Queueing Syst
DOI 10.1007/s11134-009-9109-x

Effects of system parameters on the optimal policy structure in a class of queueing control problems

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Received: 18 August 2008 / Revised: 29 January 2009

Abstract This paper studies a class of queueing control problems involving commonly used control mechanisms such as admission control and pricing. It is well established that in a number of these problems, there is an optimal policy that can be described by a few parameters. From a design point of view, it is useful to understand how such an optimal policy varies with changes in system parameters. We present a general framework to investigate the policy implications of the changes in system parameters by using event-based dynamic programming. In this framework, the control model is represented by a number of common operators, and the effect of system parameters on the structured optimal policy is analyzed for each individual operator. Whenever a queueing control problem can be modeled by these operators, the effects of system parameters on the optimal policy follow from this analysis.

Keywords Control of queueing systems · Event-based dynamic programming · Structured optimal control policies · Effects of system parameters

Mathematics Subject Classification (2000) 49L20 · 93E20

1 Introduction

This paper focuses on a class of queueing control problems that frequently arise in modeling service systems, telecommunications systems, or production and inventory

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Published online: 07 March 2009

systems. The objective in such systems is, in general, managing the queue length or the inventory level in order to maximize a reward function. This can be achieved by controlling the arrival rates using admission control or dynamic pricing, or by controlling the service rate by slowing down or speeding up service. A significant literature exists on the investigation of such problems. One of the common important findings is that in a number of situations the optimal control policy has a simple structure which facilitates simple operational policies. We study how changes in the system parameters such as arrival rates, service rates, number of servers or the buffer size influence the optimal policy.

There has been some work on developing general approaches for obtaining results on the structure of optimal policies. For instance, Veatch and Wein [27] investigate a class of queueing problems with this perspective. Smith and McCardle [24] present some results for general Markov Decision Processes (MDPs). In particular, a fruitful general approach for queueing problems is proposed by Koole [12] which introduces the so-called event-based dynamic programming framework. Event-based dynamic programming expresses the value function of a given control problem in terms of the composition of different operators corresponding to individual events. Establishing the structure of an optimal policy can then be performed by verifying that the different operators constituting the problem satisfy certain properties. This decomposition of the problem is extremely useful, since the properties of commonly used operators in queueing control problems can be investigated individually once and collected in a library as in Koole [12] and [13]. Then, if a queueing control problem is composed of known operators, establishing the structure of the optimal policy is usually trivial by checking the operator library. Our approach is similar. We employ the event-based dynamic programming framework and focus on a number of frequently employed operators and the effects of problem parameters on these operators. This has two purposes: first it gives a clearer view of how and why the optimal policy is influenced by system parameters. Second, as in Koole [12], whenever a queueing control problem can be expressed as a composition of these operators, understanding the effects of a given system parameter on the optimal policy becomes an easy problem.

We first need to establish certain properties of the value functions with respect to the system parameters. Koole [13], independently, has considered these properties for a set of operators which describes arrivals to and departures from a queueing system as exogenous processes. These results address certain optimal design issues such as the optimal number of servers, but do not include any conclusions on how optimal control policies change with system parameters. We, on the other hand, concentrate on the effects of these properties on optimal control policies. To our knowledge, there are only a few specific studies which concentrate on this issue. For instance, Ku and Jordan [14] study an admission control problem in a two-stage multi-server loss system. They characterize the structure of the optimal policy, and then establish the effects of the system parameters on this policy. Gans and Savin [7] consider a joint admission control and dynamic pricing problem in a multi-server loss system and analyze the effects of the parameters on the optimal policy. The optimal admission policy is shown to have a threshold structure, and they prove the monotonicity of the optimal thresholds and the optimal prices in the system parameters. Aktaran-Kalaycı and Ayhan [1] and Çil, Karaesmen and Örmeci [4] examine the effects of the parameters on the optimal decisions in a multi-server queueing system with limited capacity.

They show the monotonicity of the optimal prices as in [7]. Another related stream is queueing control problems with non-stationary or periodically varying parameters. Lewis, Ayhan and Foley [16] and Yoon and Lewis [29] establish monotonicity properties of admission control and pricing problems in time. Most of these papers and a number of other important models in the literature fit into our framework, and the influence of the parameters on these models can be determined easily by our results. In order to demonstrate the application of the general approach, we also investigate two models in some detail.

Our main contribution is providing a general framework to investigate the effects of system parameters on the optimal policy in queueing control problems. To our knowledge, this problem has not been considered in this generality before. Müller [19] proposes an approach to investigate the effects of transition probabilities on the value function of a Markov Decision Process in a general setting. However, his focus is not on policy implications. A fairly rich class of problems, including several earlier studied ones, fall within the framework developed here. There are, of course, certain limitations to the generality. First a given problem must be within the scope of event-based programming and the value iteration principle must hold. Moreover, for infinite-horizon problems the existence of optimal value functions and optimal policies should be checked separately. We do not address these issues here (see [12] and [13] for more on these issues). These issues are sometimes easy to verify and follow by known general results such as the ones in Puterman [22], but in some cases an analysis of the individual problem may be necessary. Second, we investigate a certain number of operators which are frequently used and cover a significant scope, but other problem-specific operators should be investigated individually. Using the results on the operators here, we think it should not be difficult to extend the investigation to other operators.

The other contributions of this paper are as follows: we investigate a subset of the dynamic programming operators from Koole [12] but in certain cases we modify or generalize these operators. In addition, we also present a number of new operators. Some of these are relevant for recent applications of interest such as dynamic pricing, others are introduced since they are used in the control of make-to-stock queues that model production/inventory systems. We present structural properties for these parameters as well as investigating the effects of system parameters. The full strength of event-based dynamic programming appears when the structure of the optimal policy and the effects of the parameters have to be determined for a new problem. To this end, we explore an inventory rationing problem for a make-to-stock queue with multiple classes of demand arriving in batches. Using the operators and the framework, we present a characterization of the optimal policy and determine the effects of the system parameters.

The rest of the paper is organized as follows. We present the framework and the operators modeling the events in some queueing control problems in Sect. 2. Then, in Sect. 3, we establish certain properties of the operators to guarantee the existence of optimal monotone policies and to characterize their behavior with respect to parameter changes. Section 4 presents several models that can be generated by combining the operators we have defined, in order to illustrate how to apply our results. Finally, we conclude the study and mention our future research objectives in Sect. 5.

2 The operators

Event-based dynamic programming [12, 13], proposes to represent the value function of a stochastic dynamic program as the composition of individual event operators. These operators can first be analyzed individually and then be combined to investigate the structure of optimal policies. The approach has been very fruitful in queueing control problems. Koole [13] presents a complete overview.

In the rest of the paper, x will denote the state of the system (the number of customers in the queue or the inventory level of an item). Similarly, we will always denote the customer or demand arrival rate by λ, the service or production rate by μ, and the number of parallel servers by m. We consider only the maximization objective, but all our results are valid for minimization problems when the properties are re-defined properly. Moreover, we define $v(x)$ as a generic value function which coincides with the total maximal expected discounted profit of the system over an infinite horizon, when the initial state of the system is x. Naturally, the existence of the infinite-horizon value functions should be shown for each different model.

We define the operators mainly for one-dimensional systems. However, the framework can accommodate Markov modulation by defining an operator, $T_{\mathrm{ENV}(j)}$, to represent transitions to the exogenous environment j as in [12]. The considered systems can have finite or infinite buffer capacity. The operators below will mainly be defined for the infinite capacity case. However, the specifics of the finite-buffer case are addressed explicitly whenever necessary.

In principle, we can classify the operators into two types: the controllable operators are those which involve a maximization, while the uncontrollable operators are all the others. We provide the definitions of the operators used in the rest of the paper.

The *cost operator* represents the system incurring a non-negative holding cost, $h(x)$, which is a function of the state of the system:

$$T_{\mathrm{COST}} v(x) = v(x) - h(x),$$

where the holding cost function $h(x)$ is increasing and convex in the inventory level x. Here, we note that the words "increasing" and "decreasing" mean "non-decreasing" and "non-increasing," respectively, in the whole paper.

The *arrival* operator, T_{ARR}, represents the arrival process to a queueing system:

$$T_{\mathrm{ARR}} v(x) = a(x)v(x+1) + \big[1 - a(x)\big]v(x).$$

The function $a(x)$ is the probability that an arriving customer joins the system when there are x customers, which we refer to as the joining probability. We assume that $a(x)$ is decreasing in x. When $a(x)$ is constant, T_{ARR} models a system where customers enter the system with a fixed probability, say a, independent of the state. Such arrivals, which do not depend on the state, are referred to as regular arrivals. Moreover, we can model a finite system with capacity K by setting $a(x) = 0$ for all $x \geq K$.

The effects of parameter changes on the operators can be of two kinds. They either influence the probability of the corresponding event or modify the definition of the operator. In particular for T_{ARR}, a change in the arrival rate λ will always change the probability of observing an arrival. On the other hand, the definition of T_{ARR} will change with a change in a parameter α, only if $a(x)$ depends on α for $\alpha \in \{\lambda, \mu, m\}$. Finally, an increase in the system capacity K always changes the definition of this arrival operator. It should be noted that when $a(x)$ depends on a parameter α with $\alpha \in \{\lambda, \mu, m, K\}$, most structural results require certain conditions on how the function $a(x)$ varies with α, an issue that will be addressed later.

The *departure operator*, T_{DEP}, represents the departure of an existing customer from the system, where the service rate may depend on the state of the system:

$$T_{\text{DEP}} v(x) = b(x) v\big((x-1)^+\big) + \big[1 - b(x)\big] v(x),$$

where $a^+ = \max\{0, a\}$. The function $b(x)$ corresponds to the probability of a service completion when the system has x customers. We assume that $b(x)$ is an increasing function of x. A change in the service rate μ will always change the probability of observing a departure. Whenever we investigate the effects of the number of servers, m, we assume that the system has m identical parallel servers. Then, $b(x)$ is specified by:

$$b(x) = \frac{\min\{x, m\}}{M}. \tag{1}$$

Hence, a change in m will always change the definition of T_{DEP}. In general, $b(x)$ may depend on parameters λ, μ or m, which will alter the definition of T_{DEP}. Then, as with the function $a(x)$ in T_{ARR}, we need to consider how $b(x)$ varies with the parameter under consideration.

The *controlled departure* and the *controlled production* operators, T_{CD} and $T_{\text{C\_PRD}}$, represent the choice of the best service rate in queueing and inventory systems, respectively. When the system uses π portion of the service rate, a non-negative cost of c_π is incurred. We assume that $c_0 = \min_{\pi \in [0,1]} c_\pi$. Then:

$$T_{\text{CD}} v(x) = \begin{cases} \max_{\pi \in [0,1]}\{-c_\pi + \pi v(x-1) + (1-\pi) v(x)\} & \text{if } x > 0, \\ -c_0 + v(x) & \text{if } x = 0, \end{cases}$$

$$T_{\text{C\_PRD}} v(x) = \max_{\pi \in [0,1]} \big\{-c_\pi + \pi v(x+1) + (1-\pi) v(x)\big\}.$$

T_{CD} ($T_{\text{C\_PRD}}$) is affected by a change in the service rate (production rate) μ, where μ changes the probability of finishing the service of a customer (of producing an item). For a capacitated system, an increase in the parameter K will change the definition of $T_{\text{C\_PRD}}$. $T_{\text{C\_PRD}}$ is the main production control operator employed in the literature on make-to-stock queues [9, 17].

The *queue pricing* and the *inventory pricing* operators, $T_{\text{Q\_PRC}}$ and $T_{\text{I\_PRC}}$ represent the optimal price, p, to be charged for the arriving customers in queueing and inventory systems, respectively. We assume that $p \in \mathcal{P}$ where $\mathcal{P}$ is a compact set and that the revenue rate $\lambda \bar{F}(p) p$ is bounded. Let R be the random variable corresponding to the maximum price a customer is willing to pay and let $F(.)$ denote its cumulative distribution function. The operators can be expressed as:

$$T_{\text{Q\_PRC}} v(x) = \max_p \big\{\bar{F}(p)\big[v(x+1) + p\big] + F(p) v(x)\big\},$$

$$T_{\text{I\_PRC}}v(x) = \begin{cases} \max_p\{\bar{F}(p)[v(x-1)+p] + F(p)v(x)\} & \text{if } x > 0, \\ v(x) & \text{if } x = 0, \end{cases}$$

where $\bar{F}(p) = 1 - F(p)$. For both of these pricing operators, an increase in the arrival rate λ increases the probability of observing the event of pricing due to an arrival. Changes in the capacity of the system K change the definition of the operator $T_{\text{Q\_PRC}}$.

