# NONCOMPLEX SMOOTH 4-MANIFOLDS WITH GENUS-2 LEFSCHETZ FIBRATIONS 

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#### Abstract

We construct noncomplex smooth 4-manifolds which admit genus-2 Lefschetz fibrations over $S^{2}$. The fibrations are necessarily hyperelliptic, and the resulting 4-manifolds are not even homotopy equivalent to complex surfaces. Furthermore, these examples show that fiber sums of holomorphic Lefschetz fibrations do not necessarily admit complex structures.


In the following we will prove the following theorem.
Theorem 1. There are infinitely many (pairwise nonhomeomorphic) 4-manifolds which admit genus2 Lefschetz fibrations but do not carry complex structure with either orientation.

Matsumoto showed that $S^{2} \times T^{2} \# 4 \overline{\mathbb{C} P^{2}}$ admits a genus-2 Lefschetz fibration over $S^{2}$ with global monodromy $\left(\beta_{1}, \ldots, \beta_{4}\right)^{2}$, where $\beta_{1}, \ldots, \beta_{4}$ are the curves indicated by Figure (For definitions and details regarding Lefschetz fibrations see M], [GS].)


Figure 1.

Let $B_{n}$ denote the smooth 4-manifold which admits a genus-2 Lefschetz fibration over $S^{2}$ with global monodromy

$$
\left(\left(\beta_{1}, \ldots, \beta_{4}\right)^{2},\left(h^{n}\left(\beta_{1}\right), \ldots, h^{n}\left(\beta_{4}\right)\right)^{2}\right)
$$

where $h=D\left(a_{2}\right)$ is a positive Dehn twist about the curve $a_{2}$ indicated in Figure 2 .
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Theorem 2. For the 4-manifold $B_{n}$ given above we have $\pi_{1}\left(B_{n}\right)=\mathbb{Z} \oplus \mathbb{Z}_{n}$.
Proof. Standard theory of Lefschetz fibrations gives that

$$
\pi_{1}\left(B_{n}\right)=\pi_{1}\left(\Sigma_{2}\right) /<\beta_{1}, \ldots, \beta_{4}, h^{n}\left(\beta_{1}\right), \ldots, h^{n}\left(\beta_{4}\right)>
$$

Let $\left\{a_{1}, b_{1}, a_{2}, b_{2}\right\}$ be the standard generators for $\pi_{1}\left(\Sigma_{2}\right)$ (Figure (2).


Figure 2.

Then we observe that

$$
\begin{aligned}
& \beta_{1}=b_{1} b_{2} \\
& \beta_{2}=a_{1} b_{1} a_{1}^{-1} b_{1}^{-1}=a_{2} b_{2} a_{2}^{-1} b_{2}^{-1} \\
& \beta_{3}=b_{2} a_{2} b_{2}^{-1} a_{1} \\
& \beta_{4}=b_{2} a_{2} a_{1} b_{1} \\
& h^{n}\left(\beta_{1}\right)=b_{1} b_{2} a_{2}^{n} \\
& h^{n}\left(\beta_{2}\right)=\beta_{2} \\
& h^{n}\left(\beta_{3}\right)=\beta_{3} \\
& h^{n}\left(\beta_{4}\right)=b_{2} a_{2}^{n+1} a_{1} b_{1} .
\end{aligned}
$$

Hence
$\pi_{1}\left(B_{n}\right)=<a_{1}, b_{1}, a_{2}, b_{2} \mid b_{1} b_{2},\left[a_{1}, b_{1}\right],\left[a_{2}, b_{2}\right], b_{2} a_{2} b_{2}^{-1} a_{1}, b_{2} a_{2} a_{1} b_{1}, b_{1} b_{2} a_{2}^{n}, b_{2} a_{2}^{n+1} a_{1} b_{1}>$ $=<a_{2}, b_{2} \mid\left[a_{2}, b_{2}\right], a_{2}^{n}>=\mathbb{Z} \oplus \mathbb{Z}_{n}$, and this concludes the proof.

Theorem 3. $B_{n}$ does not admit a complex structure.

