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Dynamic Pricing and Scheduling in a Multi-Class Single-Server Queueing System

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Abstract This paper investigates an optimal sequencing and dynamic pricing problem for a two-class queueing system. Using a Markov Decision Process based model, we obtain structural characterizations of optimal policies. In particular, it is shown that the optimal pricing policy depends on the entire queue length vector but some monotonicity results prevail as the composition of this vector changes. A numerical study finds that static pricing policies may have significant suboptimality but simple dynamic pricing policies perform well in most situations.

1 Introduction

Dynamic pricing opportunities for production and service systems have attracted significant attention recently. In particular, there is a rapidly growing literature on dynamic pricing problems for queueing-based models of such systems. Most of this literature focuses on single-class queues consisting of customers from a single segment. This paper focuses on the dynamic pricing problem for a single-server queue that serves two

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distinct classes of customers. These segments are assumed to differ in their price sensitivity as well as their waiting costs. The resulting problem involves offering different prices to these customer segments upon their arrival and sequencing the server in order to maximize the long-run profits. The multi-dimensional aspect of the problem brings new challenges for establishing the structure of optimal pricing and sequencing policies.

The analysis of the pricing problem in a queueing context goes back to Naor [17] who considers a static pricing problem for controlling the arrival rate in a finite buffer queueing system. A rich literature on the static pricing problem has evolved since then. Our main interest in this paper is dynamic pricing where different prices can be charged at different times. Low [14] is the first to focus on the dynamic pricing problem of a multi-server queue with finite waiting room capacity. In particular, Low proves the monotonicity of the optimal prices in the queue length. These results are then extended to a multiserver queue with infinite waiting room capacity in [15]. Paschalidis and Tsitsiklis [18] consider the pricing problem of a service provider, which provides access to communication network, by modeling the problem as a dynamic pricing problem of multiserver loss system with N different customer classes. They establish the monotonicity of the optimal prices in the number of customers in the system. Chen and Frank [6] consider a queuing system where a monopolist charges an entrance fee depending on the number of customers in the system. They establish the existence of monotone optimal prices for this problem. Yoon and Lewis [22] obtain some monotonicity results for a queueing-system with periodically varying parameters. Ziya, Ayhan and Foley [23, 24] investigate the related static pricing problem for a finite-buffer queue and obtain some structural results. Son [20] considers a pricing problem for discrete-time queue with the additional option to serve a second class of customers that are always available. Finally, similar monotonicity results are also obtained for the make-to-stock queue model of a production/inventory system in Li [13], Gayon et al. [8].

A related recent stream of work investigates the effect of problem parameters, such as arrival rates, service rates, and number of servers on optimal dynamic pricing policies. Gans and Savin [7] consider a joint admission control and dynamic pricing problem of a multi-server loss system. They not only establish the structure of the optimal policies maximizing the expected long-run average reward, but also investigate the effects of the parameters on these policies. Çil, Karaesmen and Örmeci [4] and Aktaran-Kalaycı and Ayhan [1] study the multi-server finite buffer queueing system and investigate the monotonicity of the optimal pricing policy as a function of problem parameters. Çil, Örmeci and Karaesmen [5] propose a framework that addresses a class of queueing/inventory problems with dynamic pricing from a parameter monotonicity perspective.

Most of the above papers consider either a single segment of customers or a queue without waiting space or identical holding costs which makes the state space of the problem single dimensional. The most closely related paper to ours is [16] which investigates a multi-class single server queue with different holding costs for each class. Recognizing the difficulty of the underlying control problem, Maglaras [16] proposes a fluid approximation and analyzes its solution to construct plausible policies. It is shown through numerical examples that this approximate solution is extremely effective for the original problem. In this paper, using a Markov Decision Process (MDP) framework, we obtain results on the structure of the optimal sequencing and pricing policy for this problem. In particular, we show that the optimal sequencing policy is a strict priority policy and establish a number of monotonicity results for the optimal prices in terms of the queue lengths.

The organization of the paper is as follows. Section 2 introduces the assumptions and the model. Section 3 presents the structural results on the optimal policy. The numerical results are presented in Section 4 and the conclusions can be found in Section 5. Finally, the proofs and the detailed numerical results are deferred to the Appendix.

2 The Model

We consider a single-server queue with infinite waiting room capacity and two classes of customers. Arrivals occur according to independent Poisson processes with rate λ_j, $j = 1, 2$. Whenever a class-j customer arrives, he either enters the system if his reservation price, R_j, is higher than the announced price or leaves the system without bringing any reward. It is assumed that R_j's are random variables with a cumulative distribution function of $F_j(.)$. We denote the probability density function by $f_j(.)$, and let $\bar{F}_j(p) = 1 - F_j(p)$. The service times of all customers are independent and exponentially distributed with mean $1/\mu$ regardless of the customer class. Moreover, the queue owner incurs a holding cost per customer per unit time, h_j, and without loss of generality it is assumed that $h_1 > h_2$. The objective is to obtain dynamic pricing and sequencing policies that maximize the total expected discounted profit with a continuous discount rate β over an infinite horizon as well as the long-run average profit.

At any time, the decision maker has to decide which class of customer is served and to choose a price from a compact set $[p_{min}, p_{max}]$. For technical reasons, which will be apparent in the text, it is assumed that $F_j(.)$ has a strictly increasing generalized failure rate, i.e., $\bar{F}_j(.)/(pf_j(.))$ is strictly decreasing, and $\lambda_1\bar{F}_1(p_{max}) + \lambda_2\bar{F}_2(p_{max}) \leq \mu$. Under any given feasible scheduling and pricing policy π, the system evolves as a continuous-time Markov chain with state $(X_1(t), X_2(t))$, where $X_j(t)$ is the number of class-j customers in the system at time t. Due to the Markovian property, it is clear that the optimal policy depends only on the current state regardless of t, and thus we simply denote the current state of the system by (x_1, x_2), where $(x_1, x_2) \in \mathbb{N}^2$ with $\mathbb{N} = \{0, 1, \cdots\}$.

In order to find the optimal policy π^* that maximizes the total expected discounted profit, we construct a discrete-time equivalent of the original system by using the standard tools of uniformization and normalization. To this end, we assume that the time between two consecutive transitions is exponentially distributed with rate $\gamma = \mu + \lambda_1 + \lambda_2 + \beta$, and, assume without loss of generality that $\gamma = 1$.

To obtain the structural properties of a system which operates over an infinite horizon, we first prove these structural properties with the objective of maximizing the expected total β-discounted reward for a finite number of transitions, n. The finite horizon problems allow us to use the induction to prove the structural properties for all finite n. To start the induction, we set $v_0(x_1, x_2) = 0$ for all states (x_1, x_2). Furthermore, $v_n(x_1, x_2)$ is the maximum expected total β-discounted reward of the system starting in state (x_1, x_2) with n transitions remaining in the future and the optimality equation of the finite horizon problem is:

$$v_{n+1}(x_1, x_2) = \mu T_{SEQ} v_n(x_1, x_2) + \sum_{j=1,2} \lambda_j T_{PRC_j} v_n(x_1, x_2) - h_1 x_1 - h_2 x_2,$$

where,

$$T_{SEQ}v(x_1,x_2) = \begin{cases} \max\{v(x_1-1,x_2), v(x_1,x_2-1)\} & \textit{if } x_1 > 0,\ x_2 > 0 \\ \max\{v(x_1-1,0), v(x_1,0)\} & \textit{if } x_1 > 0,\ x_2 = 0 \\ \max\{v(0,x_2-1), v(0,x_2)\} & \textit{if } x_1 = 0,\ x_2 > 0 \\ v(0,0) & \textit{if } x_1 = 0,\ x_2 = 0, \end{cases}$$

$$T_{PRC_1}v(x_1,x_2) = \max_p \left\{\bar{F}_1(p)[v(x_1+1,x_2)+p] + F_1(p)v(x_1,x_2)\right\},$$

$$T_{PRC_2}v(x_1,x_2) = \max_p \left\{\bar{F}_2(p)[v(x_1,x_2+1)+p] + F_2(p)v(x_1,x_2)\right\}.$$

As we assume an increasing generalized failure rate for the reservation price distribution, the maximization problem in the pricing operator has a unique solution for any given state (x_1,x_2). A brief discussion on this point can be found in Appendix A. It should be noted that most monotonicity results extend to the case where the monotone generalized failure rate assumption does not hold and there may be multiple optima. But this requires defining more complicated set-based orders.

Using well-established standard arguments (see Chapter 6 of Puterman [19] for example), there exists an optimal stationary policy for the infinite horizon problem and $v(x_1,x_2) = \lim_{n\to\infty} v_n(x_1,x_2)$ whenever $\beta > 0$ where $v(x_1,x_2)$ is the value function of the infinite horizon problem. Therefore, structural results obtained for $v_n(x_1,x_2)$ hold for $v(x_1,x_2)$. In order to address the long-run average profit criterion (i.e., no discounting with $\beta = 0$), we use another well-known result in queueing control (Weber and Stidham [21]) which establishes that the value function of the average reward problem can be obtained as the limit of the value function in the discounted problem under certain conditions. It can easily be verified that all conditions of Weber and Stidham [21] hold for our problem. The most challenging of these conditions requires the existence of a policy with a finite average reward. In our case, this can be seen as follows: consider the policy that always charges p_{max}, the maximum price allowable. Due to the assumption that $\lambda_1\bar{F}_1(p_{max}) + \lambda_2\bar{F}_2(p_{max}) \leq \mu$, the resulting system is a stable two-class queueing system, which generates finite average queue-lengths and finite average rewards. Therefore, the value function of the average profit criterion can be obtained as the limit of the value function of the discounted cost problem as the discount factor goes to zero. In addition, as shown in [21], this limit preserves all structural properties of the discounted value function, and the average reward problem possesses the identical structural properties as the discounted cost problem. In other words, all structural properties are true for the relative value function of the long-run average reward problem, which we denote by $v'(x_1,x_2)$.

