

MINIMAL NUMBER OF SINGULAR FIBERS IN A LEFSCHETZ FIBRATION

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ABSTRACT: There exists a (relatively minimal) genus g Lefschetz fibration with only one singular fiber over a closed (Riemann) surface of genus h iff $g \geq 3$ and $h \geq 2$. The singular fiber can be chosen to be reducible or irreducible. Other results are that every Dehn twist on a closed surface of genus at least three is a product of two commutators and no Dehn twist on any closed surface is equal to a single commutator.

1. INTRODUCTION

A Lefschetz fibration is a fibration of a smooth 4-manifold over a surface with general fiber another closed orientable surface, which may admit certain singular fibers. The isomorphism class of a Lefschetz fibration is determined by its global monodromy. This leads to a combinatorial approach to study the topology of smooth 4-manifolds (which admit Lefschetz fibrations) by their monodromy representations in the mapping class groups. In particular, the number of singular fibers in a Lefschetz fibration can not be arbitrary and depends on the genus of the fiber, the genus of the base and the algebraic structure of the mapping class group of the generic fiber — as we will illustrate below.

Let $N(g, h)$ denote the minimal number of singular fibers in a relatively minimal genus g Lefschetz fibration (with at least one singular fiber) over a closed (Riemann) surface of genus h .

Our first result is the following theorem.

Theorem 1. $N(g, h) = 1$ iff $g \geq 3$ and $h \geq 2$.

Let Σ_g be a closed orientable surface of genus g . The mapping class group \mathcal{M}_g of Σ_g is defined to be the group of isotopy classes of orientation preserving diffeomorphisms $\Sigma_g \rightarrow \Sigma_g$.

For a simple closed curve a on an oriented surface, let us denote by t_a the right Dehn twist about a .

We also prove the following theorem, which is needed for the proof of Theorem 1.

Theorem 2. *Let Σ_g be a closed connected oriented surface of genus g and let a be a simple closed curve on Σ_g . If $g \geq 3$, then t_a can be written as a product of two commutators.*

2. DEFINITIONS AND PROOFS

Definition 3. *Let M be a closed, connected, oriented smooth four manifold. A Lefschetz fibration is a map $\pi : M \rightarrow \Sigma$, where Σ is a closed, connected, oriented surface, such that π is injective on the set of critical points $C = \{x_1, \dots, x_n\}$ and about each x_i and $\pi(x_i)$ there are complex local coordinate charts agreeing with the orientations of M and Σ on which π is of the form*

$$\pi(z_1, z_2) = z_1^2 + z_2^2.$$

Any fiber containing a critical point is called a singular fiber. Clearly all regular fibers are closed surfaces and are of the same diffeomorphism type. We will assume that the generic fiber is connected and our fibration is relatively minimal, i.e., no fiber contains a (-1) sphere. If the genus of the fiber is at least two, then a Lefschetz fibration is determined by its monodromy representation

$$\pi_1(\Sigma - \{x_1, \dots, x_n\}) \rightarrow \mathcal{M}_g.$$

(See [GS] for more about Lefschetz fibrations.)

Recall that for a group G , the first homology group with integral coefficient is $H_1(G) = G/[G, G]$, where $[G, G]$ is the commutator subgroup of G , the subgroup generated by the commutators $[x, y] = xyx^{-1}y^{-1}$ for all $x, y \in G$. Let Σ_g be a closed oriented surface. It is well known that $H_1(\mathcal{M}_g)$ is generated by the class of a Dehn twist about a nonseparating simple closed curve and is equal to \mathbb{Z}_{12} if $g = 1$, \mathbb{Z}_{10} if $g = 2$ and trivial if $g \geq 3$. We say that a simple closed curve on a closed surface is nontrivial if it does not bound a disc.

Lemma 4. *For every $h \geq 0$, $N(1, h) = 12$.*

Proof. All nontrivial circles on Σ_1 are nonseparating and the right Dehn twists map to the same generator under the natural map $\mathcal{M}_1 \rightarrow H_1(\mathcal{M}_1)$. Hence, if a product of right Dehn twists is trivial (or equal to a product of commutators), then the number of twists is divisible by 12. This implies that the number of singular fibers in a genus one Lefschetz fibration must be divisible by 12. Moreover, this number is clearly realized by the elliptic surfaces $E(1) \# \Sigma_1 \times \Sigma_h$ for any $h \geq 0$.