The *batch admission* and the *batch rationing* operators, $T_{\text{B\_ADM}_{i,B}}$ and $T_{\text{B\_RT}_{i,B}}$, represent the choice of the number of the customers to be admitted from an arriving batch of class-i customers with batch size B in queueing and inventory systems, respectively. We assume that some of the customers in a batch can be admitted while the remaining ones are rejected, which is defined as partial acceptance in [20]. κ_i is the number of class-i customers admitted from this batch, and R_i is the reward obtained by admitting one class-i customer. Therefore:

$$T_{\text{B\_ADM}_{i,B}}v(x) = \max_{\kappa_i \le B}\left\{\kappa_i R_i + v(x+\kappa_i)\right\},$$

$$T_{\text{B\_RT}_{i,B}}v(x) = \max_{\kappa_i \le \min\{x,B\}}\left\{\kappa_i R_i + v(x-\kappa_i)\right\}.$$

The *batch* operators have a similar spirit to pricing operators, with respect to the effects of parameters. That is, they are both affected by an increase in the arrival rate λ, which increases the probability of observing the event of batch admission or rationing, while the operator $T_{\text{B\_ADM}_{i,B}}$ is additionally affected by an increase in the capacity of the system K, which changes the definition of the operator.

The *uniformization* operator puts all events together with their corresponding probabilities:

$$T_{\text{UNIF}}\left(\{f_i\}_{j=1}^l; \{p_i\}_{j=1}^l\right)(x) = \sum_{j=1}^l p_j f_j(x), \quad \text{with } \sum_{j=1}^l p_j \le 1.$$

T_{UNIF} is a convex combination of functions f_j, and it only reflects the changes in the parameters so that it is not affected by any parameter directly. $\sum_{j=1}^l p_j < 1$ models the discounting criterion, whereas $\sum_{j=1}^l p_j = 1$ refers to long-run average criterion.

The *fictitious* operator, T_{FIC}, represents all the fictitious events, which affect neither the state nor the reward of the system:

$$T_{\text{FIC}}v(x) = v(x).$$

This operator is affected by both λ and μ, since increases in these parameters increase the probability of an arrival or a departure (or production), which decreases the probability of a fictitious event.

3 Properties of the operators

3.1 Structural properties preserved by the operators

We first focus on first-order monotonicity defined as a decreasing value function i.e. $v(x) \geq v(x+1)$, for all x. This property implies a non-negative burden, or opportunity cost, of an additional customer or an additional unit of inventory, denoted by $\Delta v(x) \equiv v(x) - v(x+1)$. For most plausible queueing systems, this burden is non-negative. But this need not be the case in inventory models.

In many queueing and inventory problems, the opportunity costs, $\Delta v(x)$, affect optimal decisions, and monotonicity of $\Delta v(x)$ (e.g., concavity of the value function in a maximization problem), implies the monotonicity of optimal policies. For instance, the optimality of threshold policies in admission control problems is due to the monotonicity of opportunity costs. Therefore, the monotonicity of $\Delta v(x)$ (concavity/convexity of $v(x)$) also requires investigation.

We also consider upper and lower bounds on the opportunity costs, $\Delta v(x)$, in queueing and inventory problems, respectively. The existence of upper bounds in queueing systems implies that the opportunity cost of a new customer, $\Delta v(x)$, may be lower than the reward of one or more demand classes for all states x, and thus these classes are admitted to the system whenever possible. Similarly, the existence of lower bounds in inventory systems implies that opportunity cost of an additional inventory, $\Delta v(x)$, may be higher than the reward of one or more demand classes for all states x, and thus the demands from these classes are satisfied whenever possible. Such classes are defined to be preferred classes in [21] and [23]. The bounds imply the existence of preferred class(es). We define new properties *Lower-Bounded Differences* (LBD) and *Upper-Bounded Differences* (UBD) as follows: f is an LBD function if there exists $L \in \mathbb{R}$ such that $f(x) - f(x+1) \geq L$ for all x, and it is a UBD function if there exists $U \geq 0 \in \mathbb{R}$, $U \in \mathbb{R}$ with $f(x) - f(x+1) \leq U$ for all x.

The formal definitions of the properties stated above are:

$$\begin{aligned}
&\mathrm{Dec}(x) : f(x+1) \leq f(x), \\
&\mathrm{Inc}(x) : f(x) \leq f(x+1), \\
&\mathrm{Conc}(x) : \Delta f(x) \leq \Delta f(x+1), \\
&\mathrm{Conv}(x) : \Delta f(x+1) \leq \Delta f(x), \\
&\mathrm{LBD}(L) : f(x) - f(x+1) \geq L, \\
&\mathrm{UBD}(U) : f(x) - f(x+1) \leq U.
\end{aligned}$$

In the above, while $U \geq 0$, we do not impose any restriction on L because in inventory models the opportunity cost of an additional inventory may be negative.

The results on the properties of the operators are presented in Table 1. The ticks in Table 1 represent that the corresponding operators preserve the desired properties (Dec, LBD, UBD or Conc) for the function f. Consider e.g., operator T_{CD} and property Dec(x): If a function f is decreasing in x, then $T_{\mathrm{CD}} f$ is also decreasing in x. For the operator T_{ARR}, on the other hand, to preserve concavity of a function f, the

Table 1 Properties preserved by the operators when the function $f(x)$ has the corresponding property

Operator	Preserved properties				Additional
	Dec(x)	LBD(L)	UBD(U)	Conc(x)	condition(s)
T_{UNIF}	√	√	√	√	–
T_{COST}	√	√		√	–
T_{ARR}	√*		√*	√*†	* $a(x)$: Dec(x) † $a(x)$: Conv(x)
T_{DEP}	√*		√*	√*†	* $b(x)$: Inc(x) † $b(x)$: Conc(x)
T_{CD}	√		√	√	–
$T_{\mathrm{C\_PRD}}$		√		√	–
$T_{\mathrm{Q\_PRC}}$	√		√*	√	* $f(x)$: Conc(x)
$T_{\mathrm{I\_PRC}}$		√*		√	* $f(x)$: Conc(x)
$T_{\mathrm{B\_ADM}_{i,B}}$	√		√*	√	* $f(x)$: Conc(x)
$T_{\mathrm{B\_RT}_{i,B}}$		√*		√	* $f(x)$: Conc(x)
$T_{\mathrm{ENV}(j)}$	√	√	√	√	–
T_{FIC}	√	√	√	√	–

function $a(x)$ needs to be both decreasing and concave in x. The proofs of these properties are given in Appendix A.

We note that whenever a finite system receives regular arrivals ($a(x) = a$ for all $x < K$ and $a(x) = 0$ for all $x \geq K$), T_{ARR} does not preserve concavity since $a(x)$ is not convex in x. Hence, regular arrivals preserve concavity only in infinite systems. Finally, T_{COST} does not preserve the UBD property because the holding cost function h is increasing in x. Hence, a queueing system incurring a holding cost cannot have a preferred class.

Whenever a system can be represented by a combination of these operators, we can use Table 1 to check whether a specific property is satisfied in this system easily: Setting $v^0 \equiv 0$ satisfies all the properties considered in the table, which starts an induction for any of these properties. Now assume v^n satisfies one of these properties. If all the operators in the model preserve this specific property, then the induction step is verified so that v^{n+1} satisfies this property as well, from which we can conclude the system has that specific property.

An investigation of Table 1 reveals that all queueing operators preserve monotonicity and concavity (with only mild assumptions on $a(x)$ in operator T_{ARR} and $b(x)$ in operator T_{DEP}), whereas all inventory operators preserve concavity. Therefore, all control problems that can be represented by a combination of these operators will have concave value functions, which ensures the existence of a monotone optimal control policy, usually described by a set of appropriate thresholds. Moreover, all queueing operators, except for T_{COST}, preserve the UBD property, whereas all inventory operators preserve the LBD property. Thus, systems represented by a combination of these operators will have preferred customers, which can also be described by appropriate thresholds. These observations are important for the effect of parameters on the structure of optimal policies. Sections 3.2 and 3.3 will analyze how the value functions and optimal control policies, respectively, vary with changes in the parameters.

3.2 Effects of parameters on the properties of the value functions

In this section, we assume that optimal control policies are monotone, or equivalently the functions entering the operators are decreasing and concave in queueing systems and concave in inventory systems. Since the goal is to compare systems when certain parameters change, considering two systems with two different parameters will be considered simultaneously. To make the dependence on the parameter explicit, $v_\alpha(x)$ denotes the value function when the parameter under consideration has the value of α.

To outline the general approach, let us consider a system that can be represented by two event operators T_{E1} and T_{E2} with respective event rate parameters α_1 and α_2. The system also incurs a holding cost of $h(x)$. We are interested in the effects of an increase in α_1 by $\varepsilon > 0$. In order to compare the two systems, we need to ensure that both systems live in the same time scale. For this purpose, we add the fictitious operator T_{FIC} to the model with an event rate parameter of $\theta > \varepsilon$. Then, using the well-known method of uniformization [18], we express the corresponding optimality equation for $v_\alpha^{n+1}(x)$ as follows:

$$v_{\alpha_1}^{n+1}(x) = T_{\mathrm{COST}}\big(T_{\mathrm{UNIF}}\big(\big\{T_{\mathrm{E1}} v_{\alpha_1}^n(x)\big\}, \big\{T_{\mathrm{E2}} v_{\alpha_1}^n(x)\big\}, \big\{T_{\mathrm{FIC}} v_{\alpha_1}^n(x)\big\}; \{\alpha_1, \alpha_2, \theta\}\big)\big).$$

Now, consider the altered system with the rate of E1 as $\alpha_1 + \varepsilon$. The rate of the fictitious event in this system is decreased to $\theta - \varepsilon$ to ensure that uniformization still holds at the same time scale. In addition, let us suppose that we anticipate the expected burden of an additional customer would decrease as α_1 increases: $\Delta v_{\alpha_1}(x) \geq \Delta v_{\alpha_1+\varepsilon}(x)$ for all x. This anticipation can be verified by an induction argument since $v_{\alpha_1}(x)$ can be computed by the value iteration algorithm using $v_{\alpha_1}^n(x)$ as $v_{\alpha_1}(x) = \lim_{n\to\infty} v_{\alpha_1}^n(x)$. Setting $v_{\alpha_1+\varepsilon}^0(x) = v_{\alpha_1}^0(x) \equiv 0$ starts the induction, and it is assumed that $\Delta v_{\alpha_1}^n(x) \geq \Delta v_{\alpha_1+\varepsilon}^n(x)$ for all x. Next, it is required to show that this inequality is preserved for $n+1$:

$$\begin{aligned}
\Delta v_{\alpha_1}^{n+1}(x) &= \begin{bmatrix} -\Delta h(x) \\ +\alpha_1\, \Delta T_{\mathrm{E1}} v_{\alpha_1}^n(x) \\ +\alpha_2\, \Delta T_{\mathrm{E2}} v_{\alpha_1}^n(x) \\ +\theta\, \Delta T_{\mathrm{FIC}} v_{\alpha_1}^n(x) \end{bmatrix} \geq \begin{bmatrix} -\Delta h(x) \\ +\alpha_1\, \Delta T_{\mathrm{E1}} v_{\alpha_1+\varepsilon}^n(x) \\ +\alpha_2\, \Delta T_{\mathrm{E2}} v_{\alpha_1+\varepsilon}^n(x) \\ +\theta\, \Delta T_{\mathrm{FIC}} v_{\alpha_1+\varepsilon}^n(x) \\ +\varepsilon\, \Delta[T_{\mathrm{E1}} v_{\alpha_1+\varepsilon}^n(x) - v_{\alpha_1+\varepsilon}^n(x)] \end{bmatrix} \\
&= \Delta v_{\alpha_1+\varepsilon}^{n+1}(x).
\end{aligned} \tag{2}$$

In the first line of (2) the costs cancel each other. The next three lines require operators T_{E1}, T_{E2} and T_{FIC} to preserve the inequality $\Delta v_{\alpha_1}^n(x) \geq \Delta v_{\alpha_1+\varepsilon}^n(x)$, whereas the last line needs $\Delta[T_{\mathrm{E1}} v_{\alpha_1+\varepsilon}^n(x) - v_{\alpha_1+\varepsilon}^n(x)] \leq 0$, which can be expanded further as:

$$\begin{aligned}
\Delta\big[T_{\mathrm{E1}} v_{\alpha_1+\varepsilon}^n(x) - v_{\alpha_1+\varepsilon}^n(x)\big] &= T_{\mathrm{E1}} v_{\alpha_1+\varepsilon}^n(x) - v_{\alpha_1+\varepsilon}^n(x) \\
&\quad - \big(T_{\mathrm{E1}} v_{\alpha_1+\varepsilon}^n(x+1) - v_{\alpha_1+\varepsilon}^n(x+1)\big) \leq 0.
\end{aligned}$$

Notice that this inequality implies that the completion of E1 is more valuable when the number of customers is higher.