3 Structure of the Optimal Sequencing and Pricing Policy

In order to explore the structure of the optimal policies, it is required to investigate the properties of the value function $v(x_1,x_2)$. In addition to basic directional monotonicity (decreasingness) (denoted by $Dec1$, $Dec2$), we also establish other properties such as diagonal dominance ($Dec21$), submodularity ($SubM$) and subconcavity ($SubC$) and concavity. For the sake of completeness, the definition of all these properties are presented in Appendix B.

In order to establish that the value function preserves the properties above, we first show that all operators preserve these properties. Assume that a function $f(x_1,x_2)$ has a certain property A. An operator T preserves property A if $Tf(x_1, x_2)$ also has this property. The results are summarized in Table 1. The ticks in Table 1 represent that the corresponding operators preserve the desired properties ($Dec1$, $Dec2$, $Dec21$, $SubM$, $SubC$) for the function f. Table 1 should be read as follows: consider e.g., operator T_{SEQ} and property $Dec1$: If a function f has properties $Dec2$ and $Dec21$ in addition to $Dec1$, then $T_{SEQ}f$ is decreasing in x_1 so that T_{SEQ} preserves property $Dec1$. The proofs of these results can be found in Appendices C-E.

Table 1 Properties preserved by the operators when the function $f(x)$ has the corresponding property

Preserved Properties	**Operators** T_{SEQ}	T_{PRC_j}	**Additional Condition(s)**
$Dec1$	$\surd^*$	$\surd$	* f: $Dec2$, $Dec21$
$Dec2$	$\surd^\dagger$	$\surd$	† f: $Dec1$, $Dec21$
$Dec21$	$\surd^\diamond$	$\surd$	$^\diamond$ f: $Dec1$, $Dec2$
$SubM$	$\surd^\blacklozenge$	$\surd$	$^\blacklozenge$ f: $Dec1$, $Dec2$, $Dec21$, $SubC$
$SubC$	$\surd^\star$	$\surd^\triangle$	* f: $Dec1$, $Dec2$, $Dec21$ $^\triangle$ f: $SubM$

Remark: Koole [12] presents weaker conditions for the propagation of $SubM$ and $SubC$ for the T_{SEQ} operator. For this operator, we present a simpler proof in the appendix using other properties here for completeness. On the other hand, to the best of our knowledge, the multi-dimensional properties of T_{PRC_j}, have not been studied before.

3.1 Structure of the Optimal Sequencing Policy

In a number of queueing control problems involving processor sequencing/scheduling between multiple customer classes, the $c\mu$ rule is known to be optimal. This rule gives higher priority to those classes that have higher weighted service rates (weighted by the unit holding cost). This result is especially well-established for single-server queues with state-independent arrivals (Baras, Ma, and Makowski [2], Buyukkoc, Varaiya and Walrand [3]). In our setting, if the pricing policy is static (i.e. does not depend on queue lengths), the arrival processes do not depend on the queue lengths. By the former results, this implies that giving strict preemptive priority to class 1 whose holding cost is higher is optimal.

On the other hand, when the arrival process depends on the queue lengths, as is the case with dynamic pricing, the situation is known to be more complicated. Hordijk and Koole [10] appears to present the most comprehensive analysis for optimal scheduling in the case of general state-dependent arrival processes. They consider systems in which arrivals are generated by Markov Decision Arrival Processes (MDAPs), where MDAPs subsume several important state-dependent arrival process models. For the queue-length dependent arrival process, they show that there are counter-examples to the optimality of the $c\mu$ rule and establish that the generic optimality of this rule is not guaranteed.

Given that the optimality of the $c\mu$ priority rule is not guaranteed by the existing results when the arrival process is queue-length dependent (as in dynamic pricing), we present a concise proof of optimality in Appendix F. The main result can be summarized in the following theorem:

Theorem 1 *The optimal sequencing policy is a preemptive-priority policy which serves class 1 first whenever there is a class-1 customer in the queue in a non-idling manner.*

In addition, using Lemma 2 (from Appendix F) we can further characterize optimal pricing policies when the two customer classes are only differentiated by their holding costs:

Corollary 1 *Let the reservation price distributions F_1 and F_2 satisfy the following two conditions:*

(1) $F_2(p) - F_1(p)$ *is weakly decreasing in p:* $F_2(p+\varepsilon) - F_1(p+\varepsilon) \leq F_2(p) - F_1(p)$ *for any* $\varepsilon > 0$.
(2) $F_2(p)p - F_1(p)p$ *is weakly increasing in p:* $F_2(p)p - F_1(p)p \leq F_2(p+\varepsilon)(p+\varepsilon) - F_1(p+\varepsilon)(p+\varepsilon)$. *for any* $\varepsilon > 0$.

Then the optimal price to charge class 1, $p_1^*(x_1, x_2)$*, is greater than or equal to the optimal price to charge class 2,* $p_2^*(x_1, x_2)$*, for all* (x_1, x_2).

Remark: Using the corresponding results from [12], Theorem 1 can be extended to class-dependent service rates μ_1 and μ_2 such that $h_1\mu_1 \geq h_2\mu_2$. It appears, however, that the other properties in the rest of the paper cannot be easily extended to that case.

Theorem 1 and Corollary 1 establish two properties uncovered by Maglaras [16] using the fluid approximation: the optimality of a strict priority policy for the class with higher holding cost along with preferred pricing for the class with lower holding cost. The service provider thus seems to be encouraging a higher percentage of arrivals from the lower holding cost class but is obliged to give priority to the class with higher holding costs that join the queue.

We now examine when the conditions of Corollary 1 are satisfied. Both conditions hold when F_1 and F_2 are identical. When they are not identical, Condition *(1)* is very restrictive. This condition can be satisfied only when F_2 has a mass at the lower bound of its domain, i.e., $F_2(p_{min}) > 0$. This is true because if $F_2(p_{min}) = 0$, we have $F_2(p_{min}) = F_1(p_{min}) = 0$, $F_2(p_{max}) = F_1(p_{max}) = 1$ and $F_2 - F_1$ is continuous. In addition, the previous argument can be extended to the case when p_{min} and p_{max} are class-specific. Finally, we note that neither of the conditions seems to have a direct relation with any of the well-known stochastic orders.

3.2 Structure of the Optimal Pricing Policy

The optimal sequencing policy is obtained to be a priority policy. However, we still need to determine the optimal pricing policy. To this end, we focus on submodularity and subconcavity properties of the value function.

We only present the proof of submodularity in detail, since the arguments used to establish submodularity and subconcavity are similar. We define submodularity as:

$$\textit{Submodularity (SubM):}\ \Delta_{01}v(x_1, x_2) \leq \Delta_{01}v(x_1, x_2+1),$$

where

$$\Delta_{01}v(x_1, x_2) = v(x_1, x_2) - v(x_1 + 1, x_2).$$

We denote the opportunity cost of having an additional class-1 customer in state (x_1, x_2) by $\Delta_{01}v(x_1, x_2)$. Hence, submodularity implies that the opportunity cost of a class-1 customer is increasing in x_2.

We prove the submodularity of $v_n(x_1, x_2)$ for all finite n by induction. The initial condition is trivially true. Then, we assume the submodularity of $v_n(x_1, x_2)$. We can write the submodularity inequality for $v_{n+1}(x_1, x_2)$ as:

$$\begin{array}{rl} & \mu\Delta_{01}T_{SEQ}v_n(x_1, x_2) \\ + & \sum\limits_{j=1,2} \lambda_j \Delta_{01}T_{PRC_j}v_n(x_1, x_2) \end{array} \leq \begin{array}{rl} & \mu\Delta_{01}T_{SEQ}v_n(x_1, x_2 + 1) \\ + & \sum\limits_{j=1,2} \lambda_j \Delta_{01}T_{PRC_j}v_n(x_1, x_2 + 1). \end{array} \quad (1)$$

We know that $v_n(x_1, x_2)$ has properties $Dec1$, $Dec2$ and $Dec21$ by Lemma 2. Moreover, the induction hypothesis implies that $v_n(x_1, x_2)$ is submodular. Thus, inequality (1) is true due to Table 1, so that $v_{n+1}(x_1, x_2)$ is submodular, i.e., $\Delta_{01}v_n(x_1, x_2) \leq \Delta_{01}v_n(x_1, x_2 + 1)$, for all finite n. Then, $v(x_1, x_2)$ and $v'(x_1, x_2)$ are also submodular as $v_n(x_1, x_2)$ converges to $v(x_1, x_2)$. Similarly, $v(x_1, x_2)$ and $v'(x_1, x_2)$ have the subconcavity property. Furthermore, combining submodularity and subconcavity properties implies the concavity of $v(x_1, x_2)$ in both x_1 and x_2. Lemma 1 summarizes the structure of the value functions.

Lemma 1 *The value functions $v(x_1, x_2)$ and $v'(x_1, x_2)$ satisfy submodularity and subconcavity conditions, and they are concave in both x_1 and x_2.*

As in the one-dimensional models of dynamic pricing problems, such as Low [14], Çil et al. [4] or Gayon et al. [8], monotone opportunity costs lead to monotone optimal prices in this model and we present the implied structure of the optimal prices in the following theorem (see Appendix G for its proof).

Theorem 2 *For all (x_1, x_2), we have:*
(i) for $j = 1, 2$, $p_j^*(x_1, x_2) \leq p_j^*(x_1 + 1, x_2)$ and $p_j^*(x_1, x_2) \leq p_j^*(x_1, x_2 + 1)$,
(ii) $p_1^*(x_1, x_2 + 1) \leq p_1^*(x_1 + 1, x_2)$,
(iii) $p_2^*(x_1 + 1, x_2) \leq p_2^*(x_1, x_2 + 1)$.

