A CHARACTERIZATION OF QUASIPOSITIVE TWO-BRIDGE KNOTS

BURAK OZBAGCI
(with an appendix by STEPAN OREVKOV)

ABSTRACT. We prove a simple necessary and sufficient condition for a two-bridge knot $K(p,q)$ to be quasipositive, based on the continued fraction expansion of p/q. As an application, coupled with some classification results in contact and symplectic topology, we give a new proof of the fact that smoothly slice two-bridge knots are non-quasipositive. Another proof of this fact using methods within the scope of knot theory is presented in the Appendix.

1. INTRODUCTION

Notions of quasipositivity and strong quasipositivity for links were introduced and explored by Rudolph in a series of papers (see, for example, [28, 29, 30, 31, 32, 33]). Let $\sigma_1, \ldots, \sigma_{n-1}$ denote the standard generators of the braid group B_n, and let $\sigma_{i,j} = (\sigma_i \ldots \sigma_{j-2})(\sigma_{j-1})(\sigma_i \ldots \sigma_{j-2})^{-1}$. Strongly quasipositive links are the links which can be realized as the closure of braids of the form $\prod_{k=1}^m \sigma_{i_k,j_k}$. The weaker notion of quasipositive link is any link which can be realized as the closure of a braid of the form $\prod_{k=1}^m w_k \sigma_{i_k} w_k^{-1}$, where $w_k \in B_n$ for all $1 \leq k \leq m$. An oriented link is called positive if it has a positive diagram, i.e., a diagram in which all crossings are positive.

Throughout this paper, we assume that $p > q$ are relatively prime positive integers. The oriented lens space $L(p,q)$ is defined by $-p/q$ surgery on the unknot in S^3 and it is well-known that $L(p,q)$ is the double cover of S^3 branched along the two-bridge link $K(p,q)$, which we depicted in Figure 1. When p is odd, $K(p,q)$ is a knot, and otherwise it has two components.

The *negative* continued fraction of p/q is defined by

$$\frac{p}{q} = [a_1, a_2, \ldots, a_k]^- = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_k}}}, \qquad a_i \geq 2 \text{ for all } 1 \leq i \leq k,$$

where the coefficients $a_1, a_2, \ldots, a_k$ are uniquely determined by p/q. We say that the negative continued fraction of p/q is even if a_i is even for all $1 \leq i \leq k$.

Theorem 1.1. *If the two-bridge link $K(p, q)$ is quasipositive then the negative continued fraction of p/q is even.*

If the negative continued fraction of p/q is even, then pq must be even. Corollary 1.2 immediately follows from this simple observation coupled with Theorem 1.1.

Corollary 1.2. *If pq is odd, then the two-bridge knot $K(p, q)$ is non-quasipositive.*

Conversely, we can express Tanaka's quasipositivity criterion [34, Proposition 3.1] for two-bridge knots in terms of negative continued fractions as follows:

Proposition 1.3 (Tanaka [34])**.** *The two-bridge knot $K(p, q)$ is strongly quasipositive, provided that the negative continued fraction of p/q is even.*

We would like to point out that Proposition 1.3 cannot possibly hold for arbitrary two-bridge links. For example, $16/3 = [6, 2, 2]^-$ and $16/13 = [2, 2, 2, 2, 4]^-$, but the two-bridge link $K(16, 3)$ and its mirror $K(16, 13)$ cannot both be quasipositive since Hayden [15, Corollary 1.5] proved that if a link and its mirror are both quasipositive, then the link is an unlink.

Combining Theorem 1.1 and Proposition 1.3, we have the following characterization of quasipositive two-bridge knots.

Corollary 1.4. *The two-bridge knot $K(p, q)$ is quasipositive if and only if the negative continued fraction of p/q is even.*

Remark 1.5. It is clear that strong quasipositivity implies quasipositivity by definition. Conversely, Boileau and Rudolph [4, Proposition 3.6 and Corollary 3.7] proved for a family of alternating arborescent links, including 2-bridge links, that quasipositivity implies strong quasipositivity. On the other hand, by the work of Rudolph [32] and Nakamura [24, Lemma 4.1], positive diagrams represent strongly quasipositive links. Moreover, strongly quasipositive two-bridge links are positive, since any two-bridge link can be obtained as the boundary of plumbings of annuli and strong quasipositivity behaves natural with respect to the plumbing operation [31]. The upshot is that, positivity, quasipositivity and strong quasipositivity are in fact equivalent for two-bridge links.

The proof of Theorem 1.1 is based on the work of Plamenevskaya [26], coupled with the following result on contact topology.