The comparisons in (2) can be grouped into two: one compares two systems with two different parameters (α_1 versus $\alpha_1 + \varepsilon$); the other one, on the other hand, is involved only with the altered system, where the consequences of having the event E1 instead of having no event in states x and $x+1$ are compared. The first kind of comparison is carried out for all the operators constituting the model, whereas the second kind applies only to the operator directly affected by the altered parameter. It is clear that monotonicity of $\Delta v_{\alpha_1}(x)$ in α_1 requires both types of comparisons to hold.

The properties which will induce monotonicity on the optimal policy structure with respect to parameter changes are supermodularity, submodularity, and increasing and decreasing marginal benefit, i.e.,

$$\begin{aligned}
&\mathrm{SpM}(\alpha, x): \Delta f_\alpha(x) \geq \Delta f_{\alpha+\varepsilon}(x),\\
&\mathrm{SbM}(\alpha, x): \Delta f_\alpha(x) \leq \Delta f_{\alpha+\varepsilon}(x),\\
&\mathrm{IMB}(x): Tf(x) - f(x) \leq Tf(x+1) - f(x+1),\\
&\mathrm{DMB}(x): Tf(x) - f(x) \geq Tf(x+1) - f(x+1).
\end{aligned}$$

Intuitively, the value function, $v_\alpha(x)$, is supermodular with respect to α and x if the opportunity cost, $\Delta v_\alpha(x)$, is decreasing in α, and it is submodular if $\Delta v_\alpha(x)$ is increasing in α. The properties $\mathrm{IMB}(x)$ and $\mathrm{DMB}(x)$ refer to the second type of comparison above. An increase in parameter α generates an extra term $Tv_{\alpha+\varepsilon}(x) - v_{\alpha+\varepsilon}(x)$ in the optimality equation, whenever this increase affects the probability of observing the event represented by T.

We will consider systems with infinite waiting room (storage), in the next subsection. The finite waiting room case requires additional care, and its analysis is deferred to Sect. 3.2.2.

3.2.1 Systems with infinite waiting room (storage)

Table 2 summarizes our results, whose proofs are given in Appendix B. The columns labeled as $\mathrm{SpM}(\alpha, x)$ and $\mathrm{SbM}(\alpha, x)$ assume that the function f is supermodular and submodular in α and x, respectively. As can be observed, properties $\mathrm{SpM}(\alpha, x)$ and $\mathrm{SbM}(\alpha, x)$ hold for all parameters $\alpha \in \{\mu, \lambda, m\}$, whereas $\mathrm{IMB}(x)$ and $\mathrm{DMB}(x)$ apply only to the parameters indicated in parentheses.

We first analyze the effects of the service (production) rate, μ, and the arrival rate, λ, (i.e. $\alpha \in \{\mu, \lambda\}$). The changes in these parameters induce both kinds of comparisons, hence we need to consider preservation of super/submodularity, as well as the monotonicity of marginal benefits.

We start with super/submodularity properties: First, assume that $a_\alpha(x)$ in T_{ARR} and $b_\alpha(x)$ in T_{DEP} do not change with α. Then, an increase in α does not change the definitions of the operators. Therefore, Table 2 ensures that all operators preserve supermodularity and submodularity of a function, f, whenever f is supermodular and submodular with respect to a certain parameter α and x, where α can be μ or λ.

Table 2 Properties of the operators in systems with infinite waiting room (storage)

Operator	Properties				Additional
	SpM(α, x)	SbM(α, x)	IMB(x)	DMB(x)	condition(s)
T_{UNIF}	√	√	–	–	–
T_{COST}	√	√	–	–	–
T_{ARR}	√*	√†		√§ (λ)	* $a_\alpha(x)$: Dec(α); SbM(α, x) † $a_\alpha(x)$: Inc(α); SpM(α, x) § $a_\alpha(x) = a_\alpha$ for all $x \geq 0$
T_{DEP}	√*	√†	√* (μ)		* $b_\alpha(x)$: Inc(α); SpM(α, x) † $b_\alpha(x)$: Dec(α); SbM(α, x)
T_{CD}	√	√	√ (μ)		–
$T_{\text{C\_PRD}}$	√	√		√ (μ)	–
$T_{\text{Q\_PRC}}$	√	√		√ (λ)	–
$T_{\text{I\_PRC}}$	√	√	√ (λ)		–
$T_{\text{B\_ADM}_{i,B}}$	√	√		√ (λ)	–
$T_{\text{B\_RT}_{i,B}}$	√	√	√ (λ)		–
$T_{\text{ENV}(j)}$	√	√	–	–	–
T_{FIC}	√	√	–	–	–

If $a_\alpha(x)$ and $b_\alpha(x)$ vary with α, then we need additional conditions on how α affects these functions. For example, consider a system where the arrivals and departures are modeled by operators T_{ARR} and T_{DEP}, respectively. Now the service rate μ is increased by ε so that we expect the opportunity cost of an additional customer in a system to decrease, or equivalently supermodularity of $v_\mu(x)$ with respect to μ and x. Hence, among others, we need to check whether the operator T_{ARR} preserves supermodularity with respect to the service rate μ. If $a_\mu(x)$, the joining probability, increases with μ in all states x, then the overall load of the system may increase, which may, in turn, increase the opportunity costs, contradicting supermodularity. However, if $a_\mu(x)$ is decreasing in μ and submodular in μ and x, as indicated in Table 2, then T_{ARR} preserves supermodularity. Table 2 specifies exact conditions on $a_\alpha(x)$ and $b_\alpha(x)$ for T_{ARR} and T_{DEP} to preserve supermodularity and submodularity.

As a result, all operators preserve both supermodularity and submodularity with respect to α and x, where $\alpha \in \{\mu, \lambda\}$ (under mild conditions). Hence, it is the monotonicity of the marginal benefits, which determines whether the value function $v_\alpha(x)$ is super/submodular with respect to α and x. Table 2 shows that all operators associated with parameters $\{\mu, \lambda\}$ have either increasing or decreasing marginal benefits, corresponding to supermodularity or submodularity, respectively, of $v_\alpha(x)$ with respect to α and x.

In the context of queueing control, Table 2 states that departure-related operators have an increasing marginal benefit, i.e., the departure of an existing customer is more valuable when there is one more customer in the system. This implies the supermodularity of the value functions with respect to μ and x. Therefore, an increase in μ decreases the opportunity costs, as expected. On the other hand, arrival-related operators have a decreasing marginal benefit, so the value functions are submodular with respect to λ and x. It should be noted that, this property is guaranteed for T_{ARR} only when the joining probability, $a_\lambda(x)$, is constant in x. If $a_\lambda(x)$ is strictly decreasing in x, the system faces two opposing effects: On the one hand, as the arrival rate increases, the load of the system becomes higher so that the arrival of a new customer has a higher burden on the system. On the other hand, a new customer also relieves this burden because s/he induces a decrease in the joining probability.

An increase in m changes the definition of operator T_{DEP} (see (1)), but does not induce a term for marginal benefit. Since $b_m(x)$ is increasing in m and supermodular with respect to m and x, $b_m(x)$ satisfies the condition for preserving supermodularity with respect to m and x, but not submodularity. This is intuitive, since increasing the number of servers, or service capacity, cannot increase the opportunity costs. Finally, we note that the joining probability in the T_{ARR}, $a_m(x)$, may also depend on m. However, only supermodularity with respect to m and x is relevant, and for this it is sufficient to have $a_m(x)$ to be decreasing in m and submodular in m and x from Table 2.

In short, all queueing models that can be represented as a combination of the operators described here satisfy supermodularity with respect to μ and m. Queueing models satisfying the submodularity property with respect to λ can be built as a combination of all operators, whenever $a_\alpha(x)$ of T_{ARR} is constant for all x.

A similar investigation of Table 2 for inventory models shows that the value functions satisfy supermodularity with respect to λ and submodularity with respect to μ for all possible combinations of the operators considered here.

3.2.2 Systems with finite waiting room (storage)

Table 3 summarizes our results whose proofs are given in Appendix B. All assumptions and notations we have described for Table 2 are also valid for Table 3.

We first note that operator T_{ARR} does not appear in the table, since the behavior of T_{ARR} is rather complicated in systems with finite waiting room. For details, we refer to Appendix C.

The effects of the service (production) rate, μ, the arrival rate, λ, and the number of servers m in systems with finite waiting room (storage) are the same with those in infinite capacity systems, as long as $m < K$. Hence, we will concentrate on the effects of K and m when $m = K$ which is the case for the well-known loss system.

The value functions preserve supermodularity and submodularity with respect to K and x for all operators, except for those directly affected by K. These operators are the two controllable arrival-related operators in queueing systems, $T_{\mathrm{Q\_PRC}}$, $T_{\mathrm{B\_ADM}_{i,B}}$, and the production operator of inventory systems, $T_{\mathrm{C\_PRD}}$. They preserve only supermodularity of the value functions with respect to K and x since an increase in K increases the action spaces of these operators. For example, consider $T_{\mathrm{B\_ADM}_{i,B}}$ in a queueing system with waiting room capacity K: if it is optimal to reject the arrivals, the optimal policy does not change when the capacity increases, whereas if the arrivals are rejected due to the limited capacity, a new customer may be accepted when there are K customers after an increase in the capacity. Then, the opportunity cost of a new customer can only decrease by an increase in K. A similar argument can be given for operators $T_{\mathrm{Q\_PRC}}$ and $T_{\mathrm{C\_PRD}}$. Thus, these operators preserve only

Table 3 Properties of the operators in systems with finite waiting room (storage) of size K

Operator	Properties				Additional condition(s)
	SpM(α, x)	SbM(α, x)	IMB(x)	DMB(x)	
T_{UNIF}	$\surd$	$\surd$	–	–	–
T_{COST}	$\surd$	$\surd$	–	–	–
T_{DEP}	$\surd^*$	$\surd^\dagger$	$\surd^*$ (μ)		* $b_\alpha(x)$: Inc(α); SpM(α, x) † $b_\alpha(x)$: Dec(α); SbM(α, x)
T_{CD}	$\surd$	$\surd$	$\surd$ (μ)		–
$T_{\text{C\_PRD}}$	$\surd$	$\surd^*$		$\surd$ (μ)	* not valid for $\alpha = K$
$T_{\text{Q\_PRC}}$	$\surd$	$\surd^*$		$\surd$ (λ)	* not valid for $\alpha = K$
$T_{\text{I\_PRC}}$	$\surd$	$\surd$	$\surd$ (λ)		–
$T_{\text{B\_ADM}_{i,B}}$	$\surd$	$\surd^*$		$\surd$ (λ)	* not valid for $\alpha = K$
$T_{\text{B\_RT}_{i,B}}$	$\surd$	$\surd$	$\surd$ (λ)		–
$T_{\text{ENV}(j)}$	$\surd$	$\surd$	–	–	–
T_{FIC}	$\surd$	$\surd$	–	–	–

supermodularity with respect to K and x. Appendix D presents an example where submodularity is not preserved with respect to K and x by $T_{\text{B\_ADM}_{i,B}}$. Finally, the effects of the number of servers, m, when $m = K$, are similar to those of K.