Figure 1 illustrates the conclusions of Theorem 2. It is worth discussing the implications of Lemma 1 and Theorem 2. Due to submodularity, the optimal prices for both classes are increasing as the number of customers from the other class increases. The pricing policy therefore must take into account the total queue length (i.e. workload) in the system as in the fluid approximation of Maglaras [16]. The implications of Theorem 2 are more intriguing however. Keeping the overall total length constant, increasing the number of class-2 customers results in lower optimal prices for class 1. This is understandable since class-1 customers have higher priority: their wait is not affected by class-2 customers but only by class-1 customers ahead of themselves. Thus, the effective queue length ahead of them is reduced. On the other hand, keeping the total queue length constant, increasing the number of class-2 customers, results in higher optimal prices for class 2 customers. The intuition here seems more subtle, class-2 customers have to wait for the total queue length in front of them at their arrival plus the future arrivals from class-1 customers. But by the previous property, class-1 customers

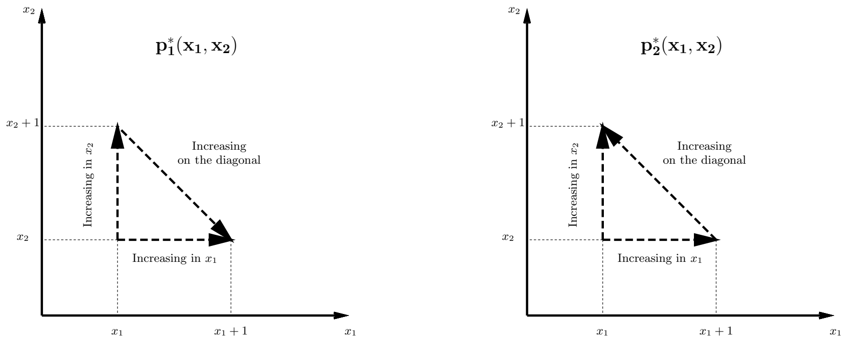

Fig. 1 Structure of the optimal prices in the 2-dimensional model

are charged lower prices when there are more class-2 customers, which increases their expected future arrival rate. Therefore, the potential queue length ahead of class-2 customers is increased, and they need to be charged higher prices.

The above results suggest that the optimal pricing policy is affected not only by the total queue length in the system but also by the class composition of the queue in a subtle manner.

4 Numerical Examples

The structural properties established in the previous section characterize the optimal scheduling and pricing policies. However, specifying a pricing policy is a challenging problem. In this section, we investigate the performance of the optimal dynamic pricing policy and compare it with two alternative benchmark policies. The first alternative is a simple static pricing policy which charges a unique price to each customer class regardless of the queue length. Appendix H presents the computational details for this policy. The second alternative is a simpler dynamic policy where the optimal prices depend on the total queue length $x_1 + x_2$ rather than (x_1, x_2). This type of policy is suggested by the fluid analysis in Maglaras [16]. The implementation details for this policy can be found in Appendix H.

In this section, we use the long-run average profit criterion (so the discount rate β is 0), and denote the optimal average profit by g^*. The optimal policy is computed numerically by truncating the state space and using the value iteration algorithm. The computation yields the optimal prices $p_i^*(x_1, x_2)$ and the expected optimal profit g^* which will be reported below.

4.1 Example 1: Symmetric arrival rates and prices, low holding costs

We begin with an example in a regime that is similar to the numerical examples in Maglaras [16]. Let $\lambda_1 = \lambda_2 = 8$ and $\mu = 4$, $h_1 = 0.4$ and $h_2 = 0.1$. The reservation prices R_1 and R_2 are independently uniformly distributed in the interval $(0, 8)$.

Under the optimal policy, the utilization of the server turns out to be 98% in this case (the detailed results can be found in Table 5 of Appendix H). The static pricing policy has a suboptimality greater than 4%. On the other hand, the theoretical results in Maglaras imply that considering the total queue length (referred to as the workload approximation in that paper) should perform well in this regime. Indeed, the suboptimality reported in Maglaras for this example is 0.2%. Our implementation of the workload approximation yields a suboptimality of 0.16%.

The optimal prices for this example are reported in Table 2. The implications of Theorem 2 can be observed in this table. Optimal prices are increasing in x_1 and x_2 for both customer classes, and the optimal price for class 1 is higher than that of class 2 in any given state. The optimal static prices are 6.22 and 6.10 for classes 1 and 2 respectively (see Table 5 in Appendix H). It can be observed that the state-dependent prices can deviate significantly from these values. More interestingly, as a consequence of subconcavity, the optimal prices were established to be monotone on the total queue length line. This is clearly manifested in Table 2 for class 1. For instance, $p_1^*(0,5) \leq p_1^*(1,4) \leq ... \leq p_1^*(5,0)$. The reverse is also true for class 2 in a weaker sense. For instance, $p_2^*(0,5) = p_2^*(1,4) = ... > p_2^*(5,0)$ (it should be noted that we only report results up to a two-digit accuracy). The latter observation partially explains the excellent performance of the total queue length heuristic. For class 2, the optimal prices are sensitive to the total queue length but not extremely sensitive to individual queue lengths. The heuristic should therefore retrieve the right prices for class 2. For class 1, the situation is different in that the optimal prices differ for the same total queue length depending on the composition. It appears, however, that the optimal profit is not very sensitive to these differences.

	$p_1^*(x_1,x_2)$						$p_2^*(x_1,x_2)$					
$x_2 \setminus x_1$	0	1	2	3	4	5	0	1	2	3	4	5
0	4.72	5.12	5.52	5.76	5.92	6.08	4.64	4.96	5.28	5.44	5.60	5.68
1	5.12	5.44	5.68	5.92	6.08	6.24	5.04	5.28	5.44	5.60	5.76	5.84
2	5.36	5.60	5.84	6.00	6.16	6.32	5.28	5.44	5.60	5.76	5.84	5.92
3	5.52	5.76	5.92	6.08	6.24	6.40	5.44	5.60	5.76	5.84	5.92	6.00
4	5.68	5.92	6.08	6.16	6.32	6.48	5.60	5.76	5.84	5.92	6.00	6.08
5	5.84	6.00	6.16	6.24	6.4	6.48	5.76	5.84	5.92	6.00	6.08	6.16

Table 2 Optimal Prices with $\lambda_1 = \lambda_2 = 8$ and $\mu = 4$, $h_1 = 0.4$ and $h_2 = 0.1$.

Table 5 (in Appendix H) presents additional detailed results as the service rate and the price parameters are varied. At low effective utilization rates, static pricing appears to be very effective but its performance degrades as the effective utilization rates increase. It should be noted that the optimal static prices do not differ significantly between classes. On the other hand, the total queue length heuristic uniformly performs extremely well for this range of parameters.

4.2 Example 2: Symmetric arrival rates and prices, high holding costs

Next, we present the same example as in Section 4.1 where we take $h_1 = 4$ and $h_2 = 1$ which generates a stronger trade-off between utilization and holding costs. The resulting optimal policy has a utilization of 88%. As can be observed in Table 3, all consequences of Theorem 2 appear in a sharper manner in this case. In particular, there seems to be a stronger dependence on individual prices for any given total queue length. Table 6 in Appendix H, which presents the detailed results for this example, shows that the static pricing policy has a suboptimality higher than 10% in this case. With respect to Example 1, the optimal static prices differ more significantly between classes. The total queue length heuristic continues to perform remarkably in this case resulting in a suboptimality of only 0.72%. In addition, varying the service rates, and thus the effective utilization does not have a negative effect on the suboptimality of the total queue length heuristic.

	$p_1^*(x_1,x_2)$						$p_2^*(x_1,x_2)$					
$x_2 \setminus x_1$	0	1	2	3	4	5	0	1	2	3	4	5
0	5.36	6.32	7.12	7.68	8.00	8.00	4.80	5.36	5.68	6.00	6.24	6.40
1	5.92	6.72	7.36	7.92	8.00	8.00	5.36	5.76	6.00	6.24	6.48	6.64
2	6.24	6.96	7.60	8.00	8.00	8.00	5.76	6.08	6.32	6.56	6.72	6.88
3	6.56	7.20	7.76	8.00	8.00	8.00	6.08	6.32	6.56	6.72	6.88	7.04
4	6.80	7.44	8.00	8.00	8.00	8.00	6.40	6.56	6.80	6.96	7.12	7.28
5	7.04	7.60	8.00	8.00	8.00	8.00	6.64	6.80	6.96	7.12	7.28	7.44

Table 3 Optimal Prices with $\lambda_1 = \lambda_2 = 8$ and $\mu = 4$, $h_1 = 4$ and $h_2 = 1$.

4.3 Asymmetrical Cases

We tested several cases with asymmetric arrival rates and reservation price distributions and observed the strong performance of the total queue length heuristic in general, whereas the performance of the static pricing policy varies. On the other hand, it appears that there are some particular situations which are difficult to capture by either the static or the total queue length heuristic. In particular, a challenging case is when one of the classes can afford higher prices (i.e. has stochastically larger reservation prices) but also incurs relatively high holding costs. Under such a condition, the customers from this class should be admitted relatively rarely and at the right time to enhance the expected profit.

Next, we discuss such a case in detail. In this example $\lambda_1 = 2$, $\lambda_2 = 0.5$, $\mu = 1.5$, $h_1 = 2$ and $h_2 = 0.1$. The reservation prices R_1 and R_2 are independently uniformly distributed in the interval $(0, 2)$ and $(0, 0.5)$ respectively.