Proposition 5. *For every $h \geq 0$, $5 \leq N(2, h) \leq 8$.*

Proof. Since every right Dehn twist about a nontrivial separating simple closed curve on Σ_2 is the product of 12 right Dehn twists about nonseparating simple closed curves, the order of its image under the map $\mathcal{M}_2 \rightarrow$

$H_1(\mathcal{M}_2)$ is 5. Hence, if a product of right Dehn twists about n nonseparating and s separating simple closed curves is trivial (or equal to a product of commutators), then

$$n + 2s \equiv 0 \pmod{10}.$$

Therefore, there are at least 5 singular fibers in a genus two Lefschetz fibration over a closed surface. Moreover, a genus 2 Lefschetz fibration with 8 singular fiber is constructed in [M2].

For a 4-manifold Y , let $e(Y)$ and $\sigma(Y)$ denote the Euler characteristic and the signature of Y , respectively. The next proposition is proved by Stipsicz [S1].

Proposition 6. *Let M be a 4-manifold which admits a relatively minimal genus $g > 0$ Lefschetz fibration over a surface of nonzero genus. Then $0 \leq c_1^2(M) \leq 10\chi(M)$, where $c_1^2(M) = 3\sigma(M) + 2e(M)$ and $\chi(M) = (\sigma(M) + e(M))/4$.*

Proposition 7. *$N(g, 1) > 1$ for all $g \geq 1$.*

Proof. Suppose that $N(g, 1) = 1$ for some g . Then there is a 4-manifold Y which admits a relatively minimal genus g Lefschetz fibration over a torus with only one singular fiber. Hence $e(Y) = 1$. The inequality $0 \leq c_1^2(Y) = 3\sigma(Y) + 2e(Y)$ implies that $\sigma(Y) \geq -2/3$. This gives $\sigma(Y) \geq 0$ since $\sigma(Y)$ is an integer.

On the other hand, $\chi(Y) = \frac{\sigma(Y) + e(Y)}{4} = \frac{\sigma(Y) + 1}{4}$. Therefore $\sigma(Y) = 4\chi(Y) - 1$. So we get $c_1^2(Y) = 3\sigma(Y) + 2e(Y) = 12\chi(Y) - 1 \leq 10\chi(Y)$. Hence $\chi(Y) \leq 0$ and $\sigma(Y) \leq -1$ which is a contradiction. This proves the proposition.

Corollary 8. *The Dehn twist about a simple closed curve on a closed surface cannot be equal to a single commutator.*

Proof. Suppose that a right Dehn twist is a commutator. Then there is a relatively minimal Lefschetz fibration of genus g with only one singular fiber over the torus. Hence, $N(g, 1) = 1$, which contradicts to Proposition 7.

We are now ready to prove our main results.

Proof of Theorem 2. Consider a sphere X with four holes with boundary components a, a_1, a_2, a_3 . By the lantern relation [J] there are three simple closed curves b_1, b_2, b_3 on X such that

$$t_a t_{a_1} t_{a_2} t_{a_3} = t_{b_1} t_{b_2} t_{b_3}$$

or

$$t_a = t_{b_1} t_{a_1}^{-1} t_{b_2} t_{a_2}^{-1} t_{b_3} t_{a_3}^{-1}.$$

Here for two diffeomorphisms α and β , the composition $\alpha\beta$ means that α is applied first.

Since the genus of Σ is at least three, X can be embedded in Σ in such a way that $a_1, a_2, a_3, b_1, b_2, b_3$ are all nonseparating. The simple closed curve a can be chosen either nonseparating or separating bounding a subsurface of arbitrary genus (cf. Fig.1 and Fig.2). Furthermore, the complement of $a_1 \cup b_1$ and that of $a_2 \cup b_2$ are connected. Hence, there is an orientation preserving diffeomorphism F of Σ such that $F(a_1) = b_2$ and $F(b_1) = a_2$. Let K be another orientation preserving diffeomorphism of Σ such that $K(b_3) = a_3$ and let f and k be the isotopy classes of F and K respectively. Then

$$\begin{aligned} t_a &= t_{b_1} t_{a_1}^{-1} t_{F(a_1)} t_{F(b_1)}^{-1} t_{b_3} t_{K(b_3)}^{-1} \\ &= t_{b_1} t_{a_1}^{-1} f^{-1} t_{a_1} t_{b_1}^{-1} f t_{b_3} k^{-1} t_{b_3}^{-1} k \\ &= [t_{b_1} t_{a_1}^{-1}, f^{-1}] [t_{b_3}, k^{-1}]. \end{aligned}$$

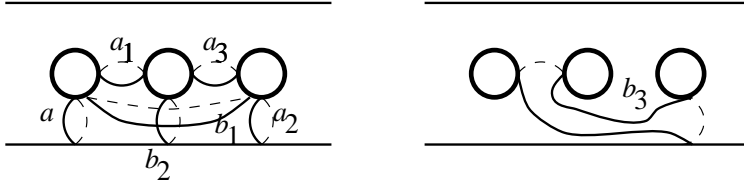


FIGURE 1. a is nonseparating.