To summarize, all finite-capacity queueing models that can be represented as a combination of the operators given in Table 3 satisfy supermodularity with respect to μ and m, and submodularity with respect to λ (under mild conditions on b_α). All such inventory models satisfy submodularity with respect to μ and supermodularity with respect to λ. Moreover, all systems represented as a combination of above operators preserve supermodularity with respect to K. Since all queueing systems require an arrival process (either one of the two operators $T_{\text{Q\_PRC}}$ and $T_{\text{B\_ADM}_{i,B}}$) and all inventory systems require a production process ($T_{\text{C\_PRD}}$), none of the systems that can be constructed by a combination of these operators have submodular value functions with respect to K and x.

3.3 Effects of parameters on structural properties of optimal control policies

Now we summarize our results on queueing systems by the following theorem:

Theorem 1 *In a queueing system represented as a combination of operators introduced above, the opportunity cost, $\Delta v_\alpha(x)$, is increasing in α when $\alpha = \lambda$, and decreasing in α when $\alpha \in \{\mu, m, K\}$.*

Proof First consider an increase in an arrival rate λ: Sects. 3.2.1 and 3.2.2 show that $\Delta v_\lambda(x)$ is submodular with respect to λ and x, so that $\Delta v_\lambda(x)$ is increasing in λ.

Now consider an increase in service capacity by letting α be either μ or m. Then, by Sects. 3.2.1 and 3.2.2, $v_\alpha(x)$ is supermodular with respect to α and x, and $\Delta v_\alpha(x)$ is decreasing in α.

Finally, consider an increase in K. Section 3.2.2 ensures that $\Delta v_K(x)$ is supermodular with respect to K and x, and $\Delta v_K(x)$ is decreasing in K. □

A consequence of Theorem 1 is that optimal prices are increasing and optimal admission thresholds are decreasing in λ, while an increase in μ and K decreases the prices and increases the optimal thresholds.

Theorem 2 *In an inventory system represented as a combination of the operators introduced above, the opportunity cost, $\Delta v_\alpha(x)$, is decreasing in α when $\alpha \in \{\lambda, K\}$ and increasing in α when $\alpha = \mu$.*

Proof The proof is similar to that of Theorem 1 and follows from the results in Sects. 3.2.1 and 3.2.2. □

The consequences of Theorem 2 are as follows: an increase in any demand rate or in storage capacity will increase optimal prices and optimal rationing threshold levels. An increase in production rate, on the other hand, will have the opposite effects on each of these controls. In addition, the optimal used portion π of the potential production rate increases by an increase in the demand rate or storage capacity and decreases by production rate.

4 Illustration of results and examples

4.1 Illustration of results on admission control problems

We consider an admission control problem in a queueing system with m identical parallel servers and infinite waiting room. The system receives N different types of customers, where class-i customers arrive at the system according to a Poisson process with a rate λ_i bringing a reward of R_i, and require an exponential service time with rate μ. Then, the state of the system can be defined as the number of customers in the system x, where $x \in \mathbb{Z}^+$, $\mathbb{Z}^+ = \{0, 1, \dots\}$. A non-negative holding cost of $h(x)$ is incurred per unit time, where $h(x)$ is a convex and increasing function of x. At each arrival epoch, the decision maker either accepts the incoming customer or rejects her in order to maximize the discounted expected profit over an infinite horizon (with a discount rate of β). We assume that a rejected customer is lost forever. This is a simplified version of the model in [5], where all batches have unit sizes. Similar admission control problems for queueing systems have been studied by for instance Stidham [25] and Blanc et al. [3].

By standard arguments (see [18]), we employ uniformization to form the equivalent discrete-time model. In particular, we assume that $\sum_{i=1}^{N} \lambda_i + M\mu + \theta + \beta = 1$ without loss of generality, where $M \geq m + 1$ and $\theta > 0$. We note that $\theta + \mu(M - m)$ corresponds to the rate of fictitious events, introduced to ensure that the time scale will not be affected when one of the parameters, μ or λ_i or m, is varied.

Let us denote by $v^n(x)$ the total expected β-discounted profit of such a system when there are n remaining transitions in the horizon. Using event-based operators, we can write:

$$v^{n+1}(x) = T_{\text{COST}}\big(T_{\text{UNIF}}\big(\big\{T_{\text{DEP}}v^n(x), \big\{T_{\text{B\_ADM}_i}v^n(x)\big\}_i, T_{\text{FIC}}v^n(x)\big\}; \\ \big\{M\mu, \{\lambda_i\}_i, \theta\big\}\big)\big),$$

where, in this case, the generic operator $T_{\text{B\_ADM}_{i,B}}$ is employed with a fixed batch size of $B = 1$.

In this particular model, having $\beta > 0$ guarantees the existence of a solution for a discounted infinite horizon problem and the infinite horizon value function can be obtained by: $v(x) = \lim_{n\to\infty} v^n(x)$. It should be noted that, taking $\beta \to 0$ in $v(x)$ converts the problem to maximizing the long-run average profit in the usual way, see e.g., [22] and the results directly carry over to that case as well. In the rest of this section, the criterion of discounted infinite horizon profit maximization is used.

From Table 1, it follows that all the operators T_{UNIF}, $T_{\text{B\_ADM}_i}$, T_{DEP}, T_{COST}, T_{FIC} preserve monotonicity and concavity, which guarantees the existence of an optimal threshold policy leading to the well-known threshold structure as summarized below:

Proposition 1 *There exists an optimal policy of threshold type, i.e., there exist numbers* $l_i^* \geq 0$ *for* $i = 1, \ldots, N$*, such that: If* $x \geq l_i^*$*, it is optimal to reject an incoming class-i customer; otherwise it is optimal to admit her. Moreover, if the rewards are ordered as* $R_1 \geq R_2 \geq \cdots \geq R_N$*, then* $l_1^* \geq l_2^* \geq \cdots \geq l_N^*$.

As for the effects of system parameters, an increase in the service rate μ has a direct effect on T_{DEP}. By Table 2, T_{DEP} has the increasing marginal benefits property, and supermodularity propagates for all operators of the model. As for an increase in the arrival rates λ_i, it can be seen from Table 2, that $T_{\text{B\_ADM}_i}$ has the property of decreasing marginal benefits, and all operators support submodularity in λ_i and x. The following theorem summarizes the policy implications of the results:

Theorem 3 *In this admission control problem, the optimal thresholds* l_i^* *are increasing in the service rate,* μ*, and the number of servers,* m*, and decreasing in the arrival rates* λ_i.

4.2 Other examples from the literature

Several known models from the literature can easily be analyzed for the effects of changes in the parameters using the framework provided.

We start with the basic queueing control models introduced in Lippman [18]. The first model is an admission control problem, which is very similar to the above example. This model can be represented as a combination of the batch admission operator, $T_{\text{B\_ADM}_{i,B}}$ (with batch sizes of 1 and modified for the finite buffer size with $K = m$), the departure operator T_{DEP} (with $b(x)$ given by (1)), T_{FIC}, and T_{UNIF}. It is shown, similarly, that optimal thresholds, l_i^*, exist. Moreover, since the operator T_{COST} is excluded from this model, all operators preserve the UBD property, so that class 1 is always preferred if the rewards are ordered $R_1 > R_2 > \cdots > R_N$. In addition to these existing results, using the properties given in Table 3, we can conclude that the optimal thresholds l_j^* are increasing in μ and m, and decreasing in λ_i for all i, j. We finally note that these results are also valid for an extension of this model to a loss system with batch arrivals as considered by Örmeci and Burnetas [20].

The second model of Lippman is an $M/M/1$ queue with a controllable service rate consisting of the operators T_{CD}, T_{ARR} and T_{FIC}. It is shown that the optimal service rate is increasing in the queue length. We can complement these results by establishing, for example, that the optimal service rate is increasing in the arrival rate. The third model of Lippman considers a dynamic pricing problem in an $M/M/m$ queue. This problem employs the operators $T_{\mathrm{Q\_PRC}}$, T_{DEP} and T_{FIC}. The optimal price to be charged to an arriving customer is shown to be increasing in the queue length. The effects of parameters on this problem have recently been analyzed independently by Aktaran-Kalaycı and Ayhan [1] and Çil, Karaesmen and Örmeci [4]. Our framework also easily yields that the optimal prices are decreasing in the service rate and in the number of servers, whereas they are increasing in the arrival rate.

Most basic control problems for make-to-stock queues can also be analyzed within this framework. Consider the standard problem of production control of a single processor with exponential processing times, Poisson demand arrivals, linear holding costs and lost sales (see Veatch and Wein [28]). The optimal production policy for this problem is a threshold policy called a base stock policy: it is optimal to produce whenever the inventory level is below the base stock level and not to produce otherwise. The optimality equation is composed of the operators $T_{\mathrm{C\_PRD}}$, and T_{DEP}, in addition to T_{COST} and T_{FIC}. It then follows by Table 2 that the optimal base stock level is decreasing in μ. Li [17] considers a make-to-stock queue with dynamic pricing of inventories. In this model, the Poisson demand rate depends on the price chosen. Li shows that the optimal production policy is of base-stock type, and optimal prices are decreasing in the inventory level. The optimality equation of this model is composed of the operators $T_{\mathrm{C\_PRD}}$, and $T_{\mathrm{I\_PRC}}$, in addition to T_{COST} and T_{FIC}. Using the lemmas, we can establish that the base stock level is decreasing and the optimal prices are decreasing in the processing rate. Even an extension of Li, when the demand arrivals occur according to a Markov modulated Poisson process (MMPP), studied by Gayon et al. [8], falls into our framework by adding the operator $T_{\mathrm{ENV}(j)}$. For instance, an increased arrival rate in any one of the environment states leads to higher optimal prices.

4.3 A stock rationing problem for a make-to-stock queue with batch arrivals

Our final example investigates a stock rationing problem with batch demand arrivals. To our knowledge, this version of the problem has not been analyzed before and the framework is employed for both obtaining new structural results and investigating the effects of parameters.

Different customer classes with difference rewards have to be satisfied from a common stock. It may be optimal to not fulfill certain demand types in anticipation of future demand from more valuable classes. The basic stock rationing problem goes back to Topkis [26] who studies the optimal ordering and rationing policies for a single-product inventory system with several demand classes under periodic review. Our focus here is on a recent stream of research that uses the make-to-stock queueing framework to model limited production capacity. For instance, Ha [9], [10], and de Véricourt et al. [6] examine such inventory systems under different assumptions. In particular, Ha [9] considers the stock rationing problem in a make-to-stock production system with several demand classes and lost sales and characterizes the structure of the optimal policy.

Consider an extension of the model introduced by Ha [9]: a make-to-stock production system that produces a single product with N demand classes with batch demand and lost sales. Demand arrivals of class i occur according to a Poisson process with rate λ_i, and each class-i customer requests B units of products with probability p_{iB}, where $i = 1, \dots, N$ and $B = 1, 2, \dots$. We assume that the demands may be partially satisfied. Whenever a class-i demand is satisfied, a reward of $R_i > 0$ per unit is obtained, and without loss of generality we assume that $R_1 \geq \cdots \geq R_N$. The production time is assumed to be exponentially distributed with mean $1/\mu$. Moreover, the inventory holding cost per unit time, $h(x)$, is an increasing and convex function of the on-hand inventory.

At any time, the decision maker has to decide whether to produce or not, and the number of products to be rationed to class-i customers, κ_i. We assume that the variable production cost is c, which is set to 0 in [9]. Then, we represent the arrivals and rationing by $T_{\mathrm{B\_RT}_{i,B}}$, the production control by $T_{\mathrm{C\_PRD}}$, where $\pi \in \{0, 1\}$ and $c_\pi = \pi c$, and the fictitious events by T_{FIC}.