Table 4 reports the optimal prices for this example. The detailed summary results can be found in Table 7 in Appendix H. The static pricing heuristic has a very significant suboptimality (over 25%). Moreover, the total queue length heuristic also results in a significant suboptimality of 17%. To see why this occurs, we note, from Table 4, that class-1 customers are only accepted whenever $x_1 = 0$. This makes the optimal prices for class 2 significantly different for small levels of total queue length (i.e., $x_1 + x_2 = 1$, 2, or 3). But neither the static pricing policy nor the total queue length heuristic can take this dependence into account. The optimal static prices are 1.84 and 0.31 for classes 1 and 2 respectively. But the dynamic pricing policy completely rejects arrivals of class 1 in certain states. In Table 7, we continue the same experimentation by varying the service rate between 1 and 2, and the suboptimality is consistently around 15% and 25% for the total queue length and static pricing heuristics, respectively. It appears that this regime is troublesome for both of the heuristics.

$x_2 \setminus x_1$	$p_1^*(x_1,x_2)$ 0	1	2	3	4	5	$p_2^*(x_1,x_2)$ 0	1	2	3	4	5
0	1.70	2.00	2.00	2.00	2.00	2.00	0.30	0.34	0.37	0.41	0.44	0.48
1	1.74	2.00	2.00	2.00	2.00	2.00	0.34	0.38	0.41	0.45	0.48	0.50
2	1.78	2.00	2.00	2.00	2.00	2.00	0.38	0.42	0.45	0.48	0.50	0.50
3	1.80	2.00	2.00	2.00	2.00	2.00	0.42	0.45	0.49	0.50	0.50	0.50
4	1.84	2.00	2.00	2.00	2.00	2.00	0.46	0.49	0.50	0.50	0.50	0.50
5	1.88	2.00	2.00	2.00	2.00	2.00	0.49	0.50	0.50	0.50	0.50	0.50

Table 4 Optimal Prices with $\lambda_1 = 2$, $\lambda_2 = 0.5$, $\mu = 1.5$, $h_1 = 2$ and $h_2 = 0.1$.

5 Conclusion

We investigated the structure of optimal dynamic pricing and sequencing policies in a two-class queueing system. As in most similar cases, the sequencing problem turns out to be easy and the optimal sequencing policy gives priority to the customer with the higher holding cost. We were also able to obtain monotone characterizations of the optimal prices as the queue lengths and their compositions vary. On the other hand, despite the characterization results, developing an approximate pricing policy remains a difficult problem. Our numerical results indicate that static pricing policies do not perform well especially when there are strong asymmetries between the customer types. This is in contrast with the results in [24] where static pricing is very effective in a similar situation but with FIFO sequencing. The holding cost asymmetry and the resulting priority sequencing policy seems to work against static pricing policies.

It appears, however, that simple dynamic pricing policies may be effective. In particular, the numerical evidence supports that the total queue length based approach proposed by Maglaras [16] performs remarkably well in general. Yet there are certain situations where the individual queue lengths matter significantly in terms of the pricing policy, and dynamic pricing policies should take this information into account. Such cases require a special attention and an approximate solution of the multi-dimensional MDP seems worthwhile when the customer mix includes lucrative but expensive to hold customers.

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A Uniqueness of the Optimal Prices

In our pricing operator for class-1, we have the following maximization problem:

$$\max_p \left\{ \bar{F}_1(p)[v(x_1+1, x_2+p] + F_1(p)v(x_1, x_2) \right\}.$$

When we rearrange the first order optimality condition of the above problem, we have:

$$p - \frac{\bar{F}_1(p)}{f_1(p)} = v(x_1, x_2) - v(x_1+1, x_2). \tag{2}$$

In order to ensure that there is a unique p solving (2), it is sufficient that the right-hand side of (2) is increasing in p. This is a indeed a conservative condition because (2) may have a unique solution even though $p - \bar{F}_j(p)/f_j(p)$ is not monotone. For notational convenience, let $g(p) = p - \bar{F}_1(p)/f_1(p)$. Below, we show that $g(p)$ is increasing in p when F_1 has the Increasing Generalized Failure Rate ($IGFR$) property, i.e., $\bar{F}_1(p)/pf_1(p)$ is decreasing in p.

$$\begin{aligned}
\frac{\bar{F}_1(p)}{pf_1(p)} \textit{ is decreasing in } p \Rightarrow & \left[1 - \frac{\bar{F}_1(p)}{pf_1(p)}\right] \textit{ is increasing in } p \\
\Rightarrow & \; p\left[1 - \frac{\bar{F}_1(p)}{pf_1(p)}\right] \textit{ is increasing in } p \\
\Rightarrow & \; g(p) \textit{ is increasing in } p
\end{aligned}$$

The same argument naturally also applies to the pricing operator for class-2.

B Definitions of Properties

We start by the basic monotonicity properties of the value function $v(x_1, x_2)$. We note that the words "increasing", "decreasing" and "positive" mean "non-decreasing", "non-increasing" and "non-negative", respectively, in the whole paper. These properties are defined as follows:

$$\textit{Decreasing in } x_1 \textit{ (Dec1):} \quad v(x_1, x_2) \geq v(x_1+1, x_2) \tag{1}$$

$$\textit{Decreasing in } x_2 \textit{ (Dec2):} \quad v(x_1, x_2) \geq v(x_1, x_2+1) \tag{2}$$

$$\textit{Decreasing on the diagonal (Dec21):} \quad v(x_1, x_2+1) \geq v(x_1+1, x_2). \tag{3}$$

Inequality (1) implies that when a new class-1 customer enters the system, the expected discounted profit decreases. In other words, an additional class-1 customer incurs a positive opportunity cost. Similarly, inequality (2) implies a positive opportunity cost of an additional class-2 customer. Inequality (3), on the other hand, implies that the value function decreases when a class-2 customer is exchanged by a class-1 customer. We introduce the following notation:

$$\begin{aligned}
\Delta_{01} v(x_1, x_2) &= v(x_1, x_2) - v(x_1+1, x_2), \\
\Delta_{02} v(x_1, x_2) &= v(x_1, x_2) - v(x_1, x_2+1), \\
\Delta_{21} v(x_1, x_2) &= v(x_1, x_2+1) - v(x_1+1, x_2),
\end{aligned}$$

where $\Delta_{0j} v(x_1, x_2)$ represents the opportunity cost of having an additional class-j customer in state (x_1, x_2), and $\Delta_{21} v(x_1, x_2)$ the opportunity cost of having an additional class-1 customer rather than an additional class-2 customer in state (x_1, x_2).

Now we focus on the concavity properties of the value function. Concavity represents the monotonicity of the opportunity costs, which directly affects the optimal policy structure. Concavity of $v(x_1, x_2)$ in x_1 and x_2 are given by inequalities (4) and (5), respectively:

$$\Delta_{01} v(x_1, x_2) \leq \Delta_{01} v(x_1+1, x_2), \tag{4}$$

$$\Delta_{02} v(x_1, x_2) \leq \Delta_{02} v(x_1, x_2+1). \tag{5}$$

Concavity of the value function in x_j implies that the opportunity cost of a class-j customer is increasing in x_j. Although concavity properties are quite intuitive, it is difficult to prove these inequalities directly. Therefore, we employ the supporting properties submodularity and subconcavity (see [11]) in order to prove concavity. These two properties are of interest in themselves and when combined they imply concavity. Submodularity implies that the opportunity cost of a class-1 (class-2) customer is increasing in x_2 (x_1):

$$\begin{aligned}
\textit{Submodularity (SubM):} \ \Delta_{01} v(x_1, x_2) &\leq \Delta_{01} v(x_1, x_2+1), \ \textit{or equivalently} \\
\Delta_{02} v(x_1, x_2) &\leq \Delta_{02} v(x_1+1, x_2).
\end{aligned}$$

Subconcavity, on the other hand, is the monotonicity of the opportunity cost on the diagonal. Since we have two classes of customers, subconcavity consists of two conditions: The first condition states that the opportunity cost of changing a class-2 customer to a class-1 is decreasing in the number of class-2 customers, x_2, whereas the second condition states that it is increasing in the number of class-1 customers, x_1:

$$\textit{Subconcavity (SubC):} \ \Delta_{21} v(x_1, x_2+1) \leq \Delta_{21} v(x_1, x_2) \leq \Delta_{21} v(x_1+1, x_2).$$

As a result, when we add the inequality of submodularity and the second inequality of subconcavity, we obtain the concavity of $v(x_1, x_2)$ in x_1, and similarly adding the submodularity inequality and the first inequality of the subconcavity property yields to the concavity of $v(x_1, x_2)$ in x_2. Lemma 1 summarizes the structure of the value functions.

C Proof: Monotonicity of the Operators

We show that the operators that we consider, T_{SEQ}, T_{PRC_1} and T_{PRC_2}, preserve the monotonicity properties (1), (2) and (3) of the function $v(x_1, x_2)$ to which they are applied.

C.1 Monotonicity of T_{SEQ}

In the proof of the monotonicity of $T_{SEQ}v(x_1, x_2)$, we assume without loss of generality that $v(x_1, x_2 + 1) \geq v(x_1 + 1, x_2)$ and this property implies that serving the expensive customer is more valuable than serving a cheap customer. Therefore, we can redefine this operator as:

$$T_{SEQ}v(x_1, x_2) = \begin{cases} v(x_1 - 1, x_2) & \textit{if } x_1 > 0,\ x_2 > 0 \\ v(0, x_2 - 1) & \textit{if } x_1 = 0,\ x_2 > 0 \\ v(0, 0) & \textit{if } x_1 = 0,\ x_2 = 0. \end{cases} \quad (7)$$

Then, we investigate whether the departure operator preserves all of the three monotonicity properties. Since the operator is partially defined, we consider all possible cases, i.e., $(x_1 > 0,\ x_2 > 0)$, $(x_1 = 0,\ x_2 > 0)$ and $(x_1 = 0,\ x_2 = 0)$, separately for each property.