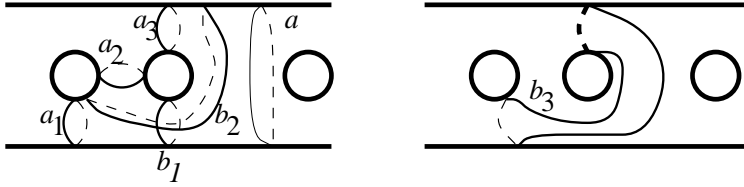


FIGURE 2. a is separating.

This concludes the proof of Theorem 2.

Proof of Theorem 1. The fact that $N(g, h) > 1$ is proved above for $g < 3$ or $h = 1$. Clearly, $N(g, 0) > 1$ for all $g \geq 1$.

Suppose that $g \geq 3$. Using the relation $t_a = [t_{b_1} t_{a_1}^{-1}, f^{-1}] [t_{b_3}, k^{-1}]$, we can construct a genus g Lefschetz fibration with only one singular fiber (whose vanishing cycle is the given curve a) over a closed surface of genus 2 as follows:

The fundamental group of Σ_2 has a presentation

$$\pi_1(\Sigma_2) = \langle x_1, y_1, x_2, y_2 \mid [x_1, y_1][x_2, y_2] = 1 \rangle.$$

By the standard bundle theory, there exist a genus g surface bundle over Σ_2 with the following monodromy representation:

$$\Psi : \pi_1(\Sigma_2) \rightarrow \mathcal{M}_g,$$

where $\Psi(x_1) = t_{b_1} t_{a_1}^{-1}$, $\Psi(y_1) = f^{-1}$, $\Psi(x_2) = t_{b_3}$, $\Psi(y_2) = k^{-1}$.

Consequently there exists a genus g surface bundle over the surface $\Sigma_{2,1}$ of genus two with one hole, such that the monodromy over the boundary is given by the product

$$[t_{b_1} t_{a_1}^{-1}, f^{-1}][t_{b_3}, k^{-1}].$$

We can extend this surface bundle to obtain a Lefschetz fibration over the closed genus two surface Σ_2 just by inserting a singular fiber with monodromy t_a . To obtain a Lefschetz fibration with one singular fiber over a surface of genus $h \geq 3$, we can take a fiber sum with $\Sigma_g \times \Sigma_{h-2}$. This proves that $N(g, h) = 1$ if $g \geq 3$ and $h \geq 2$.

This finishes the proof the theorem.

Remarks.

1) Theorem 2 answers a question of Mess in negative. See Problem 2.13 in [K]. If the genus of the surface is at least three, then the proof of the Theorem 2 shows that t_a^n can be expressed as a product of $(3n+1)/2$ (resp. $3n/2$) commutators if n is odd (resp. even). Using the lantern relation, we can also prove that if the genus is at least four, then t_a^n can be written as a product of n (resp. $n+1$) commutators if n is even (resp. odd).

2) Theorem 2 also shows that the unique singular fiber can be chosen reducible or irreducible. (A singular fiber is called reducible if the corresponding vanishing cycle is separating.)

3) ESTIMATES ON $N(g, 0)$.

In [C], Cadavid proves that $N(g, 0) \leq 2g + 4$ if g is even and $N(g, 0) \leq 2g + 10$ if g is odd. (This result was also discovered independently by the authors [KO]). Recently, Stipsicz [S2] proved that $\frac{1}{5}(4g + 2) \leq N(g, 0)$.

4) ESTIMATES ON $N(2, h)$.

In [M2], it is proven that $5 \leq N(2, h) \leq 8$ for all $h \geq 0$. The second author showed that $N(2, 0) = 7$ or 8 in his thesis [O]. It is also clear that there is a $k \geq 1$ such that $N(2, h) = 5$ for each $h \geq k$ (cf. [E]).

5) Examples of a Lefschetz fibration with a unique singular fiber and prescribed fundamental group were constructed in [ABKP].

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