The objective of the problem is to maximize the expected total β-discounted reward over an infinite horizon. As before, we use uniformization and assume that assume that $\mu + \sum_{i=1}^{N} \lambda_i + \theta + \beta = 1$. The value function $v(x)$, which is defined as the total expected β-discounted profit over an infinite horizon when the system starts in state x can then be expressed as:

$$v(x) = T_{\mathrm{COST}}\big(T_{\mathrm{UNIF}}\big(\big\{T_{\mathrm{C\_PRD}} v(x), \big\{T_{\mathrm{B\_RT}_{i,B}} v(x)\big\}_{i,B}, T_{\mathrm{FIC}} v(x)\big\}; \big\{\mu, \{\lambda_i p_{iB}\}_{i,B}, \theta\big\}\big)\big).$$

4.3.1 Structure of the optimal policy

In the original model [9], Ha shows that the optimal production control and rationing policies are of threshold type, and class 1 is preferred, i.e., it is always optimal to satisfy class-1 demands whenever it is possible. We investigate the existence of similar structures in the extended model which would be implied by concavity and the LBD property of the $v(x)$.

The concavity of $v(x)$ is directly verified by Table 1. This, in turn, has the following implication for the optimal production policy. Let:

$$S^* = \min\big\{x : v(x) - v(x+1) > -c\big\}. \tag{3}$$

Due to the concavity of $v(x)$, for all states $x \geq S^*$, it is not worth producing a new unit; while for all $x < S^*$ it is always optimal to produce. S^* given by (3) is therefore the optimal production threshold.

Now we consider the effect of concavity on the rationing policy. Let:

$$l_i^* = \max\big\{x : v(x-1) - v(x) < -R_i\big\}, \tag{4}$$

with $l_i^* = 0$ if there is no such x. Therefore, when $x = l_i^*$, it is optimal to reject the whole class-i batch since $v(l_i^* - 1) + R_i < v(l_i^*)$, while for $x = l_i^* + 1$ it is optimal to satisfy one unit from a class-i batch because $v(l_i^*) + R_i \geq v(l_i^* + 1)$, by the definition of l_i^*. Then for all $x \leq l_i^*$, $v(x-1) + R_i < v(x)$ due to the concavity of $v(x)$, so that it will be always optimal to reject the whole batch in all states $x \leq l_i^*$, i.e., $\kappa_i^*(x) = 0$. Similarly, for all $x \geq l_i^* + 1$, $v(x-1) + R_i \geq v(x)$ by concavity, so that it will be optimal to satisfy class-i demand until either the inventory level drops down to l_i^* or the whole batch is satisfied. Hence, $\kappa_i^*(x) = \min\{x - l_i^*, B\}$ for all $x \geq l_i^* + 1$. In other words, the optimal rationing policy will reject the whole class-i batch if $x \leq l_i^*$, partially satisfy the demand if $l_i^* < x < l_i^* + B$, and satisfy the entire batch if $x \geq l_i^* + B$. Therefore, a low threshold l_i^* leads to a higher fulfillment rate of class-i demand. In fact, if the reward obtained by admitting a class-i customer is higher than the reward of a class-j customer, then the optimal threshold of class-i will be lower than that of class-j as a result of the definition of l_i^*, i.e., when $R_i \geq R_j$, $l_i^* \leq l_j^*$. Moreover, it is always optimal to satisfy class-1 customers whenever it is possible, i.e., class 1 is preferred. This result is easily established by Table 1, since all operators of the model preserve the LBD(R_1) property (see the proof of Table 1 for details). The following theorem establishes the structure of an optimal policy for this model.

Theorem 4 *In this stock rationing model*:

- *A base-stock policy is an optimal production control policy.* S^* (*given by* (3)) *is the base-stock level so that it is optimal to produce if and only if* $x < S^*$.
- *The optimal rationing policy is a sequential threshold policy for each demand class*, *where optimal thresholds are given by* (4). *Then*, *the optimal number of class-i customers to be satisfied from an arriving batch in state* x, $\kappa_i^*(x)$, *is given by*:

$$\kappa_i^*(x) = \begin{cases} \min\{B, x - l_i^*\} & \text{if } x > l_i^*, \\ 0 & \text{if } x \leq l_i^*. \end{cases}$$

Moreover, l_i^**'s are monotone in* i *and class* 1 *is preferred, i.e.*, $l_N{}^* \geq \cdots \geq l_1{}^* = 0$.

Remark 1 The concavity of $T_{\mathrm{B\_RT}_i}$ was established by Lautenbacher and Stidham [15] in a revenue management context. A version of Theorem 4 has independently been obtained by Huang and Iravani [11]. Our approach not only facilitates the overall proof but can also be employed to generalize the model using other operators that preserve concavity.

4.3.2 Effects of the parameters: μ, λ_i

From Table 2, $v_\mu(x)$ is submodular with respect to μ and x and $v_{\lambda_i}(x)$ is supermodular with respect to λ_i and x. Since both properties are shown similarly, we only provide an expanded version of the latter one, which requires:

$$\mu \Delta T_{\text{C\_PRD}} v_{\lambda_i}(x) + \sum_{j=1}^{N} \lambda_j \sum_B p_{jB} \Delta T_{\text{B\_RT}_{j,B}} v_{\lambda_i}(x) + \theta \Delta T_{\text{FIC}} v_{\lambda_i}(x) - \Delta h(x)$$

$$\geq \mu \Delta T_{\text{C\_PRD}} v_{\lambda_i+\varepsilon}(x) + \sum_{j=1}^{N} \lambda_j \sum_B p_{jB} \Delta T_{\text{B\_RT}_{j,B}} v_{\lambda_i+\varepsilon}(x)$$

$$+ \theta \Delta T_{\text{FIC}} v_{\lambda_i+\varepsilon}(x) - \Delta h(x)$$

$$+ \varepsilon \left[\sum_B p_{iB} \left(\Delta T_{\text{B\_RT}_{i,B}} v_{\lambda_i+\varepsilon}(x) - \Delta v_{\lambda_i+\varepsilon}(x) \right) \right],$$

where all four lines satisfy the inequality by Table 2, with first three lines due to SpM properties of all operators and the last line due to IMB property of operator $T_{\text{B\_RT}_{i,B}}$. Hence, $v_{\lambda_i}(x)$ is supermodular with respect to λ_i and x. The overall result is summarized by the following theorem.

Theorem 5 *In this stock rationing model, the optimal base-stock level, S^*, and the optimal rationing thresholds, l_i^*, are decreasing in μ and increasing in λ_i for all $i = 1, \dots, N$.*

5 Conclusion

We presented a general framework for investigating the effects of system parameters on the optimal policy for a class of queueing and inventory control problems. Our approach is based on an exploration of the properties of the operators that constitute the value function of the given control problem. The approach gives a clear guideline of how system parameters affect the structure of the operators and the consequent effects on the optimal policy. In addition, for a rich class of problems, the framework enables a direct assessment of how changes in different system parameters affect the optimal policy.

There are several interesting research directions. Extending the scope of application of the general approach is one potential direction. Discrete-time models with non-stationary parameters can be handled with some modifications. An example is provided by Aydin, Akcay and Karaesmen [2]. Additional control operators could be investigated and possibly a more general class of problems could be addressed. Exploring multi-dimensional problems is also a major challenge. As always these problems pose additional difficulties and an immediate general extension seems very hard but it would be interesting to understand to what extent the results here could be generalized to multi-dimensional problems.

Acknowledgements We thank Yalçın Akçay, Refik Güllü and Evrim Güneş for their helpful comments, and Ger Koole for sharing his related work. This work has been partially supported by grants from TÜBA and TÜBİTAK.

Appendix A: Proofs of the statements given in Table 1

In this appendix, we present the proofs of the properties presented earlier. In particular, we present the complete proofs for the key operators and omit them for other operators when the approach is similar. In addition, the proofs are presented mainly for systems with infinite buffer capacity. The differences in the finite capacity case are shortly discussed at the end of each proof. Finally, in all proofs below and in Appendix B, we implicitly assume that the optimal action is unique. Non-uniqueness of the optimal action can be resolved by using additional conditions or defining appropriate set-based orders for each operator. For instance, a hazard-rate based condition ensures the uniqueness of the optimal price for the pricing operator but we do not elaborate on this issue further.

A.1 Monotonicity

This section focuses on the monotonicity properties of the operators introduced in Sect. 3. The desired property is:

$$Tf(x) \geq Tf(x+1), \tag{5}$$

whenever $f(x)$ is a decreasing function of x. Below, we present the proofs for $T_{\mathrm{B\_ADM}_i}$ and $T_{\mathrm{Q\_PRC}}$. The other proofs are similar and can be obtained from the authors.

A.1.1 Monotonicity preserved by $T_{\mathrm{Q\_PRC}}$

Let p_x and p_{x+1} be the optimal prices for the states x and $x+1$, respectively. Then, inequality (5) for $T_{\mathrm{Q\_PRC}}$ is as follows:

$$\begin{aligned}&\bar{F}(p_x)\big[f(x+1)+p_x\big]+F(p_x)f(x)\\&\quad\geq \bar{F}(p_{x+1})\big[f(x+2)+p_{x+1}\big]+F(p_{x+1})f(x+1).\end{aligned} \tag{6}$$

We have the following inequalities by the definition of the pricing operator and the monotonicity of $f(x)$:

$$\begin{aligned}&\bar{F}(p_x)\big[f(x+1)+p_x\big]+F(p_x)f(x)\\&\quad\geq \bar{F}(p_{x+1})\big[f(x+1)+p_{x+1}\big]+F(p_{x+1})f(x),\\&\bar{F}(p_{x+1})\big[f(x+1)+p_{x+1}\big]+F(p_{x+1})f(x)\\&\quad\geq \bar{F}(p_{x+1})\big[f(x+2)+p_{x+1}\big]+F(p_{x+1})f(x+1).\end{aligned}$$

Combining these inequalities ensures that inequality (6) holds, and so $T_{\mathrm{Q\_PRC}}$ is decreasing in x. The pricing operator can also be used in the capacitated queues. In this case, we need to observe the boundary effects in state $x=K-1$. However, since we will use the optimality of p_{K-1}, the foregoing proof is still true. Note that p_K can be taken as a large enough price to set $\bar{F}(p_K)=0$.

A.1.2 Monotonicity preserved by $T_{\text{B\_ADM}_i}$

Let $\kappa_x^{iB_i}$ and $\kappa_{x+1}^{iB_i}$ be the optimal numbers of class-i customers to be admitted from an arriving batch in states x and $x+1$. Then, we can write inequality (5) for the operators as follows:

$$\kappa_x^{iB_i} R_i + f\big(x + \kappa_x^{iB_i}\big) \geq \kappa_{x+1}^{iB_i} R_i + f\big(x + 1 + \kappa_{x+1}^{iB_i}\big). \tag{7}$$

Since $\kappa_x^{iB_i}$ is the optimal action for the state x and $f(x)$ is decreasing in x, we have:

$$\kappa_x^{iB_i} R_i + f\big(x + \kappa_x^{iB_i}\big) \geq \kappa_{x+1}^{iB_i} R_i + f\big(x + \kappa_{x+1}^{iB_i}\big), \quad \text{and}$$
$$\kappa_{x+1}^{iB_i} R_i + f\big(x + \kappa_{x+1}^{iB_i}\big) \geq \kappa_{x+1}^{iB_i} R_i + f\big(x + 1 + \kappa_{x+1}^{iB_i}\big).$$

As in the previous case, combining these inequalities completes the proof. Therefore, $T_{\text{B\_ADM}_i} f(x)$ is decreasing in x if $f(x)$ is a decreasing function of x. As in the pricing operator, the proof for the monotonicity in systems with finite capacity does not change due to the optimality of $\kappa_x^{iB_i}$.