C.1.1 Monotonicity in x_1

We can write the first monotonicity inequality for T_{SEQ} as follows:

Cases	$T_{SEQ}v(x_1, x_2) \geq T_{SEQ}v(x_1 + 1, x_2)$
$(x_1 > 0,\ x_2 > 0)$	$v(x_1 - 1, x_2) \geq v(x_1, x_2)$
$(x_1 = 0,\ x_2 > 0)$	$v(0, x_2 - 1) \geq v(0, x_2)$
$(x_1 = 0,\ x_2 = 0)$	$v(0, 0) \geq v(0, 0)$

It is obvious that all three cases are true: the first case is true by the monotonicity of $v(x_1, x_2)$ in x_1, the second case is true by the monotonicity of $v(x_1, x_2)$ in x_2, and the left-hand side and the right-hand side is equal in the third case. Thus, the departure operator preserves the first monotonicity property of $v(x_1, x_2)$.

The proof for the second monotonicity property is similar and is omitted.

C.1.2 Monotonicity on the diagonal

The third monotonicity inequality for T_{SEQ} is as follows:

Cases	$T_{SEQ}v(x_1, x_2 + 1) \geq T_{SEQ}v(x_1 + 1, x_2)$
$(x_1 > 0,\ x_2 > 0)$	$v(x_1 - 1, x_2 + 1) \geq v(x_1, x_2)$
$(x_1 = 0,\ x_2 > 0)$	$v(0, x_2) \geq v(0, x_2)$
$(x_1 = 0,\ x_2 = 0)$	$v(0, 0) \geq v(0, 0)$

In this monotonicity property, all of the three cases are also true: the first one is true by the monotonicity of $v(x_1, x_2)$ on the diagonal and the remaining ones are trivially true. Hence, the departure operator also preserves the third monotonicity property of $v(x_1, x_2)$, and it preserves all of the monotonicity properties of $v(x_1, x_2)$.

C.2 Monotonicity of T_{PRC_1}

C.2.1 Monotonicity in x_1

Let p_1^* and p_2^* be the optimal prices for the states (x_1, x_2) and $(x_1 + 1, x_2)$, respectively. Then, we show that T_{PRC_1} preserves the monotonicity of $v(x_1, x_2)$ in x_2. Inequality (1) for this operator can be written as:

$$\bar{F}(p_1)[p_1 + v(x_1 + 1, x_2)] + F(p_1)v(x_1, x_2) \geq \bar{F}(p_2)[p_2 + v(x_1 + 2, x_2)] + F(p_2)v(x_1 + 1, x_2). \quad (8)$$

Since p_1^* is the optimal price for the state (x_1, x_2), we have that:

$$\bar{F}(p_1)[p_1 + v(x_1 + 1, x_2)] + F(p_1)v(x_1, x_2) \geq \bar{F}(p_2)[p_2 + v(x_1 + 1, x_2)] + F(p_2)v(x_1, x_2), \quad (9)$$

and by the monotonicity of $v(x_1, x_2)$ in x_1,

$$\bar{F}(p_2)[p_2+v(x_1+1, x_2)]+F(p_2)v(x_1, x_2) \geq \bar{F}(p_2)[p_2+v(x_1+2, x_2)]+F(p_2)v(x_1+1, x_2). \quad (10)$$

When we combine inequalities (9) and (10), it is obvious that inequality (8) is true and thus T_{PRC_1} preserves the monotonicity of $v(x_1, x_2)$ in x_1.

The proof for the second monotonicity property are similar and is omitted.

C.2.2 Monotonicity on the diagonal

Similar to the previous monotonicity proofs of the pricing operator, we let p_1^* and p_2^* be the optimal prices for the states $(x_1, x_2 + 1)$ and $(x_1 + 1, x_2)$, respectively, and write inequality (3) for the pricing operator as:

$$\bar{F}(p_1)[p_1+v(x_1+1, x_2+1)]+F(p_1)v(x_1, x_2+1) \geq \bar{F}(p_2)[p_2+v(x_1+2, x_2)]+F(p_2)v(x_1+1, x_2). \quad (11)$$

Since p_1^* is the optimal price for the state $(x_1, x_2 + 1)$, we have that,

$$\bar{F}(p_1)[p_1+v(x_1+1, x_2+1)]+F(p_1)v(x_1, x_2+1) \geq \bar{F}(p_2)[p_2+v(x_1+1, x_2+1)]+F(p_2)v(x_1, x_2+1), \quad (12)$$

and by the monotonicity of $v(x_1, x_2)$ on the diagonal,

$$\bar{F}(p_2)[p_2+v(x_1+1, x_2+1)]+F(p_2)v(x_1, x_2+1) \geq \bar{F}(p_2)[p_2+v(x_1+2, x_2)]+F(p_2)v(x_1+1, x_2). \quad (13)$$

When we combine inequalities (12) and (13), it is obvious that inequality (11) is true and thus T_{PRC_1} preserves the monotonicity of $v(x_1, x_2)$ on the diagonal.

The monotonicity proofs for T_{PRC_2} are similar to these proofs.

D Proof: Submodularity of the Operators

Here, we prove that T_{SEQ}, T_{PRC_1} and T_{PRC_2} preserve the submodularity of a function on which they are applied $v(x_1, x_2)$:

$$\Delta_{01} T v(x_1, x_2) \leq \Delta_{01} T v(x_1, x_2 + 1) \quad (14)$$

D.1 Submodularity of T_{SEQ}

While considering the departure operator, we assume that $v(x_1, x_2)$ is decreasing in x_1, x_2 and on the diagonal, and it satisfies the submodularity and subconcavity inequalities, i.e., $v(x_1, x_2)$ is concave in x_1 and x_2. Since we assume the monotonicity of $v(x_1, x_2)$, serving an expensive customer is more valuable than serving a cheap customer. Thus, we can again use the redefined version of the departure operator, which is introduced in Equation (7). With the help of this modification, we need to examine the submodularity inequality for only these three possible cases: $(x_1 > 0, x_2 > 0)$, $(x_1 = 0, x_2 > 0)$ and $(x_1 = 0, x_2 = 0)$, separately. Inequality (14) can be written as follows for each case:

Cases	$\Delta_{01} T_{SEQ} v(x_1, x_2) \leq \Delta_{01} T_{SEQ} v(x_1, x_2 + 1)$
$(x_1 > 0, x_2 > 0)$	$v(x_1 - 1, x_2) - v(x_1, x_2) \leq v(x_1 - 1, x_2 + 1) - v(x_1, x_2 + 1)$
$(x_1 = 0, x_2 > 0)$	$v(0, x_2 - 1) - v(0, x_2) \leq v(0, x_2) - v(0, x_2 + 1)$
$(x_1 = 0, x_2 = 0)$	$v(0, 0) - v(0, 0) \leq v(0, 0) - v(0, 1)$

The first case is true by the submodularity of $v(x_1, x_2)$, the second case is true by the concavity of $v(x_1, x_2)$ in x_2, and the last case holds by the monotonicity of $v(x_1, x_2)$ in x_2. Hence, T_{SEQ} preserves the submodularity of $v(x_1, x_2)$ under given assumptions.

D.2 Submodularity of T_{PRC_1}

We let the optimal prices for the states (x_1, x_2), (x_1+1, x_2), (x_1, x_2+1) and (x_1+1, x_2+1) as follows:

$p^*(.,.)$	x_2	x_2+1
x_1	$p_{1,1}$	$p_{1,2}$
x_1+1	$p_{2,1}$	$p_{2,2}$

Then, we write the submodularity inequality for the pricing operator as:

$$\begin{array}{c} \bar{F}(p_{1,1})[p_{1,1}+v(x_1+1,x_2)] \\ +F(p_{1,1})v(x_1,x_2) \\ -\bar{F}(p_{2,1})[p_{2,1}+v(x_1+2,x_2)] \\ -F(p_{2,1})v(x_1+1,x_2) \end{array} \leq \begin{array}{c} \bar{F}(p_{1,2})[p_{1,2}+v(x_1+1,x_2+1)] \\ +F(p_{1,2})v(x_1,x_2+1) \\ -\bar{F}(p_{2,2})[p_{2,2}+v(x_1+2,x_2+1)] \\ -F(p_{2,2})v(x_1+1,x_2+1). \end{array} \tag{15}$$

Using the optimality of $p_{2,2}$ at the state (x_1+1, x_2+1) and after some algebra, we have that:

$$\begin{array}{c} \bar{F}(p_{1,1})[p_{1,1}+v(x_1+1,x_2)] \\ +F(p_{1,1})v(x_1,x_2) \\ -\bar{F}(p_{2,1})[p_{2,1}+v(x_1+2,x_2)] \\ -F(p_{2,1})v(x_1+1,x_2) \end{array} \leq \begin{array}{c} \bar{F}(p_{1,1})p_{1,1}-\bar{F}(p_{2,1})p_{2,2} \\ +F(p_{1,1})[v(x_1,x_2)-v(x_1+1,x_2)] \\ +\bar{F}(p_{2,2})[v(x_1+1,x_2)-v(x_1+2,x_2)]. \end{array} \tag{16}$$

Similarly, since $p_{1,2}$ is the optimal price for the state (x_1, x_2+1),

$$\begin{array}{c} \bar{F}(p_{1,2})[p_{1,2}+v(x_1+1,x_2+1)] \\ +F(p_{1,2})v(x_1,x_2+1) \\ -\bar{F}(p_{2,2})[p_{2,2}+v(x_1+2,x_2+1)] \\ -F(p_{2,2})v(x_1+1,x_2+1) \end{array} \geq \begin{array}{c} \bar{F}(p_{1,1})p_{1,1}-\bar{F}(p_{2,1})p_{2,2} \\ +F(p_{1,1})[v(x_1,x_2+1)-v(x_1+1,x_2+1)] \\ +\bar{F}(p_{2,2})[v(x_1+1,x_2+1)-v(x_1+2,x_2+1)]. \end{array} \tag{17}$$

Finally, using the fact that $v(x_1, x_2)$ satisfies submodularity, and inequalities (16) and (17), inequality (15) holds and T_{PRC_1} preserves the submodularity of $v(x_1, x_2)$. The proof for T_{PRC_2} is similar.