Proposition 1.6. *The lens space $L(p, q)$ admits a tight contact structure with trivial first Chern class if and only if the negative continued fraction of p/q is even.*

Recall that a knot in S^3 is called *smoothly slice* if it bounds a smooth properly embedded disk in B^4. Rudolph [30, Proposition 2] showed that the only smoothly slice strongly quasipositive knot is the unknot, as a corollary to the celebrated work of Kronheimer and Mrowka [17]. As we pointed out in Remark 1.5, quasipositivity and strong quasipositivity are equivalent for a two-bridge link. Therefore, we conclude that smoothly slice two-bridge knots are non-quasipositive. We will provide a new proof of this fact as an application of Corollary 1.4 coupled with the following result on symplectic topology.

Proposition 1.7. *Let (Y, ξ) be the contact double cover of the standard contact 3-sphere (S^3, ξ_{st}) branched along the knot K which we assume to be transverse to ξ_{st}. If K is smoothly slice and quasipositive, then (Y, ξ) admits a rational homology ball Stein filling.*

After we finished a first draft of this paper, it came to our attention that in [25, Remark 2], Orevkov observed that quasipositivity implies (by results of [25] combined with [7]) strong quasipositivity for two-bridge links, and for a larger class of links including all alternating Montesinos links—which does not directly follow from the arguments in [4, Proposition 3.6 and Corollary 3.7]. Since we convinced him that his result would be of interest to knot enthusiasts, he kindly agreed to write the details of his [25, Remark 2] in the Appendix (especially in the two-bridge case where [21] can be used instead of [7]), which in turn, gives yet another proof of the fact that smoothly slice two-bridge knots are non-quasipositive.

2. Applications of contact and symplectic topology

We begin with the proof of Proposition 1.6.

Proof of Proposition 1.6. Suppose that $p/q = [a_1, a_2, \ldots, a_k]^-$, where $a_i \geq 2$ is even for all $1 \leq i \leq k$. Note that $L(p, q)$ can be obtained by surgery on a chain of unknots with framings $-a_1, -a_2, \ldots, -a_k$, respectively. Since $a_i \geq 2$ is even for all $1 \leq i \leq k$, we can Legendrian realize each unknot in the chain, with respect to the standard contact structure, so that the rotation number of each Legendrian unknot is zero. It follows by [14, Proposition 2.3] that the first Chern class of the Stein fillable (and hence tight) contact structure on $L(p, q)$ obtained by Legendrian surgery on the resulting Legendrian link is zero.

To prove the only if direction, suppose that $p/q = [a_1, a_2, \ldots, a_k]^-$, where $a_i \geq 2$ for all $1 \leq i \leq k$ and a_j is odd for some $1 \leq j \leq k$. Let ξ be the Stein fillable contact structure on $L(p, q)$ obtained by Legendrian surgery along an arbitrary Legendrian realization $\mathcal{L}$ of the chain of unknots with smooth framings $-a_1, -a_2, \ldots, -a_k$, respectively. Let $\overline{\mathcal{L}}$ be the Legendrian link obtained from $\mathcal{L}$ by taking the mirror image of each component and let $\bar{\xi}$ be the Stein fillable contact structure on $L(p, q)$ obtained by Legendrian surgery along $\overline{\mathcal{L}}$. It follows that rotation number of each component of $\overline{\mathcal{L}}$ is the negative of the rotation number of the corresponding component of $\mathcal{L}$ and hence $\overline{\xi}$ is obtained from ξ by reversing the orientation of the contact planes. By Honda's classification [16, Theorem 2.1] of tight contact structures on $L(p,q)$, the contact structures ξ and $\overline{\xi}$ are not isotopic because of the assumption that a_j is odd for some $1 \leq j \leq k$. Note that non-isotopic tight contact structures on $L(p,q)$ are non-homotopic [16, Proposition 4.24]. This implies that $c_1(\xi) \neq 0$ since according to Gompf [14, Corollary 4.10], an oriented plane field in any closed oriented 3-manifold is homotopic to itself with reversed orientation if and only if it has trivial first Chern class. □

We now give a proof of Theorem 1.1, based on Proposition 1.6.

Proof of Theorem 1.1. The double cover of S^3 branched along $K(p,q)$ is $L(p,q)$. We can make $K(p,q)$ transverse to the standard contact structure ξ_{st} by isotopy. Let $(L(p,q),\xi)$ be the contact double cover of (S^3,ξ_{st}) branched along the transverse link $K(p,q)$. Suppose that $K(p,q)$ is quasipositive. By the work of Plamenevskaya [26, Proposition 1.4 and Lemma 5.1], we conclude that ξ is Stein fillable and moreover $c_1(\xi) = 0$. It follows by Proposition 1.6 that the negative continued fraction of p/q must be even. □

The proof of Proposition 1.3 is essentially obtained by rephrasing Proposition 3.1 in Tanaka's paper [34], where he uses regular continued fractions to describe a sufficient condition for a two-bridge knot to be strongly quasipositive.