Proof. Assume that $B_{n}$ admits a complex structure. Let $M_{n}$ denote its $n$-fold cover for which $\pi_{1}\left(M_{n}\right) \cong \mathbb{Z}$ and $M_{n}^{\prime}$ the minimal model of $M_{n}$. By the theorem of Gompf GS $B_{n}$ admits a symplectic structure, hence so does $M_{n}$ and (by combining results of Taubes and Gompf [1], [G2]) $M_{n}^{\prime}$. Consequently, if $B_{n}$ is a complex surface, then we have a symplectic, minimal complex surface $M_{n}^{\prime}$ with $\pi_{1}\left(M_{n}^{\prime}\right) \cong \mathbb{Z}$. In the following we will show that this leads to a contradiction.

By the Enriques-Kodaira classification of complex surfaces BPV, (since $b_{1}\left(M_{n}^{\prime}\right)=1$ ) $M_{n}^{\prime}$ is either a surface of class $V I I$ (in which case $b_{2}^{+}\left(M_{n}^{\prime}\right)=0$ ), a secondary Kodaira surface (in which case $b_{2}\left(M_{n}^{\prime}\right)=0$ ) or a (minimal) properly elliptic surface.

Since $M_{n}^{\prime}$ is a symplectic 4-manifold, $b_{2}^{+}\left(M_{n}^{\prime}\right)$ (and so $\left.b_{2}\left(M_{n}^{\prime}\right)\right)$ is positive; this observation excludes the first two possibilities.

Suppose now that $M_{n}^{\prime}$ admits an elliptic fibration over a Riemann surface. If the Euler characteristic of $M_{n}^{\prime}$ is 0 , then (following form the fact that $b_{1}\left(M_{n}^{\prime}\right)=b_{3}\left(M_{n}^{\prime}\right)=1$ ) we get that $b_{2}\left(M_{n}^{\prime}\right)=0$, which leads to the above contradiction. Suppose finally that $M_{n}^{\prime}$ is a minimal elliptic surface with positive Euler characteristic. Since $b_{1}\left(M_{n}^{\prime}\right)=1$, it can only be fibered over $S^{2}$ (see for example (FM). In that case (according to G1, for example) its fundamental group is

$$
\pi_{1}\left(M_{n}^{\prime}\right)=<x_{1}, \ldots x_{k} \mid x_{i}^{p_{i}}=1, i=1, \ldots, k ; x_{1} \cdots x_{k}=1>
$$

This cannot be isomorphic to $\mathbb{Z}$, since if $\pi_{1}\left(M_{n}\right) \cong \mathbb{Z}=<a>$, then $x_{1}=a^{m_{1}}$ for some $m_{1} \in \mathbb{Z}$, so $a$ has finite order, which is a contradiction. Consequently the assumption that $B_{n}$ is complex leads us to a contradiction, hence the theorem is proved.

Remark . The above proof, in fact, shows that $B_{n}$ is not even homotopy equivalent to a complex surface - our arguments used only homotopic invariants (the fundamental group, $b_{2}$ and $b_{2}^{+}$) of the 4-manifold $B_{n}$. Note that basically the same idea shows that $\bar{B}_{n}$ (the manifold $B_{n}$ with the opposite oreintation) carries no complex structure: The arguments involving the fundamental group, $b_{2}$ and the Euler characteristic only, apply without change. Since the fiber of the Lefschetz fibration on $B_{n}$ is homotopically essential and provides a class with square 0 , the intersection form of $B_{n}$ and so of $M_{n}$ are not definite - consequently these manifolds cannot be homotopy equivalent (with either orientation) to the blow-up of a surface of Class VII.

Proof of Theorem 1. By the definition of the 4-manifolds $B_{n}$ we get infinitely many manifolds admitting genus-2 (consequenlty hyperelliptic) Lefschetz fibrations which are (by Theorem 2.) nonhomeomorphic. As Theorem 3. and the above remark show, the manifolds $B_{n}$ do not carry complex structures with either orientation, hence the proof of the Theorem 1. is complete.

Remark . We would like to point out that similar examples have been found by Fintushel and Stern [FS - they used Seiberg-Witten theory to prove that their (simply connected) genus-2 Lefschetz fibrations are noncomplex.

Note that $B_{n}$ is given as the fiber sum of two copies of $S^{2} \times T^{2} \# 4 \overline{\mathbb{C} P^{2}}$, hence provides an example of the phenomenon that the fiber sum of holomorphic Lefschetz fibrations is not necessarily complex.

Acknowledgement. Examples of genus-2 Lefschetz fibrations with $\pi_{1}=\mathbb{Z} \oplus \mathbb{Z}_{n}$ were also constructed (as fiber sums) independently by Ivan Smith S].

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