E Proof: Subconcavity of the Operators

Below, we prove that T_{SEQ}, T_{PRC_1} and T_{PRC_2} preserve both of the subconcavity conditions of $v(x_1, x_2)$. In other words, we show that the conditions below hold for T_{SEQ}, T_{PRC_1} and T_{PRC_2} under the assumptions mentioned in the lemma.

$$\Delta_{01}Tv(x_1, x_2+1) \leq \Delta_{01}Tv(x_1+1, x_2) \tag{18}$$
$$\Delta_{02}Tv(x_1+1, x_2) \leq \Delta_{02}Tv(x_1, x_2+1) \tag{19}$$

E.1 Subconcavity of T_{SEQ}

We can express the first condition of subconcavity for each of the cases as follows:

Cases	$\Delta_{01}T_{SEQ}v(x_1, x_2+1) \leq \Delta_{01}T_{SEQ}v(x_1+1, x_2)$
$(x_1 > 0,\ x_2 > 0)$	$v(x_1-1, x_2+1) - v(x_1, x_2+1) \leq v(x_1, x_2) - v(x_1+1, x_2)$
$(x_1 = 0,\ x_2 > 0)$	$v(0, x_2) - v(0, x_2+1) \leq v(0, x_2) - v(1, x_2)$
$(x_1 = 0,\ x_2 = 0)$	$v(0,0) - v(0,1) \leq v(0,0) - v(1,0)$

The first case is true by the first condition of subconcavity and the last two cases are true by the monotonicity of $v(x_1, x_2)$ on the diagonal, i.e., $v(x_1, x_2+1) \geq v(x_1+1, x_2)$. Therefore, T_{SEQ} preserves the first condition of subconcavity.

Similar to the first condition, we can write the second condition of subconcavity as:

Cases	$\Delta_{02} T_{SEQ} v(x_1+1, x_2) \leq \Delta_{02} T_{SEQ} v(x_1, x_2+1)$
$(x_1 > 0,\ x_2 > 0)$	$v(x_1, x_2) - v(x_1, x_2+1) \leq v(x_1-1, x_2+1) - v(x_1-1, x_2+2)$
$(x_1 = 0,\ x_2 > 0)$	$v(0, x_2) - v(0, x_2+1) \leq v(0, x_2) - v(0, x_2+1)$
$(x_1 = 0,\ x_2 = 0)$	$v(0,0) - v(0,1) \leq v(0,0) - v(,1)$

The last two cases are obviously true and the first case is true by the second condition of subconcavity. Hence, T_{SEQ} also preserves the second condition of subconcavity.

E.2 Subconcavity of T_{PRC_1}

In order to prove both of the subconcavity conditions, we let the optimal prices for the states (x_1, x_2+1), (x_1, x_2+2), (x_1+1, x_2), (x_1+1, x_2+1) and (x_1+2, x_2) as follows:

$p^*(.,.)$	x_2	x_2+1	x_2+2
x_1		$p_{1,2}$	$p_{1,3}$
x_1+1	$p_{2,1}$	$p_{2,2}$	
x_1+2	$p_{3,1}$		

and then focus on the subconcavity conditions.

E.2.1 1^{st} *Condition:* $\Delta_{21} v(x_1, x_2) \leq \Delta_{21} v(x_1+1, x_2)$

The proof of the first condition is similar to the proof of the submodularity of the pricing operator: We first derive two inequalities by using the optimality of $p_{2,2}$ and $p_{2,1}$, and we combine these inequalities with the fact that $v(x_1, x_2)$ satisfies the first condition of subconcavity. Therefore, the proof is omitted.

E.2.2 2^{nd} *Condition:* $\Delta_{21} v(x_1, x_2+1) \leq \Delta_{21} v(x_1, x_2)$

The proof of the second condition is not as trivial as the first condition because the second condition is related to the opportunity costs of class-2 customers and the pricing operator is defined for class-1 customers. Therefore, we distinguish two cases: $(p_{2,1} \geq p_{1,3})$ and $(p_{2,1} < p_{1,3})$. The idea of this case by case analysis comes from our computational studies. In these studies, we observe that for some holding cost parameters $p_{2,1} \geq p_{1,3}$ whereas for some other parameters $p_{2,1} < p_{1,3}$. This result implies that the opportunity cost of an additional class-1 customer at state (x_1+1, x_2) may or not may not be higher than the opportunity cost of an additional class-1 customer at state (x_1, x_2+2) according to the cost parameters. The intuition behind this result is the ratio of the holding cost of an expensive customer and a cheap customer. When this ratio is very high, i.e. cost of an expensive customer is much higher than cost of a cheap one, having 2 more class-1 customers may be more expensive than having one expensive and two cheap customers, and thus $p_{2,1} \geq p_{1,3}$.

Then, we write inequality (19) for the pricing operator as:

$$\begin{array}{c} \bar{F}(p_{2,1})[p_{2,1} + v(x_1+2, x_2)] \\ +F(p_{2,1})v(x_1+1, x_2) \\ -\bar{F}(p_{2,2})[p_{2,2} + v(x_1+2, x_2+1)] \\ -F(p_{2,2})v(x_1+1, x_2+1) \end{array} \leq \begin{array}{c} \bar{F}(p_{1,2})[p_{1,2} + v(x_1+1, x_2+1)] \\ +F(p_{1,2})v(x_1, x_2+1) \\ -\bar{F}(p_{1,3})[p_{1,3} + v(x_1+1, x_2+2)] \\ -F(p_{1,3})v(x_1, x_2+2). \end{array} \quad (20)$$

Case 1: $(p_{2,1} \geq p_{1,3})$

Using the optimality of $p_{2,2}$ at the state (x_1+1, x_2+1) and after some algebra, we have that:

$$\begin{array}{c} \bar{F}(p_{2,1})[p_{2,1} + v(x_1+2, x_2)] \\ +F(p_{2,1})v(x_1+1, x_2) \\ -\bar{F}(p_{2,2})[p_{2,2} + v(x_1+2, x_2+1)] \\ -F(p_{2,2})v(x_1+1, x_2+1) \end{array} \leq \begin{array}{c} \bar{F}(p_{2,1})[v(x_1+2, x_2) - v(x_1+2, x_2+1)] \\ +F(p_{2,1})[v(x_1+1, x_2) - v(x_1+1, x_2+1)]. \end{array} \quad (21)$$

Similarly, by using the optimality of $p_{1,2}$, we have that

$$\begin{array}{c}\bar{F}(p_{1,2})[p_{1,2}+v(x_1+1,x_2+1)]\\ +F(p_{1,2})v(x_1,x_2+1)\\ -\bar{F}(p_{1,3})[p_{1,3}+v(x_1+1,x_2+2)]\\ -F(p_{1,3})v(x_1,x_2+2)\end{array} \geq \begin{array}{c}\bar{F}(p_{1,3})[v(x_1+1,x_2+1)-v(x_1+1,x_2+2)]\\ +F(p_{1,3})[v(x_1,x_2+1)-v(x_1,x_2+2)].\end{array} \tag{22}$$

Now, we focus on the right hand side of the inequalities (21) and (22) and show that the following inequality holds:

$$\begin{array}{c}\bar{F}(p_{2,1})[v(x_1+2,x_2)-v(x_1+2,x_2+1)]\\ +F(p_{2,1})[v(x_1+1,x_2)-v(x_1+1,x_2+1)]\end{array} \leq \begin{array}{c}\bar{F}(p_{1,3})[v(x_1+1,x_2+1)-v(x_1+1,x_2+2)]\\ +F(p_{1,3})[v(x_1,x_2+1)-v(x_1,x_2+2)].\end{array} \tag{23}$$

As we know that $(p_{2,1} \geq p_{1,3})$, we also have that $\bar{F}(p_{1,3}) = \bar{F}(p_{2,1}) + \xi$, where $\xi > 0$. Then, inequality (23) becomes:

$$\begin{array}{c}\bar{F}(p_{2,1})[v(x_1+2,x_2)-v(x_1+2,x_2+1)]\\ +F(p_{2,1})[v(x_1+1,x_2)-v(x_1+1,x_2+1)]\end{array} \leq \begin{array}{c}\bar{F}(p_{2,1})[v(x_1+1,x_2+1)-v(x_1+1,x_2+2)]\\ +F(p_{2,1})[v(x_1,x_2+1)-v(x_1,x_2+2)]\\ \xi\begin{bmatrix}[v(x_1+1,x_2+1)-v(x_1+1,x_2+2)]\\ -[v(x_1,x_2+1)-v(x_1,x_2+2)]\end{bmatrix}.\end{array}$$

Here, the first two lines are true by the second condition of subconcavity and the last line is true by the submodularity of $v(x_1, x_2)$. Therefore, inequality (23) is true. When we combine (21), (22) and (23), it is obvious that equation (20) holds for the first case.