A.2 Upper-bounded difference, UBD

Here we show that our queueing operators preserve the UBD property, i.e.,

$$Tf(x) - Tf(x+1) \leq U, \tag{8}$$

whenever $f(x) - f(x+1) \leq U$ for some $U > 0$. The operators T_{ARR}, T_{DEP}, T_{CD} are not involved with any rewards so that they preserve UBD property for any $U > 0$. On the other hand, the operators $T_{\text{Q\_PRC}}$ and $T_{\text{B\_ADM}_i}$ generate revenue directly, so for these operators we will specify a positive value for U. For this purpose, we will specify a maximum price for $T_{\text{Q\_PRC}}$ and a maximum revenue for $T_{\text{B\_ADM}_i}$. We present the proofs for these two operators below, the proofs for the other operators are similar.

A.2.1 UBD property preserved by $T_{\text{Q\_PRC}}$

We make the realistic assumption that allowable prices are bounded. Then $p_{\max} = \sup\{p_x : x = 0, 1, 2, \ldots\} < \infty$, where p_x is an optimal price in state x. We set $U = p_{\max}$, so $f(x) - f(x+1) \leq p_{\max}$ for all x.

The operator $T_{\text{Q\_PRC}}$ preserves the UBD property of function f, only if f is a concave function of x. Hence, we assume that $f(x) - f(x+1) \leq f(x+1) - f(x+2)$, which ensures the monotonicity of optimal prices, i.e., $p_x \leq p_{x+1}$ for all x.

Now, we write inequality (8) for $T_{\text{Q\_PRC}}$ as:

$$\bar{F}(p_x)\big[f(x+1) + p_x\big] + F(p_x)f(x) - \bar{F}(p_{x+1})\big[f(x+2) + p_{x+1}\big]$$
$$- F(p_{x+1})f(x+1) \leq p_{\max}. \tag{9}$$

Table 4 Possible optimal actions in states x, $x+1$ with operator $T_{\mathrm{B\_ADM}_i}$

Cases	$(\kappa_x^{iB_i}, \kappa_{x+1}^{iB_i})$	Rewritten form of inequality (11)
Case I	(a, a)	$f(x+a) - f(x+a+1) \le R_1$
Case II	$(a+1, a)$	$R_i \le R_1$

Since $p_x \le p_{x+1}$, $\bar{F}(p_x) \ge \bar{F}(p_{x+1})$. Then, we can manipulate the LHS of the inequality to have:

$$\begin{aligned}&\bar{F}(p_{x+1})\big[f(x+1) + p_x - f(x+2) - p_{x+1}\big]\\&\quad+\big[\bar{F}(p_x) - \bar{F}(p_{x+1})\big]\big[f(x+1) + p_x - f(x+1)\big]\\&\quad+F(p_x)\big[f(x) - f(x+1)\big] \le p_{\max},\end{aligned} \tag{10}$$

which is always true since $f(x) - f(x+1) \le p_{\max}$ and $p_x \le p_{x+1} \le p_{\max}$ for all $x \ge 0$. Thus $T_{\mathrm{Q\_PRC}} f(x) - T_{\mathrm{Q\_PRC}} f(x+1) \le p_{\max}$.

For systems with finite capacity, we need to consider state $x = K-1$ to observe the boundary effects. For this state, only the first line in the LHS of inequality (10) changes, and the inequality is still true since $p_x \le p_{x+1}$. In fact, in finite systems, we can specify $p_{\max}$ as p_{K-1} due to the monotonicity of optimal prices.

A.2.2 UBD property preserved by $T_{\mathrm{B\_ADM}_i}$

As for $T_{\mathrm{Q\_PRC}}$, we need to specify a value of U for $T_{\mathrm{B\_ADM}_i}$. For this purpose, we assume that there are N classes of customers, where class-i customers bring a reward of R_i with $R_1 \ge R_2 \ge \cdots \ge R_N$, without loss of generality. Then, the UBD property is valid for $U = R_1$, the maximal reward that can be obtained from a customer. Hence, we assume $f(x) - f(x+1) \le R_1$, where R_1 is the reward associated with class 1. Moreover, the operator $T_{\mathrm{B\_ADM}_i}$ preserves the UBD property of function f, only when f is a concave function of x. Therefore, we also assume that $f(x) - f(x+1) \le f(x+1) - f(x+2)$.

Let $\kappa_x^{iB_i}$ be optimal number of customers to be admitted from an arriving batch of size B_i in state x. Now we need to show for all possible $(\kappa_x^{iB_i}, \kappa_{x+1}^{iB_i})$ and for all i:

$$\kappa_x^{iB_i} R_i + f\big(x + \kappa_x^{iB_i}\big) - \kappa_{x+1}^{iB_i} R_i - f\big(x + 1 + \kappa_{x+1}^{iB_i}\big) \le R_1. \tag{11}$$

It can easily be shown that concavity of f implies that $\kappa_x^{iB_i}$ and $\kappa_{x+1}^{iB_i}$, optimal decisions in state x and $x+1$, respectively, satisfy either $\kappa_x^{iB_i} = \kappa_{x+1}^{iB_i}$, or $\kappa_x^{iB_i} = \kappa_{x+1}^{iB_i} + 1$. Then, it is enough to consider the two cases: (a, a) with $0 \le a \le B_i$ and $(a, a+1)$ with $0 \le a < B_i$. We rewrite inequality (11) for each case in Table 4. Case II is true since R_1 is the highest reward offered by the customers, whereas case I is also true by the assumption $f(x) - f(x+1) \le R_1$. Thus, inequality (11) is true for all cases, so that $T_{\mathrm{B\_ADM}_i} f(x) - T_{\mathrm{B\_ADM}_i} f(x+1) \le R_1$.

In systems with finite capacity, we need to observe the states $x \ge K - B_i$ in order to see the boundary effects. However, optimal decisions in these states also will satisfy one of the cases given in Table 4. Hence, $T_{\mathrm{B\_ADM}_i}$ preserves UBD property in these systems as well.

A.3 Lower-bounded difference, LBD

Here, we establish for $T_{\text{C\_PRD}}$, $T_{\text{I\_PRC}}$ and $T_{\text{B\_RT}_i}$ that if $f(x)$ is an LBD function, then:

$$Tf(x) - Tf(x+1) \geq L, \tag{12}$$

where L is generally negative. $T_{\text{C\_PRD}}$ will preserve the LBD property for any value of L, whereas for the operators $T_{\text{I\_PRC}}$ and $T_{\text{B\_RT}_i}$ we will give specific values for L. We present the proof of $T_{\text{I\_PRC}}$ below and the other proofs are similar.

A.3.1 LBD property preserved by $T_{\text{I\_PRC}}$

The operator $T_{\text{I\_PRC}}$ preserves the LBD property of function f, if f is a concave function of x. Hence, we assume that $f(x) - f(x+1) \leq f(x+1) - f(x+2)$, which ensure the monotonicity of optimal prices, so that if p_x denotes an optimal price in state x, then $p_{x+1} \leq p_x$. We let $L = -p_1 = -\max\{p_x : x \geq 1\}$. Now, we write inequality (12) for $T_{\text{I\_PRC}}$ as follows:

$$\begin{aligned}
&\bar{F}(p_x)\big[f(x-1) + p_x\big] + F(p_x)f(x) \\
&\quad - \bar{F}(p_{x+1})\big[f(x) + p_{x+1}\big] - F(p_{x+1})f(x+1) \geq -p_1.
\end{aligned} \tag{13}$$

Since $p_x \geq p_{x+1}$, $\bar{F}(p_x) \leq \bar{F}(p_{x+1})$. Then, we can manipulate the LHS of the inequality to have:

$$\begin{aligned}
&\bar{F}(p_x)\big[f(x-1) + p_x - f(x) - p_{x+1}\big] \\
&\quad + \big[\bar{F}(p_{x+1}) - \bar{F}(p_x)\big]\big[f(x) - p_{x+1} - f(x)\big] + F(p_{x+1})\big[f(x+1) - f(x)\big] \\
&\geq -p_1,
\end{aligned}$$

which is true since $f(x) - f(x+1) \geq -p_1$ and $p_{x+1} \leq p_x \leq p_1$ for all x.

A.4 Concavity

We now show that all operators defined in Sect. 3 preserve concavity of function f. We present the proofs for T_{CD} and $T_{\text{B\_ADM}_i}$ since the other proofs are similar. However, we note that T_{ARR} preserves concavity only when $a(x)$ is decreasing and convex in x and that T_{DEP} preserves concavity when $b(x)$ is concave.

A.4.1 Concavity preserved by T_{CD}

Let π_x be optimal service rate in state x. Then, the concavity inequality of the operator is:

$$\begin{aligned}
&-c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x)f(x) + c_{\pi_{x+1}} - \pi_{x+1}f(x) - (1-\pi_{x+1})f(x+1) \\
&\quad \leq -c_{\pi_{x+1}} + \pi_{x+1}f(x) + (1-\pi_{x+1})f(x+1) \\
&\qquad + c_{\pi_{x+2}} - \pi_{x+2}f(x+1) - (1-\pi_{x+2})f(x+2).
\end{aligned} \tag{14}$$

We first consider the LHS of inequality (14). Since π_{x+1} is the optimal service rate in state $x+1$, we have:

$$\begin{aligned}
&-c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) + c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) \\
&\quad \le -c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) + c_{\pi_x} - \pi_x f(x) - (1-\pi_x) f(x+1) \\
&\quad \le f(x) - f(x+1) + \pi_x \big(\big[f(x-1) - f(x)\big] - \big[f(x) - f(x+1)\big]\big) \\
&\quad \le f(x) - f(x+1),
\end{aligned} \tag{15}$$

where the second inequality is true due to some algebra and the third by the concavity of $f(x)$.

Similarly, the RHS of inequality (14) can be manipulated to have:

$$\begin{aligned}
&- c_{\pi_{x+1}} + \pi_{x+1} f(x) + (1-\pi_{x+1}) f(x+1) \\
&\quad + c_{\pi_{x+2}} - \pi_{x+2} f(x+1) - (1-\pi_{x+2}) f(x+2) \ge f(x) - f(x+1).
\end{aligned} \tag{16}$$

When inequalities (15) and (16) are combined, it is seen that inequality (14) is true. Hence, T_{CD} preserves concavity of $f(x)$ in x.

Proofs for $T_{\mathrm{C\_PRD}}$, $T_{\mathrm{Q\_PRC}}$ and $T_{\mathrm{I\_PRC}}$ are similar. However, while considering finite capacity queues, we need to use the concavity of $T_{\mathrm{Q\_PRC}}$ for the state $x = K-2$. Since we use the optimal action for the state $x+1$, i.e., $K-1$, and it is not affected by the waiting room capacity, the foregoing proof is still valid for systems with finite capacity.

A.4.2 Concavity preserved by $T_{\mathrm{B\_ADM}_i}$

Let $\bar{\kappa}^{iB_i} = (\kappa_x^{iB_i}, \kappa_{x+1}^{iB_i}, \kappa_{x+2}^{iB_i})$ be an optimal action vector and $\kappa_x^{iB_i}$ be optimal number of class-i customers admitted from an arriving batch B_i in state x. Then, we prove that the batch admission operator will be concave in x if $f(x)$ is concave in x. In other words, we show that the following inequality is true for all possible $\bar{\kappa}^{iB_i}$:

$$\begin{aligned}
&\kappa_x^{iB_i} R_i + f\big(x + \kappa_x^{iB_i}\big) - \kappa_{x+1}^{iB_i} R_i - f\big(x + 1 + \kappa_{x+1}^{iB_i}\big) \\
&\quad \le \kappa_{x+1}^{iB_i} R_i + f\big(x + 1 + \kappa_{x+1}^{iB_i}\big) - \kappa_{x+2}^{iB_i} R_i - f\big(x + 2 + \kappa_{x+2}^{iB_i}\big).
\end{aligned} \tag{17}$$

It is enough to consider four different cases for $\bar{\kappa}^{iB_i}$ as shown in Table 5, because, as mentioned previously, the concavity of $f(x)$ implies that the optimal number of customers to be admitted in states x and $x+1$ can differ at most by 1. We rewrite inequality (17) for each case in Table 5. Case III is obviously true and case I is true due to the concavity of $f(x)$. In case II, the optimal action in state $x+1$ is admitting a customers so rejecting $(a+1)$st customer of the arriving batch, which implies that $f(x+a+1) \ge R_i + f(x+a+2)$. Thus, inequality (17) is true in case II. In a similar manner, inequality (17) is also true in case IV by the optimal action in the state $x+1$, and the result follows.

