{P},\ N_t = k\right]\right].
\end{aligned}
$$

Note that conditioned on $N_t = k$ and $\mathcal{P}$, T_n is the order statistics of k i.i.d. random variables on $[0,t]$ with cumulative distribution function

$$
\Phi(s) = \frac{\int_0^s \lambda(P_u)du}{\int_0^t \lambda(P_u)du} \tag{28}
$$

and probability density function $\phi(s) = \Phi'(s) = \lambda(P_s)/\int_0^t \lambda(P_u)du$ on $0 \le s \le t$. Then,

$$
r_t(p) = E\left[\sum_{k=0}^{\infty} P\{N_t = k \mid \mathcal{P}\} kE\left[e^{-r\widetilde{T}} f\left(P_{\widetilde{T}}\right) \mid \mathcal{P}\right]\right]
$$

where $\widetilde{T}$ is a random variable with distribution Φ given in (28) . This implies that

$$
r_t(p) = E\left[E[N_t \mid \mathcal{P}]\, E\left[e^{-r\widetilde{T}} f\left(P_{\widetilde{T}}\right) \mid \mathcal{P}\right]\right] = E\left[E[N_t \mid \mathcal{P}]\left(\frac{\int_0^t e^{-ru} f(P_u)\lambda(P_u)du}{\int_0^t \lambda(P_u)du}\right)\right].
$$

Note also that

$$
E[N_t \mid \mathcal{P}] = \int_0^t \lambda(P_u)du
$$

which implies

$$
r_t(p) = \int_0^t e^{-ru} E[f(P_u)\lambda(P_u)]\, du.
$$

Proof of Theorem 1

We proceed by induction. First note that the terminal value function $V_{M+1}(x,p)$ is concave in x for each p. Now assume that $V_{k+1}(x;p)$ is concave in x for some $k \le M-1$. Then, $\Psi_k(y;p)$ given in (7) is concave in y by the linearity of the expectation operator. Note also that for each p, one-period expected profit $g(y;p)$ in (5) is concave in y since $-py$ is linear and $b(p) \geq 0$, $h(p) \geq 0$ ensure that $-E\left[b(p)(N_T - y)^+ + h(p)(y - N_T)^+ \mid P_0 = p\right]$ is concave. This makes $G_k(y;p)$ given in (8) and consequently $V_k(x,p)$ concave functions of y and x, respectively. By induction argument, $V_k(x,p)$ and $G_k(y;p)$ are concave for each period k. Clearly, concavity of $G_k(y;p)$ ensures the optimality of a base-stock policy with a base-stock level of $S_k(p)$ which is the maximizer of $G_k(y;p)$. To calculate the base-stock level for period k, we can apply the first order optimality condition on $G_k(y;p)$. More specifically,

$$
\begin{aligned}
S_k(p) &= \inf\left\{y \geq 0 : \triangle G_k(y;p) \leq 0\right\} = \inf\left\{y \geq 0 : \triangle g(y;p) + \gamma \triangle \Psi_k(y;p) \leq 0\right\} \\
&= \inf\left\{y \geq 0 : -p + b(p) P\left\{N_T \geq y+1 \mid P_0 = p\right\}\right. \\
&\qquad \left. -h(p) P\left\{N_T \leq y \mid P_0 = p\right\} + \gamma \triangle \Psi_k(y;p) \leq 0\right\} \\
&= \inf\left\{y \geq 0 : P\left\{N_T \leq y \mid P_0 = p\right\} \geq \frac{-p + b(p) + \gamma \triangle \Psi_k(y;p)}{b(p) + h(p)}\right\}
\end{aligned}
$$

which is (9).

Proof of Corollary 1

Since $V_{M+1}(x,p) = -px^-$, using (7) we obtain

$$
\begin{aligned}
\gamma \triangle \Psi_M(y;p) &= -\gamma \triangle E\left[P_T(N_T - y)^+ \mid P_0 = p\right] \\
&= \gamma E\left[P_T(N_T - y)^+ \mid P_0 = p\right] - \gamma E\left[P_T(N_T - y - 1)^+ \mid P_0 = p\right] \\
&= \gamma E\left[P_T\left[(N_T - y)^+ - (N_T - y - 1)^+\right] \mid P_0 = p\right] \\
&= \gamma E\left[P_T 1_{\{N_T \geq y+1\}} \mid P_0 = p\right] = \gamma E\left[P_T\left(1 - 1_{\{N_T \leq y\}}\right) \mid P_0 = p\right] \\
&= z(p) - \gamma E\left[P_T 1_{\{N_T \leq y\}} \mid P_0 = p\right]. \qquad (29)
\end{aligned}
$$

Substituting (29) in (9) for $k = M$ yields (10).