Case 2: $(p_{2,1} < p_{1,3})$

We first rearrange Equation (20) as:

$$\begin{array}{c}\bar{F}(p_{2,1})[p_{2,1}+v(x_1+2,x_2)]\\ +F(p_{2,1})v(x_1+1,x_2)\\ -\bar{F}(p_{1,2})[p_{1,2}+v(x_1+1,x_2+1)]\\ +F(p_{1,2})v(x_1,x_2+1)\end{array} \leq \begin{array}{c}\bar{F}(p_{2,2})[p_{2,2}+v(x_1+2,x_2+1)]\\ -F(p_{2,2})v(x_1+1,x_2+1)\\ -\bar{F}(p_{1,3})[p_{1,3}+v(x_1+1,x_2+2)]\\ -F(p_{1,3})v(x_1,x_2+2).\end{array} \tag{24}$$

Then, using the optimality of $p_{1,2}$ and after some algebra, we have that:

$$\begin{array}{c}\bar{F}(p_{2,1})[p_{2,1}+v(x_1+2,x_2)]\\ +F(p_{2,1})v(x_1+1,x_2)\\ -\bar{F}(p_{1,2})[p_{1,2}+v(x_1+1,x_2+1)]\\ +F(p_{1,2})v(x_1,x_2+1)\end{array} \leq \begin{array}{c}\bar{F}(p_{2,1})[v(x_1+2,x_2)-v(x_1+1,x_2+1)]\\ +F(p_{2,1})[v(x_1+1,x_2)-v(x_1,x_2+1)].\end{array} \tag{25}$$

Similarly, by using the optimality of $p_{2,2}$,

$$\begin{array}{c}\bar{F}(p_{2,2})[p_{2,2}+v(x_1+2,x_2+1)]\\ -F(p_{2,2})v(x_1+1,x_2+1)\\ -\bar{F}(p_{1,3})[p_{1,3}+v(x_1+1,x_2+2)]\\ -F(p_{1,3})v(x_1,x_2+2)\end{array} \geq \begin{array}{c}\bar{F}(p_{1,3})[v(x_1+2,x_2+1)-v(x_1+1,x_2+2)]\\ +F(p_{1,3})[v(x_1+1,x_2+1)-v(x_1,x_2+2)].\end{array} \tag{26}$$

As in the previous case, we show that the following is true:

$$\begin{array}{c}\bar{F}(p_{2,1})[v(x_1+2,x_2)-v(x_1+1,x_2+1)]\\ +F(p_{2,1})[v(x_1+1,x_2)-v(x_1,x_2+1)]\end{array} \leq \begin{array}{c}\bar{F}(p_{1,3})[v(x_1+2,x_2+1)-v(x_1+1,x_2+2)]\\ +F(p_{1,3})[v(x_1+1,x_2+1)-v(x_1,x_2+2)].\end{array} \tag{27}$$

Since $(p_{2,1} \geq p_{1,3})$, we have that $\bar{F}(p_{2,1}) = \bar{F}(p_{1,3}) + \xi$, where $\xi > 0$. Then, inequality (27) becomes:

$$\begin{array}{c}\bar{F}(p_{1,3})[v(x_1+2,x_2)-v(x_1+1,x_2+1)]\\ +F(p_{1,3})[v(x_1+1,x_2)-v(x_1,x_2+1)]\\ +\xi\begin{bmatrix}[v(x_1+2,x_2)-v(x_1+1,x_2+1)]\\ -[v(x_1+1,x_2)-v(x_1,x_2+1)]\end{bmatrix}\end{array} \leq \begin{array}{c}\bar{F}(p_{1,3})[v(x_1+2,x_2+1)-v(x_1+1,x_2+2)]\\ +F(p_{1,3})[v(x_1+1,x_2+1)-v(x_1,x_2+2)].\end{array}$$

Here, the first two lines hold by the second condition of subconcavity and the last line is true by the first condition of subconcavity. Therefore, inequality (27) is true. When we combine (25), (26) and (27), it is seen that (20) holds for the second case. Therefore, T_{PRC_1} preserves the second condition of subconcavity for both of the cases.

In conclusion, we show that $T_{PRC_1}v(x_1, x_2)$ will preserve both conditions of the subconcavity if the necessary assumptions are satisfied. The proof for T_{PRC_2} is similar. While considering T_{PRC_2}, we need to investigate the first condition in two cases: $(p_{1,2}) \geq p_{3,1}$ and $(p_{1,2}) < p_{3,1}$.

F Structure of the Optimal Sequencing Policy

In order to establish the structure of the optimal sequencing policy, we need to show that inequalities (1), (2) and (3) hold for the value function, $v(x_1, x_2)$. Below, we provide a sketch of the proof that $v(x_1, x_2)$ is decreasing in x_1. From the optimality equations, we have the following for the finite horizon value function v_{n+1}:

$$\begin{array}{c} \mu T_{SEQ} v_n(x_1, x_2) \\ + \sum_{j=1,2} \lambda_j T_{PRC_j} v_n(x_1, x_2) \\ -h_1 x_1 - h_2 x_2 \end{array} \geq \begin{array}{c} \mu T_{SEQ} v_n(x_1+1, x_2) \\ + \sum_{j=1,2} \lambda_j T_{PRC_j} v_n(x_1+1, x_2) \\ -h_1(x_1+1) - h_2 x_2. \end{array}$$

The inequalities in the first two lines can easily be shown to hold and the last line is trivially true. Therefore, $v_{n+1}(x_1, x_2)$ and $v(x_1, x_2)$ are decreasing in x_1. The infinite horizon value function $v(x_1, x_2)$ can be shown to possess the other monotonicity properties $Dec2$ and $Dec21$ in a similar way. Thus, $v(x_1, x_2)$ satisfies inequalities (1), (2) (3).

Lemma 2 *The value functions $v(x_1, x_2)$ and $v'(x_1, x_2)$ have the monotonicity properties $Dec1$, $Dec2$ and $Dec21$ as specified in inequalities (1), (2), and (3).*

F.1 Proof: Order of the Optimal Prices Among Classes (Corollary 1)

We prove this result by contradiction. To this end, we suppose $p_1^*(x_1, x_2) \leq p_2^*(x_1, x_2)$ for some (x_1, x_2) on the contrary. Let $p_j^* = p_j^*(x_1, x_2)$ for notational convenience. First, note that the following inequalities hold due to the optimality of p_1^* and p_2^*:

$$\bar{F}_1(p_1^*)\big[p_1^* + v(x_1+1, x_2)\big] + F_1(p_1^*)v(x_1, x_2) > \bar{F}_1(p_2^*)\big[p_2^* + v(x_1+1, x_2)\big] + F_1(p_2^*)v(x_1, x_2),$$
$$\bar{F}_2(p_2^*)\big[p_2^* + v(x_1, x_2+1)\big] + F_2(p_2^*)v(x_1, x_2) > \bar{F}_2(p_1^*)\big[p_1^* + v(x_1, x_2+1)\big] + F_2(p_1^*)v(x_1, x_2).$$

Combining these two inequalities, we have that

$$\begin{aligned} &\big[F_1(p_2^*) - F_1(p_1^*)\big]v(x_1+1, x_2) + \big[F_2(p_1^*) - F_2(p_2^*)\big]v(x_1, x_2+1) \\ &-\Big(\big[F_2(p_1^*) - F_1(p_1^*)\big] - \big[F_2(p_2^*) - F_1(p_2^*)\big]\Big)v(x_1, x_2) \\ &+\Big(\big[F_2(p_1^*)p_1^* - F_1(p_1^*)p_1^*\big] - \big[F_2(p_2^*)p_2^* - F_1(p_2^*)p_2^*\big]\Big) > 0. \end{aligned}$$

Rearranging the above inequality, we have that

$$\begin{aligned} &\big[F_1(p_2^*) - F_1(p_1^*)\big]\big[v(x_1+1, x_2) - v(x_1, x_2+1)\big] \\ &+\Big(\big[F_2(p_1^*) - F_1(p_1^*)\big] - \big[F_2(p_2^*) - F_1(p_2^*)\big]\Big)\big[v(x_1, x_2+1) - v(x_1, x_2)\big] \\ &+\Big(\big[F_2(p_1^*)p_1^* - F_1(p_1^*)p_1^*\big] - \big[F_2(p_2^*)p_2^* - F_1(p_2^*)p_2^*\big]\Big) > 0. \end{aligned} \tag{28}$$

Since $F_1(p_2^*) - F_1(p_1^*) > 0$ by our assumption on the optimal prices, $p_1^* \leq p_2^*$, and $\big[v(x_1 + 1, x_2) - v(x_1, x_2+1)\big] \leq 0$ by the monotonicity of $v(x_1, x_2)$, the first term above is less than zero. The second term is also less than zero because $\Big(\big[F_2(p_1^*) - F_1(p_1^*)\big] - \big[F_2(p_2^*) - F_1(p_2^*)\big]\Big) \geq 0$ by our assumptions on the optimal prices ($p_1^* \leq p_2^*$), and the reservation price distributions (Condition *(1)*), and $\big[v(x_1, x_2+1) - v(x_1, x_2)\big] \leq 0$ by the monotonicity of $v(x_1, x_2)$. Furthermore, the last term is less than zero as $F_2(p)p - F_1(p)p$ is (weakly) increasing in p (Condition *(2)*) and $p_1^* \leq p_2^*$. Combining these three observations, inequality (28) cannot be true, which leads to a contradiction. Therefore, our assumption on the optimal prices cannot be correct. Hence, the optimal prices for class-1, $p_1^*(x_1, x_2)$, is greater than the optimal prices for class-2, $p_1^*(x_1, x_2)$, as long as the reservation price distributions satisfy the condition stated in the corollary.