For capacitated queues, we need to focus on states $x \ge K - B_i - 1$ to investigate the boundary effect. However, the optimal actions in these states will also fall in

Table 5 Possible optimal actions in states x, $x+1$ and $x+2$ with operator $T_{\mathrm{B\_ADM}_i}$

Cases	$\bar{\kappa}^{iB_i} = (\kappa_x^{iB_i}, \kappa_{x+1}^{iB_i}, \kappa_{x+2}^{iB_i})$	Rewritten form of inequality (17)
Case I	(a, a, a)	$f(x+a) - f(x+a+1) \le f(x+a+1) - f(x+a+2)$
Case II	$(a+1, a, a)$	$R_i \le f(x+a+1) - f(x+a+2)$
Case III	$(a+2, a+1, a)$	$R_i \le R_i$
Case IV	$(a+1, a+1, a)$	$f(x+a+1) - f(x+a+2) \le R_i$

one of the categories given in Table 5. Therefore, $T_{\mathrm{B\_ADM}_i} f(x)$ is concave in x for systems with finite capacity.

The proof of the concavity of $T_{\mathrm{B\_RT}_i} f(x)$ is similar to this proof (see also Lautenbacher and Stidham [15]).

Appendix B: Proof of the statements given in Tables 2 and 3

In this section, we will prove supermodularity properties of the value functions. We omit the proof of submodularity since the proofs of these two properties are very similar. Section B.2 proves the monotonicity properties of the marginal benefits.

B.1 Supermodularity properties

As mentioned in the paper, proving supermodularity requires us to consider two systems, one with parameter α and the other with $\alpha + \varepsilon$. Here, we simplify the notation, by denoting functions f, a, and b by f_ε, a_ε, and b_ε after the parameter α increases by ε, respectively. The parameter α will always be clear from the context, so we do not need to indicate it in the functions. Then the inequality for supermodularity of a certain operator, T, is as follows:

$$\Delta T f(x) \ge \Delta T f_\varepsilon(x). \tag{18}$$

Here, we also need to clarify the definition of the supermodularity of f in α and x when the state space of x is different for systems with parameter α and $\alpha + \varepsilon$. When $\alpha \in \{\lambda, \mu, m\}$ with $m < K$, the two systems have the same state space, whereas increasing $\alpha = K$ by 1 alters the state space of x. In the latter case, we define the supermodularity of f in K and x only for $x \le K$.

B.1.1 Supermodularity preserved by T_{ARR}

Let α be in $\{\lambda, \mu, m\}$, and the system have infinite capacity. We first note that $a(x)$ is assumed to be convex in x, decreasing in α, and submodular with respect to α and x.

We can write inequality (18) for this operator as follows:

$$\begin{aligned} &a(x) f(x+1) + \big[1 - a(x)\big] f(x) - a(x+1) f(x+2) - \big[1 - a(x+1)\big] f(x+1) \\ &\quad \ge a_\varepsilon(x) f_\varepsilon(x+1) + \big[1 - a_\varepsilon(x)\big] f_\varepsilon(x) \\ &\qquad - a_\varepsilon(x+1) f_\varepsilon(x+2) - \big[1 - a_\varepsilon(x+1)\big] f_\varepsilon(x+1). \end{aligned}$$

Now we add and subtract the terms, $a(x)[f_\varepsilon(x) - f_\varepsilon(x+1)]$ and $a(x+1)[f_\varepsilon(x+1) - f_\varepsilon(x+2)]$, to this inequality, and then rearrange it to obtain:

$$\begin{aligned} & a(x+1)\big[f(x+1) - f(x+2)\big] + \big[1 - a(x)\big]\big[f(x) - f(x+1)\big] \\ & \quad + \big[a(x+1) - a_\varepsilon(x+1)\big]\big[f_\varepsilon(x+1) - f_\varepsilon(x+2)\big] \\ & \geq a(x+1)\big[f_\varepsilon(x+1) - f_\varepsilon(x+2)\big] + \big[1 - a(x)\big]\big[f_\varepsilon(x) - f_\varepsilon(x+1)\big] \\ & \quad + \big[a(x) - a_\varepsilon(x)\big]\big[f_\varepsilon(x) - f_\varepsilon(x+1)\big]. \end{aligned} \tag{19}$$

The inequality is true for the first two terms on both sides by the supermodularity of $f(x)$ with respect to α and x. Now we need to show that the third line also satisfies the inequality. We have the following relations due to the concavity of f and the submodularity of a:

$$\begin{aligned} 0 \leq f_\varepsilon(x) - f_\varepsilon(x+1) \ &\leq\ f_\varepsilon(x+1) - f_\varepsilon(x+2), \\ 0 \leq a(x) - a_\varepsilon(x) \ &\leq\ a(x+1) - a_\varepsilon(x+1). \end{aligned}$$

When we combine these inequalities, we have:

$$\begin{aligned} 0 &\leq \big[a(x) - a_\varepsilon(x)\big]\big[f_\varepsilon(x) - f_\varepsilon(x+1)\big] \\ &\leq \big[a(x+1) - a_\varepsilon(x+1)\big]\big[f_\varepsilon(x+1) - f_\varepsilon(x+2)\big]. \end{aligned}$$

Thus, the inequality (19) is also true for the third terms. Hence, we complete the proof of the supermodularity of $T_{\text{ARR}} f(x)$ with respect to α and x in systems with infinite capacity.

Now we consider systems with a finite capacity K, and let $\alpha \in \{\lambda, \mu, m\}$ with $m < K$. We also assume that $a(x)$ is convex in x, and constant with respect to α. In order to observe the boundary effects, let $x = K - 1$. Then the first line in inequality (19) will be 0, and the second line still satisfies the inequality as before. Finally, the third line is also 0, since $a(x)$ is constant with respect to α. Therefore, T_{ARR} preserves supermodularity with respect to x and α whenever $a(x)$ does not depend on α. Recall that in finite systems whenever $a(x) = a$ for all x, T_{ARR} does not preserve supermodularity, since it cannot preserve concavity (see Appendix A.4).

Finally, we note that supermodularity of a function f with respect to x and K cannot be preserved due to the boundary effects, since the function $a(x)$ inevitably depends on K.

B.1.2 Supermodularity preserved by T_{DEP}

We omit the complete proof which is similar to that of T_{ARR}. It should be noted that the proof for this operator is valid for all $\alpha \in \{\lambda, \mu, m, K\}$ and for both finite and infinite systems.

B.1.3 Supermodularity preserved by T_{CD}

The proof for this operator is valid for all $\alpha \in \{\lambda, \mu, K\}$ and for both finite and infinite systems. Let π_x and $\tilde{\pi}_x$ be the optimal service rates for the state x before and after the parameter α increases, respectively. Then, we show that the following supermodularity inequality is true for T_{CD}.

$$\begin{aligned} &c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) - c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) \\ &\quad \geq c_{\tilde{\pi}_x} + \tilde{\pi}_x f_\varepsilon(x-1) + (1-\tilde{\pi}_x) f_\varepsilon(x) - c_{\tilde{\pi}_{x+1}} \\ &\qquad - \tilde{\pi}_{x+1} f_\varepsilon(x) - (1-\tilde{\pi}_{x+1}) f_\varepsilon(x+1). \end{aligned} \tag{20}$$

As a result of the optimality of π_x for the state x, the LHS of the inequality can be bounded from below as follows:

$$\begin{aligned} &c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) - c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) \\ &\quad \geq c_{\tilde{\pi}_x} + \tilde{\pi}_x f(x-1) + (1-\tilde{\pi}_x) f(x) \\ &\qquad - c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) \\ &\quad = c_{\tilde{\pi}_x} - c_{\pi_{x+1}} + \tilde{\pi}_x \big[f(x-1) - f(x) \big] + (1-\pi_{x+1}) \big[f(x) - f(x+1) \big], \end{aligned} \tag{21}$$

where the equality follows by some algebra. Similarly, we can obtain the following for the RHS of inequality (20) due to the optimality of $\tilde{\pi}_{x+1}$ and by some algebra:

$$\begin{aligned} &c_{\tilde{\pi}_x} + \tilde{\pi}_x f_\varepsilon(x-1) + (1-\tilde{\pi}_x) f_\varepsilon(x) - c_{\tilde{\pi}_{x+1}} - \tilde{\pi}_{x+1} f_\varepsilon(x) - (1-\tilde{\pi}_{x+1}) f_\varepsilon(x+1) \\ &\quad \leq c_{\tilde{\pi}_x} - c_{\pi_{x+1}} + \tilde{\pi}_x \big[f_\varepsilon(x-1) - f_\varepsilon(x) \big] + (1-\pi_{x+1}) \big[f_\varepsilon(x) - f_\varepsilon(x+1) \big]. \end{aligned} \tag{22}$$

Inequalities (21) and (22) together with the supermodularity of the function f imply that inequality (20) is true. Thus, we complete the proof of the supermodularity of $T_{\mathrm{CD}} f(x)$ with respect to α and x.

Proofs of the supermodularity of $T_{\mathrm{C\_PRD}}$, $T_{\mathrm{Q\_PRC}}$ and $T_{\mathrm{I\_PRC}}$ are similar to this proof.

B.1.4 Supermodularity preserved by $T_{\mathrm{B\_ADM}_i}$

We first consider systems with infinite capacity, and let $\alpha \in \{\lambda, \mu, m\}$. We denote by $\bar{\kappa}^{iB_i} = (\kappa_x^{iB_i}, \kappa_{x+1}^{iB_i}, \tilde{\kappa}_x^{iB_i}, \tilde{\kappa}_{x+1}^{iB_i})$ the optimal action vector, where $\kappa_x^{iB_i}$ and $\tilde{\kappa}_x^{iB_i}$ are the optimal number of customers to be admitted from an arriving batch in state x before and after the parameter α increases, respectively. Then, we show that the following supermodularity inequality is true for the batch admission operator:

$$\begin{aligned} &\kappa_x^{iB_i} R_i + f\big(x + \kappa_x^{iB_i}\big) - \kappa_{x+1}^{iB_i} R_i - f\big(x + 1 + \kappa_{x+1}^{iB_i}\big) \\ &\quad \geq \tilde{\kappa}_x^{iB_i} R_i + f_\varepsilon\big(x + \tilde{\kappa}_x^{iB_i}\big) - \tilde{\kappa}_{x+1}^{iB_i} R_i - f_\varepsilon\big(x + 1 + \tilde{\kappa}_{x+1}^{iB_i}\big). \end{aligned} \tag{23}$$

We have to consider all possible optimal action vectors. We know that $\kappa_x^{iB_i}$ and $\kappa_{x+1}^{iB_i}$ can differ at most by 1 due to concavity of f. Moreover, again by concavity of f, if $\kappa_x^{iB_i} = \kappa_{x+1}^{iB_i}$, we either have $\kappa_x^{iB_i} = \kappa_{x+1}^{iB_i} = 0$ or $\kappa_x^{iB_i} = \kappa_{x+1}^{iB_i} = B_i$. The supermodularity of f with respect to α and x, on the other hand, implies that $\kappa_x \leq \tilde{\kappa}_x^{iB_i}$ for all x. Hence, it is enough to consider the following cases: $(0, 0, 0, 0)$, $(0, 0, a+1, a)$,

Table 6 Possible optimal actions in states x and $x+1$ in systems with parameters α and $\alpha+\varepsilon$

Cases	$\bar{\kappa}^{iB_i}=(\kappa_x^{iB_i},\kappa_{x+1}^{iB_i},\tilde{\kappa}_x^{iB_i},\tilde{\kappa}_{x+1}^{iB_i})$	Rewritten form of inequality (23)
Case I	$(0,0,0,0)$	$f(x)-f(x+1)\geq f_\varepsilon(x)-f_\varepsilon(x+1)$
Case II	$(0,0,a+1,a)$	$f(x)-f(x+1)\geq R_i$
Case III	$(0,0,B_i,B_i)$	$f(x)-f(x+1)\geq f_\varepsilon(x+B_i)-f_\varepsilon(x+B_i+1)$
Case IV	$(a+1,a,d+1,d)$	$R_i\geq R_i$
Case V	$(a+1,a,B_i,B_i)$	$R_i\geq f_\varepsilon(x+B_i)-f_\varepsilon(x+B_i+1)$
Case VI	(B_i,B_i,B_i,B_i)	$f(x+B_i)-f(x+B_i+1)\geq f_\varepsilon(x+B_i)-f_\varepsilon(x+B_i+1)$

$(0,0,B_i,B_i)$, $(a+1,a,d+1,d)$, $(a+1,a,B_i,B_i)$ and (B_i,B_i,B_i,B_i), where $0\leq a\leq B_i-1$ and $a\leq d\leq B_i-1$. Table 6 presents inequality (23) for all these six cases. Case IV is obviously true, while cases I and VI are true by the supermodularity of $f(x)$. In case II, it is optimal to reject the entire batch in state x of the system with parameter α, so that $f(x)-f(x+1)\geq R_i$, and the inequality holds. Similarly, the optimal action of case V is to admit the whole batch in state $x+1$ for the system with parameter $\alpha+\varepsilon$, which implies that $B_iR_i+f_\varepsilon(x+B_i+1)\geq(B_i-1)R_i+f_\varepsilon(x+B_i)$, coincides with case V in Table 6. Finally, in case III optimal actions in state x of the system with parameter α and in $x+1$ of the system with parameter $\alpha+\varepsilon$ ensure that inequality (23) holds.