Proof of Proposition 1.3. A *regular* continued fraction of p/q is defined by

$$\frac{p}{q} = [c_1, c_2, \ldots, c_{2m+1}] = c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots + \cfrac{1}{c_{2m+1}}}}, \qquad c_i > 0 \text{ for all } 1 \leq i \leq 2m+1.$$

Note that there is always a regular continued fraction of p/q of odd length, as above. To see this, suppose that p/q has a regular continued fraction of even length, i.e. $\frac{p}{q} = [c_1, c_2, \ldots, c_{2m}]$ with $c_i > 0$. If $c_{2m} = 1$, then

$$\frac{p}{q} = [c_1, c_2, \ldots, c_{2m}] = [c_1, c_2, \ldots, c_{2m-2}, 1 + c_{2m-1}]$$

and otherwise,

$$\frac{p}{q} = [c_1, c_2, \ldots, c_{2m}] = [c_1, c_2, \ldots, c_{2m-1}, -1 + c_{2m}, 1].$$

Using an odd length regular continued fraction $\frac{p}{q} = [c_1, c_2, \ldots, c_{2m+1}]$, we define the two bridge link $K(p, q)$ as depicted in Figure 1, where the integer inside each box denotes the signed number of half twists to be inserted.

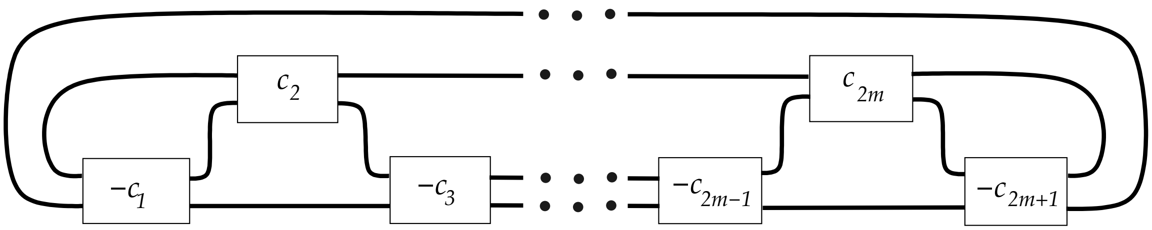

FIGURE 1. Two-bridge link $K(p, q)$, where $p/q = [c_1, c_2, \ldots, c_{2m+1}]$.

In [34], Tanaka uses the notation $\mathcal{C}(c_1, c_2, \ldots, c_{2m+1})$, with each $c_i > 0$, to describe a two-bridge knot. By comparing [34, Figure 6], for example, with our Figure 1, we see that our definition of $K(p, q)$ is the *mirror image* of the one described by Tanaka, but agrees with the one described by Lisca [18].

According to [34, Proposition 3.1], the two-bridge knot $\mathcal{C}(c_1, c_2, \ldots, c_{2m+1})$, which is the mirror image of our $K(p, q)$ defined above, is strongly quasipositive as long as c_i is even for each even index $2 \leq i \leq 2m$. To express this condition in terms of negative continued fractions, we just describe how to convert a given regular continued fraction of any p/q to the negative continued fraction of $p/(p-q)$ and vice versa in Lemma 2.1. This finishes the proof of Theorem 1.1, since by Lemma 2.1, we can easily deduce that the negative continued fraction of $p/(p-q)$ is even if and only if p/q has a regular continued fraction of odd length where each even indexed coefficient is even. $\square$

Lemma 2.1. *Suppose that* $p/q = [c_1, c_2, \ldots, c_{2m+1}]$ *with* $c_i > 0$ *for all* $1 \leq i \leq 2m + 1$. *Then*

$$\frac{p}{q} = [1 + c_1, \underbrace{2, \ldots, 2}_{c_2 - 1}, 2 + c_3, \underbrace{2, \ldots, 2}_{c_4 - 1}, \ldots, 2 + c_{2m-1}, \underbrace{2, \ldots, 2}_{c_{2m} - 1}, 1 + c_{2m+1}]^-$$

and

$$\frac{p}{p-q} = [\underbrace{2, \ldots, 2}_{c_1 - 1}, 2 + c_2, \underbrace{2, \ldots, 2}_{c_3 - 1}, 2 + c_4, \ldots, \underbrace{2, \ldots, 2}_{c_{2m-1} - 1}, 2 + c_{2m}, \underbrace{2, \ldots, 2}_{c_{2m+1} - 1}]^-$$

Proof. The negative continued fraction of p/q can be obtained from a given regular continued fraction of p/q by a straightforward induction argument, whereas the negative continued fraction of $p/(p-q)$ is obtained from that of p/q by the Riemenschneiders point diagram method [27]. □

Using Lemma 2.1, we can rephrase Corollary 1.4 as follows:

Corollary 2.2. *The two-bridge knot $K(p,q)$ is quasipositive if and only if $p/(p-q)$ has a regular continued fraction of odd length where each even indexed coefficient is even.*

Finally, we turn our attention to Proposition 1.7.