G Proof: Monotonicity of the Optimal Prices

Let p_1^* and p_2^* be the optimal prices for class-1 for the states (x_1, x_2) and (x_1+1, x_2), respectively, and assume that $p_1^* > p_2^*$. Then, we have the following as a result of the uniqueness and optimality of p_1^* and p_2^*:

$$\bar{F}_1(p_1^*)[p_1^* + v(x_1+1, x_2)] + F_1(p_1^*)v(x_1, x_2) > \bar{F}_1(p_2^*)[p_2^* + v(x_1+1, x_2)] + F_1(p_2^*)v(x_1, x_2)$$
$$\bar{F}_1(p_2^*)[p_2^* + v(x_1+2, x_2)] + F_1(p_2^*)v(x_1+1, x_2) > \bar{F}_1(p_1^*)[p_1^* + v(x_1+2, x_2)] + F_1(p_1^*)v(x_1+1, x_2).$$

When we combine these inequalities, we obtain:

$$[F_1(p_1^*) - F_1(p_2^*)][[v(x_1, x_2) - v(x_1+1, x_2)] - [v(x_1+1, x_2) - v(x_1+2, x_2)]] > 0. \quad (29)$$

Since $F_1(p_1^*) - F_1(p_2^*) > 0$ by our assumption on the optimal prices and $[v(x_1, x_2) - v(x_1+1, x_2)] - [v(x_1+1, x_2) - v(x_1+2, x_2)] \leq 0$ by the concavity of $v(x_1, x_2)$, inequality (29) cannot be true. Therefore, there is a contradiction and our assumption on the optimal prices is not correct. Hence, the optimal prices for class-1, $p_1^*(x_1, x_2)$, are increasing in the number of class-1 customers in the system. The monotonicity of the optimal prices for class-1 in x_2 and on the diagonal can be proven in a similar manner. Finally, the monotonicity of the optimal prices for class-2 can also be proven similarly.

H Detailed Numerical Results

In this section, we present the detailed numerical results pertaining to the examples of Section 4.

We denote by g^* the expected optimal profit corresponding to the solution of the MDP. We also employ two benchmark policies: a static pricing heuristic and a total queue length heuristic. A static pricing policy charges one unique price to each customer class regardless of the queue length. Under any given static pricing policy, the system becomes a priority queue whose expected queue lengths are easily obtained (see [9]). We numerically search over the two prices to obtain the corresponding expected profit g_{SP}.

The implementation of a total queue length based heuristic is more challenging. In our setting, the workload approximation logic of Maglaras [16] translates into a pricing policy that only depends on the total queue length $x_1 + x_2$ rather than on individual queue lengths. Unfortunately, beyond this basic fact, devising and implementing a well-performing total queue length heuristic is in itself extremely difficult. The implementation in [16] depends on a free parameter that needs to be optimized numerically. Since our main objective in this paper is not constructing a new total queue length based heuristic, we employ the optimal prices that were numerically computed in the MDP solution to come up with a reasonable total queue length based price. In particular, for any given total queue length $w = x_1 + x_2$, the corresponding price for class j is obtained by averaging the optimal prices: $p_j^W(w) = (p_j^*(0, w) + p_j^*(w, 0))/2$. This is not a useful practical heuristic since the optimal prices come from the numerical solution of the multi-dimensional MDP. But it yields a simple benchmark approximation that performs well. For instance, for the set of numerical examples presented in [16], this implementation yields slightly better results in terms of percentage optimality. We denote by g_W the expected optimal profit corresponding to the total queue length heuristic.

The percentage suboptimality of the static pricing and total queue length benchmarks are denoted respectively by $\Delta_{g^{SP}}$ and Δ_{g^W} and are defined as:

$$\Delta_{g^W} = \frac{g^* - g_{SP}}{g^*} \times 100 \text{ and } \Delta_{g^W} = \frac{g^* - g_W}{g^*} \times 100.$$

We are also interested in the fluctuations in the optimal price for a given workload level. As a simple measure we take the relative difference of the highest and the lowest price for the total queue length as a measure of this fluctuation. To this end, we denote percentage relative price difference for a total queue length of w for class i by $\Delta_{p_i}(w)$. More precisely,

$$\Delta_{p_i}(w) = \frac{|p_i^*(w, 0) - p_i^*(0, w)|}{\min\{p_i^*(w, 0), p_i^*(0, w)\}} \times 100.$$

Finally, we denote by ρ^* the utilization rate under the optimal policy and by p_1^* and p_2^* the optimal static prices under the optimal static pricing policy.

Table 5 reports the results that were summarized in Section 4.1. It is observed that the optimal prices do not fluctuate much for the second class. On the other hand, the price fluctuation is sometimes significant for class 1. Despite this fluctuation, the total queue length heuristic performs well in all cases including those where the utilization rate is low.

λ	g^*	ρ^*	p_1^*	p_2^*	$\Delta_{p_1}(1)$	$\Delta_{p_1}(3)$	$\Delta_{p_1}(5)$	$\Delta_{p_2}(1)$	$\Delta_{p_2}(3)$	$\Delta_{p_2}(5)$	Δ_{g^W}	$\Delta_{g^{SP}}$
1	0.43	0.23	0.68	0.52	7.02	21.67	29.69	0	0	0	0.03	0.23
2	1.82	0.46	1.11	1.04	7.14	12.28	18.64	0	0	1.78	0.03	0.33
3	4.12	0.68	1.70	1.62	3.7	7.27	13.56	0	0	0	0.06	0.53
4	7.18	0.84	2.44	2.34	3.63	6.89	11.67	1.88	0	0	0.11	1.31
5	10.73	0.92	3.32	3.21	1.75	4.92	9.52	0	0	0	0.16	2.60
6	14.51	0.96	4.27	4.15	1.69	4.69	7.46	0	0	0	0.17	3.63
7	18.39	0.98	5.24	5.12	1.64	2.99	5.71	0	0	0	0.16	4.28
8	22.31	0.98	6.22	6.10	0	4.35	4.11	1.61	0	1.41	0.16	4.65

Table 5 $\lambda_1 = \lambda_2 = \lambda$, $\mu = 4$, h_1=0.4, h_2=0.1

Table 6 reports the detailed results for the case outlined in Section 4.2. The effect of higher holding costs leads to lower optimal profits and lower utilization rates with respect to the results in Table 6. The price fluctuations for the first class appear to be more significant at higher utilization rates. Maybe surprisingly, the total queue length heuristic still performs at a suboptimality of 1%.

λ	g^*	ρ^*	p_1^*	p_2^*	$\Delta_{p_1}(1)$	$\Delta_{p_1}(3)$	$\Delta_{p_1}(5)$	$\Delta_{p_2}(1)$	$\Delta_{p_2}(3)$	$\Delta_{p_2}(5)$	Δ_{g^W}	$\Delta_{g^{SP}}$
1	0.13	0.09	1	0.65	0	0	0	0	0	0	0	1.49
2	0.89	0.29	1.68	1.24	18.65	1.01	0	0	0	0	0.86	4.61
3	2.39	0.46	2.40	1.88	16.88	13.63	3.09	1.59	1.35	2.41	0.81	5.36
4	4.53	0.6	3.20	2.61	13.51	20.48	9.89	0	1.38	2.5	0.83	6.67
5	7.16	0.71	4.06	3.42	14.7	21.95	12.36	1.59	1.39	2.53	0.8	7.88
6	10.17	0.79	4.96	4.28	9.59	23.46	13.63	1.56	2.77	2.53	0.79	9.14
7	13.44	0.84	5.90	5.19	8.22	19.51	13.63	1.54	2.74	3.79	0.79	10.15
8	16.89	0.88	6.86	6.12	6.76	17.07	13.63	0	1.33	3.75	0.72	10.91

Table 6 $\lambda_1 = \lambda_2 = \lambda$, $\mu = 4$, h_1=4, h_2=1

Finally, Table 7 reports the details of the case from Section 4.3. The utilization rates are now uniformly low since the first class is admitted only when there are no waiting customers from this class. The price fluctuation is again insignificant for class 2 but is more significant for class 1. The percentage suboptimality of the total queue length heuristic is now quite high, ranging above 10% for a wide range of service rates and nearing 20% in certain cases. The price fluctuation does not explain this bad performance in itself. It appears that in this case, the optimal policy is extremely sensitive to when class-1 customers are accepted and the acceptance rule is highly asymmetrical with respect to the total queue length. In particular, a class-1 customer may be admitted with 5 class-2 customers in the system but is rejected with a single class-1 customer (and no class-1 customer) in the system. This is a challenging situation for the total queue length heuristic.

μ	g^*	ρ^*	p_1^*	p_2^*	$\Delta_{p_1}(1)$	$\Delta_{p_1}(3)$	$\Delta_{p_1}(5)$	$\Delta_{p_2}(1)$	$\Delta_{p_2}(3)$	$\Delta_{p_2}(5)$	Δ_{g^W}	$\Delta_{g^{SP}}$
1	0.04	0.18	2	0.33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.83
1.1	0.04	0.22	1.97	0.32	1.01	0.00	0.00	0.00	0.00	0.00	1.99	9.32
1.2	0.06	0.26	1.94	0.32	5.26	1.01	0.00	1.43	0.00	0.00	12.57	19.67
1.3	0.08	0.27	1.91	0.31	8.69	4.16	0.00	1.45	2.35	0.00	16.05	25.30
1.4	0.11	0.29	1.88	0.31	12.36	7.53	3.09	1.47	2.41	3.09	18.42	29.18
1.5	0.13	0.29	1.84	0.31	14.94	11.11	6.38	1.49	3.70	3.16	17.12	26.15
1.6	0.16	0.29	1.81	0.30	17.65	13.63	9.89	1.51	3.79	4.35	15.64	27.50
1.7	0.18	0.29	1.78	0.30	20.48	16.28	12.36	1.54	3.89	4.44	13.69	24.44
1.8	0.20	0.29	1.76	0.30	21.95	20.48	13.63	1.56	3.95	5.75	11.86	21.00
1.9	0.22	0.29	1.73	0.30	25.00	20.48	16.28	1.59	2.67	4.65	10.19	18.63
2	0.25	0.28	1.70	0.29	26.58	21.95	19.04	3.22	4.11	4.76	8.54	19.60

Table 7 $\lambda_1 = 2$, $\lambda_2 = 0.5$, h_1=2, h_2=0.1