In systems with finite capacity, the decisions in x and $x+1$ of systems with parameter α and $\alpha+\varepsilon$ will also satisfy one of the six cases we consider for $\alpha\in\{\lambda,\mu,m,K\}$. Hence, $T_{\text{B\_ADM}_i}$ preserves supermodularity in finite systems as well.

The proof of $T_{\text{B\_RT}_i}$ is similar to this proof.

B.2 Monotonicity in marginal benefits

In this subsection, we show that if $f(x)$ is concave, then $Tf(x)-f(x)$ will be either increasing (inequality (24)) or decreasing (inequality (25)) in x according to the characteristics of the operator T:

$$Tf(x)-f(x)\leq Tf(x+1)-f(x+1), \tag{24}$$

$$Tf(x)-f(x)\geq Tf(x+1)-f(x+1). \tag{25}$$

B.2.1 $T_{\text{ARR}}f(x)-f(x)$ is decreasing

As we mentioned in the paper, to show inequality (25) for T_{ARR}, we must assume $a(x)$ is constant and the buffer capacity is infinite. Then, inequality (25) for T_{ARR} becomes:

$$a\big[f(x+1)-f(x)\big]\geq a\big[f(x+2)-f(x+1)\big],$$

which is true by the concavity of $f(x)$. The proof for $T_{\text{DEP}}f(x)-f(x)$ is similar and is omitted.

B.2.2 $T_{\mathrm{CD}} f(x) - f(x)$ is increasing

Let π_x be the optimal service rate for the state x. Then inequality (24) for T_{CD} becomes:

$$\begin{aligned} &c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) - f(x) \\ &\quad \le c_{\pi_{x+1}} + \pi_{x+1} f(x) + (1-\pi_{x+1}) f(x+1) - f(x+1). \end{aligned} \tag{26}$$

Now we consider the RHS of this inequality:

$$\begin{aligned} &c_{\pi_{x+1}} + \pi_{x+1} f(x) + (1-\pi_{x+1}) f(x+1) - f(x+1) \\ &\quad \ge c_{\pi_x} + \pi_x f(x) + (1-\pi_x) f(x+1) - f(x+1) \\ &\quad = c_{\pi_x} + \pi_x \big[f(x) - f(x+1) \big] \\ &\quad \ge c_{\pi_x} + \pi_x \big[f(x-1) - f(x) \big] \\ &\quad = c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) - f(x), \end{aligned}$$

where the first inequality follows from the optimality of π_{x+1} and the second is due to the concavity of f, while the equalities follow by some algebra.

The proof for $T_{\mathrm{C\_PRD}} f(x) - f(x)$ is similar and is omitted.

B.2.3 $T_{\mathrm{Q\_PRC}} f(x) - f(x)$ is decreasing

Letting p_x be the optimal price in state x, inequality (25) becomes:

$$\begin{aligned} &\bar{F}(p_x)\big[f(x+1) + p_x \big] + F(p_x) f(x) - f(x) \\ &\quad \ge \bar{F}(p_{x+1})\big[f(x+2) + p_{x+1} \big] + F(p_{x+1}) f(x+1) - f(x+1). \end{aligned} \tag{27}$$

We consider the LHS of this inequality:

$$\begin{aligned} &\bar{F}(p_x)\big[f(x+1) + p_x \big] + F(p_x) f(x) - f(x) \\ &\quad \ge \bar{F}(p_{x+1})\big[f(x+1) + p_{x+1} \big] + F(p_{x+1}) f(x) - f(x) \\ &\quad = \bar{F}(p_{x+1}) p_{x+1} + \bar{F}(p_{x+1})\big[f(x+1) - f(x) \big] \\ &\quad \ge \bar{F}(p_{x+1}) p_{x+1} + \bar{F}(p_{x+1})\big[f(x+2) - f(x+1) \big] \\ &\quad = \bar{F}(p_{x+1})\big[f(x+2) + p_{x+1} \big] + F(p_{x+1}) f(x+1) - f(x+1), \end{aligned}$$

where the first inequality follows by the optimality of p_x and the second by the concavity of f, while the equalities are due to some algebra. Hence, inequality (27) is true.

In the capacitated case, we need to consider inequality (27) for state $x = K - 1$. Then the RHS of (27) becomes 0, so that it is enough to show that the LHS of (27) is non-negative. By simple algebra, the LHS becomes $\bar{F}(p_{K-1})[f(K) + p_{K-1} - f(K-1)]$, which is clearly non-negative due to the optimality of p_{K-1}. Hence, inequality (27) is also true for the capacitated queues.

The proof for $T_{\mathrm{I\_PRC}} f(x) - f(x)$ is similar and is omitted.

Table 7 Possible optimal actions in states x and $x+1$

Cases	$\bar{\kappa}^{iB_i} = (\kappa_x^{iB_i}, \kappa_{x+1}^{iB_i})$	Rewritten form of inequality (28)
Case I	(0,0)	$0 \geq 0$
Case II	$(a+1, a)$	$R_i \geq f(x) - f(x+1)$
Case III	(B_i, B_i)	$f(x+B_i) - f(x) \geq f(x+B_i+1) - f(x+1)$

B.2.4 $T_{\text{B\_ADM}_i} f(x) - f(x)$ *is decreasing*

Let $\kappa_x^{iB_i}$ be the optimal number of customers admitted from an arriving batch in state x. Then, inequality (25) becomes:

$$\kappa_x^{iB_i} R_i + f\big(x + \kappa_x^{iB_i}\big) - f(x) \geq \kappa_{x+1}^{iB_i} R_i + f\big(x + 1 + \kappa_{x+1}^{iB_i}\big) - f(x+1). \qquad (28)$$

Due to the concavity of $f(x)$ it is enough to consider three cases for optimal actions $(\kappa_x^{iB_i}, \kappa_{x+1}^{iB_i})$: $(0, 0)$, $(a+1, a)$ and (B_i, B_i). We rewrite inequality (28) for each case in Table 7. Case I is obviously true, and case III is true due to the concavity of $f(x)$. In case II, it is optimal to admit $a+1$ customers from an arriving batch in state x, which implies that $R_i \geq f(x+a) - f(x+a+1)$. Moreover, by the concavity of $f(x)$, we have that $f(x) - f(x+1) \leq f(x+a) - f(x+a+1)$ for all $a > 0$. When we combine these two inequalities, we obtain that $R_i \geq f(x) - f(x+1)$ and thus, inequality (28) is true in case II.

For capacitated queues, we need to focus on the states $x \geq K - B_i$ in order to investigate the boundary effect. For these states, case III is not possible because admitting B_i customers in state $x+1$ is not feasible, which leaves only cases I and II, whose proofs are the same as above.

The proof for $T_{\text{B\_RT}_i}$ is similar to this proof and is omitted.

Appendix C: Properties of T_{ARR} in systems with finite waiting room

Section 3.2.1 of the paper shows that operator T_{ARR} preserves the DMB property in systems with infinite waiting room capacity only when $a(x)$ is constant in x, an assumption we would have still needed in finite systems. However, from Sect. 3.1, we know that operator T_{ARR} does not preserve concavity in finite systems when $a(x)$ is constant in x. Unfortunately, the concavity assumption would also be needed. Due to these conflicting requirements, T_{ARR} neither has a monotonicity of marginal benefits (i.e., DMB property) nor preserves supermodularity in systems with a finite capacity. Below, we present an example in which T_{ARR} does not preserve supermodularity with respect to K and x. T_{ARR} preserves submodularity with respect to α and x, as stated in the following lemma.

Lemma 1 *Assume that $a(x)$ is decreasing and convex in x, increasing in α, and supermodular with respect to α and x. Then, whenever f is decreasing and concave in x and submodular in α and x, T_{ARR} is submodular with respect to α and x, for all $\alpha \in \{\mu, \lambda, m, K\}$.*

Now we present a counterexample to the supermodularity with respect to K and x: Let $a(x) = 1$ whenever the finite room has at least one empty space, and we set:

$$f_K(x) = f_{K+1}(x) = K + 1 - x \quad \forall x.$$

Therefore, $f_K(x)$ is both decreasing and concave in x, and supermodular in K and x. Applying the operator T_{ARR} gives:

$$T_{\text{ARR}} f_K(x) = K - x \quad \forall x = 0, \dots, K - 1, \quad \text{and} \quad T_{\text{ARR}} f_K(K) = 1,$$
$$T_{\text{ARR}} f_{K+1}(x) = K - x \quad \forall x = 0, \dots, K, \quad \text{and} \quad T_{\text{ARR}} f_{K+1}(K + 1) = 0.$$

Now we can check the inequality for T_{ARR} to preserve the supermodularity for state $x = K - 1$:

$$T_{\text{ARR}} f_K(K - 1) - T_{\text{ARR}} f_K(K) = 0 \geq^? T_{\text{ARR}} f_{K+1}(K - 1) - T_{\text{ARR}} f_{K+1}(K) = 1,$$

which does not hold, so that the supermodularity is not preserved.

Appendix D: Counterexamples for submodularity

In the paper, we commented on how an increase in K affects the controllable operators, i.e., $T_{\text{Q\_PRC}}$, $T_{\text{B\_ADM}_i}$ and $T_{\text{C\_PRD}}$, where we conclude that the controllable operators cannot preserve submodularity. Here, we present a simple counterexample for the operators $T_{\text{B\_ADM}_i}$. Similar examples can be produced for all the above operators.

We first consider $T_{\text{B\_ADM}_i}$ with $B_i = 1$, so each batch consists of only one customer. We set:

$$f_K(x) = f_{K+1}(x) = 0 \quad \forall x.$$

Therefore, $f_K(x)$ is both decreasing and concave in x, and submodular in K and x. Applying the operator $T_{\text{B\_ADM}_i}$ gives:

$$T_{\text{B\_ADM}_i} f_K(x) = R_i \quad \forall x = 0, \dots, K - 1, \quad \text{and} \quad T_{\text{B\_ADM}_i} f_K(K) = 0,$$
$$T_{\text{B\_ADM}_i} f_{K+1}(x) = R_i \quad \forall x = 0, \dots, K, \quad \text{and} \quad T_{\text{B\_ADM}_i} f_{K+1}(K + 1) = 0.$$

Now we can check the inequality for $T_{\text{B\_ADM}_i}$ to preserve the submodularity for state $x = K - 1$:

$$\begin{aligned} &T_{\text{B\_ADM}_i} f_K(K - 1) - T_{\text{B\_ADM}_i} f_K(K) \\ &\quad = R_i \leq^? T_{\text{B\_ADM}_i} f_{K+1}(K - 1) - T_{\text{B\_ADM}_i} f_{K+1}(K) = 0, \end{aligned}$$

which does not hold, so submodularity is not preserved in this case.

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