Proof of Theorem 2

Consider the characterization of $S_M(p)$ in (10). Let $0 < P_0^{(1)} < P_0^{(2)}$ be two distinct initial price values and $P^{(i)}$, $\Lambda^{(i)}$ and $N^{(i)}$ be the corresponding price, arrival rate and demand processes for

$i = 1, 2$. By Assumption, 1 $P_t^{(1)} \leq_{st} P_t^{(2)}$ for any $t \in (0, T]$. Since $\lambda(.)$ is decreasing by Assumption 2, $\lambda_t^{(1)} \geq_{st} \lambda_t^{(2)}$ for any $t \in (0, T]$ (**?**). It is known that for a doubly stochastic Poisson process, if the rate measures are stochastically ordered, then the counting measures are also stochastically ordered in the same direction. More specifically, $\lambda_t^{(1)} \geq_{st} \lambda_t^{(2)}$ implies $N_t^{(1)} \geq_{st} N_t^{(2)}$ (see Theorem 1 in **?**). Then,

$$E\left[\left(b\left(P_0^{(1)}\right) + h\left(P_0^{(1)}\right) + \gamma P_T^{(1)}\right) 1_{\left\{N_T^{(1)} \leq y\right\}}\right] \geq E\left[\left(b\left(P_0^{(2)}\right) + h\left(P_0^{(2)}\right) + \gamma P_T^{(2)}\right) 1_{\left\{N_T^{(2)} \leq y\right\}}\right]$$

and by Assumption 3, $-P_0^{(1)} + b\left(P_0^{(1)}\right) + z_T(P_0^{(1)}) \leq -P_0^{(2)} + b\left(P_0^{(2)}\right) + z_T(P_0^{(2)})$. As a result, $S_M(P_0^{(1)}) \geq S_M(P_0^{(2)})$.

Proof of Theorem 3.

We prove the result by induction. First note that terminal value function $V_{M+1}(x, p)$ is trivially concave. Now assume that for any $k \leq M - 1$, $V_{k+1}(x, p)$ is concave. Note that forward differences of the one-period profit function is

$$\triangle g(y, p) = E\left[1_{\{T_{y+1} \leq T\}}\left(e^{-rT_{y+1}} f(P_{T_{y+1}}) + b(p) + h(p)\right) - p - h(p) \mid P_0 = p\right]$$

which is also decreasing in y under Assumption (4). This makes $g(y, p)$ concave in y. Since $V_{k+1}(x, p)$ is concave by induction, $\Psi_k(y; p)$ is concave which makes $G_k(y, p)$ concave. This in turn implies that $V_k(x, p) = \max_{y \geq x} G_k(y; p) + px$ is concave in x. By induction, it is true that $V_k(x, p)$ and $G_k(y; p)$ are concave for all periods k and initial price p which suggests the existence of an optimal price-dependent base-stock type policy for this inventory model. Similar to the backorder model, optimal base-stock level for any period k can be found by analyzing the forward difference of $G_k(y; p)$. More specifically,

$$\begin{aligned}
S_k(p) &= \inf\left\{y \geq 0 : \triangle G_k(y; p) \leq 0\right\} \\
&= \inf\Big\{y \geq 0 : E\left[1_{\{T_{y+1} \leq T\}}\left(e^{-rT_{y+1}} f(P_{T_{y+1}}) + b(p) + h(p)\right) \mid P_0 = p\right] \\
&\qquad\qquad -p - h(p) + \gamma \triangle \Psi_k(y; p) \leq 0\Big\} \\
&= \inf\Big\{y \geq 0 : E\left[1_{\{T_{y+1} \leq T\}} \mid P_0 = p\right]
\end{aligned}$$

$$\leq \frac{p+h(p)-\gamma \triangle \Psi_k(y;p)-E\left[1_{\{T_{y+1}\leq T\}}e^{-rT_{y+1}}f(P_{T_{y+1}}) \mid P_0=p\right]}{b(p)+h(p)}\Bigg\}$$

$$= \inf\{y \geq 0 : P\{N_T \geq y+1 \mid P_0 = p\}$$

$$\leq \frac{p+h(p)-\gamma \triangle \Psi_k(y;p)-E\left[1_{\{T_{y+1}\leq T\}}e^{-rT_{y+1}}f(P_{T_{y+1}}) \mid P_0=p\right]}{b(p)+h(p)}\Bigg\}$$

$$= \inf\{y \geq 0 : P\{N_T \leq y \mid P_0 = p\}$$

$$\geq \frac{-p+b(p)+E\left[1_{\{T_{y+1}\leq T\}}e^{-rT_{y+1}}f(P_{T_{y+1}}) \mid P_0=p\right]+\gamma \triangle \Psi_k(y;p)}{b(p)+h(p)}\Bigg\}.$$

Proof of Proposition 1

Note that $(b(p)+h(p))\,E\left[1_{\{T_n\leq T\}} \mid P_0=p\right]$ is decreasing in n as arrival times T_n's form an increasing sequence which makes $1_{\{T_n\leq T\}}$ decreasing. Now define

$$\varphi(t,p) = E\left[e^{-rt}f(P_t)\,1_{\{t\leq T\}} \mid P_0=p\right]$$

Note that if $z_t(p)$ is decreasing in t, $\varphi(t,p)$ is decreasing in t as $1_{\{t\leq T\}}$ is decreasing in t. Now, we can write

$$E\left[e^{-rT_n}f(P_{T_n})\,1_{\{T_n\leq T\}} \mid P_0=p\right] = E\left[\varphi(T_n,p)\right]$$

which is decreasing in n since T_n is increasing in n.

Proof of Theorem 4.

First note that the terminal value function $V_{M+1}(x,p)=0$ is trivially concave. Now assume that for some $k \leq M-1$, $V_{k+1}(x,p)$ is concave. Then, $\gamma\Psi_k(y;p)$ is concave. Note also that $N(y)$ is increasing in y and the same reasoning as in Proposition (1) applies here; that is, (21) is decreasing if the expected price is a decreasing function of time, i.e., if $z_t(p)$ is decreasing in t. Therefore, $r(y;p)$ given in (20) is concave. Moreover, since both h and b are positive parameters, it is clear that $-E\left[b(\overline{D}_{N_T}-y)^+ + h\left(y-\overline{D}_{N_T}\right)^+\right]$ is also a concave function and the one-period profit function $g(y;p)$ given in (19) is concave. Since both $g(y;p)$ and $\gamma\Psi_k(y;p)$ are concave, so is $G_k(y,p)$. This in turn makes $V_k(x,p)$ given in (6) concave. By induction, $V_k(x,p)$ is concave for all k. Then, it is clear that $G_k(y,p)$ is concave for all k. To characterize the optimal base-stock levels, consider the first order optimality condition for $G_k\left(y;p\right)$,

$$\begin{aligned}G_k'\left(y;p\right) &= -p + r'(y;p) + b\left(p\right)P\left\{\overline{D}_{N_T} > y \mid P_0 = p\right\} - h\left(p\right)\left\{\overline{D}_{N_T} \leq y \mid P_0 = p\right\} + \gamma\Psi_k'(y;p) \\ &= -p + b\left(p\right) + E\left[e^{-rT_{N(y)}}f(P_{T_{N(y)}})1_{\left\{T_{N(y)}\leq T\right\}}\right] \\ &\quad - \left(h\left(p\right) + b\left(p\right)\right)P\left\{\overline{D}_{N_T} \leq y \mid P_0 = p\right\} + \gamma\Psi_k'(y;p) = 0\end{aligned}$$

which can also be written as

$$P\left\{\overline{D}_{N_T} \leq y \mid P_0 = p\right\} = \frac{-p + b\left(p\right) + E\left[e^{-rT_n}f(P_{T_{N(y)}})1_{\left\{T_{N(y)}\leq T\right\}}\right] + \gamma\Psi_k'(y;p)}{b\left(p\right) + h\left(p\right)}. \tag{30}$$

However, since the distribution of $\overline{D}_{N_T}$ has a mass at $y = 0$, (30) should be corrected as (22). Note that if $D_k = 1$ for all k, then (22) reduces to (15) since $N\left(y\right) = y$ and $\overline{D}_{N_T} = N_T$.

Proof of Theorem 5

Note that the proof is valid for both backorder and lost-sale cases under Assumption 4 for the latter. Let us proceed with the backorder case. Note that $V_M(x,p) = 0$ is clearly K-concave in x. Now assume that $V_{k+1}(x,p)$ in (23) is K-concave in x for some $k \leq M-1$. Since K-concavity is preserved under expectation, $\Psi_k(y;p)$ given in (7) is also K-concave which makes $\gamma\Psi_k(y;p)$ $\gamma K-$concave. Then, $G_k(y;p)$ in (8) is K-concave since it is the sum of a concave function and a $\gamma K-$concave function where $\gamma \leq 1$. In reference to **?**, $G_k^*(x;p)$ in (24) is also $K-$concave in x and this leads to $V_k(x,p)$ in (23) being $K-$concave. Then, the optimal policy must be of the form $(s_k(p), S_k(p))$ where $S_k(p)$ is the maximizer of $G_k(y;p)$ and $s_k(p)$ is defined as the smallest inventory value x that satisfies

$$G_k(x,p) \geq G_k(S_k(p),p) - K.$$