Proof of Proposition 1.7. Rudolph [28] showed that quasipositive links arise as the transverse intersection of $S^3 \subset \mathbb{C}^2$, with a complex curve. Therefore, the quasipositivity assumption implies that K bounds a complex curve in B^4. Since K is assumed to be smoothly slice as well, there is a smooth disk in B^4 with boundary K. But by the "local Thom conjecture" [17, Corollary 1.3] , the complex curve minimizes genus, so the slice disk can be assumed to be complex. Hence, the analytic double cover of B^4 equipped with its standard complex structure, branched along this complex slice disk, is a rational homology ball Stein filling of (Y, ξ). It is well-known that the double cover is a rational homology ball and it is in fact Stein by [19, Theorem 3]. □

3. Smoothly slice two-bridge knots are non-quasipositive

As we pointed out in the Introduction, the fact that smoothly slice two-bridge knots are non-quasipositive follows by combining [30, Proposition 2] with [4, Proposition 3.6 and Corollary 3.7]. Here we provide an alternate proof based on contact and symplectic topology.

Corollary 3.1. *Smoothly slice two-bridge knots are non-quasipositive.*

Proof. Lisca [18] showed that $L(p,q)$ bounds a rational homology ball if and only if p/q belongs to a certain subset $\mathcal{R}$ of the set of positive rational numbers. Lisca also showed that, for odd p, any two-bridge knot $K(p,q)$ is smoothly slice if and only if $p/q \in \mathcal{R}$. By definition, the set $\mathcal{R}$ contains a subset, denoted by $\mathcal{O}$ here, which consists of $p/q > 0$ such that $p = m^2$ (for odd m), and $q = mh - 1$ where $0 < h < m$ and $(m,h) = 1$.

Suppose that $p/q \in \mathcal{R} \setminus \mathcal{O}$ and $K(p,q)$ is a smoothly slice, quasipositive two-bridge knot. We can isotope $K(p,q)$ to be transverse to the standard contact structure ξ_{st} in S^3. Let $(L(p,q), \xi)$ be the double cover of (S^3, ξ_{st}) branched along the transverse knot K. According to Proposition 1.7, $(L(p,q), \xi)$ admits a rational homology ball Stein filling, which in turn, implies that ξ is isomorphic to the canonical contact structure ξ_{can}, because no virtually overtwisted lens space has a rational homology ball symplectic filling by the work of Golla and Starkston [13]. (See also [10, Lemma 1.5], [11, Proposition 11]). This gives a contradiction to the fact that $(L(p,q),\xi_{can})$ admits a rational homology ball symplectic filling if and only if $p/q \in \mathcal{O}$, as shown by Lisca [18, Corollary 1.2(c)].

Now suppose that $p/q \in \mathcal{O}$, i.e. $p = m^2$, for odd $m > 1$ and $q = mh - 1$ where $0 < h < m$ and $(m,h) = 1$. If h is even, then $q = mh - 1$ is odd and $K(p,q)$ is non-quasipositive by Corollary 1.4. On the other hand, if h is odd, then $q = mh-1$ is even but $q' = m(m-h)-1$ is odd and $qq' \equiv 1(\text{mod } m^2)$. Since $K(p,q)$ is isotopic to $K(p,q')$ [5, Chapter 12], we conclude that $K(p,q')$ and hence $K(p,q)$ is non-quasipositive, again by Corollary 1.4. □

Acknowledgment: We would like to thank Sebastian Baader, Peter Feller, Marco Golla, and András Stipsicz for their useful comments on a draft of this note.

4. APPENDIX BY STEPAN OREVKOV

The Seifert graph of a connected link diagram D is the graph G_D whose vertices correspond to Seifert circles and the edges correspond to the crossings. Each edge is equipped with the sign of the corresponding crossing. We say that a diagram D is reduced if G_D does not have any edge whose removal disconnects G_D. Let $w(D)$ denote the writhe of D, which is the sum of the signs of all crossings of D.

Suppose that D is an alternating diagram of a link L. Let $b = b(L)$ denote the braid index of L and $s = s(D)$ denote the number of Seifert circles of D. We define $d^{\pm} = d^{\pm}(D)$ as the number of edges of $\pm$ sign in a spanning tree of G_D.

Suppose that β is a braid with b strands realizing L. Due to DynnikovPrasolov Theorem [9], $w(\beta)$ does not depend on the choice of b-strand β realizing L, which allows one to define the numbers $\mathbf{r}^{\pm} = \mathbf{r}^{\pm}(D)$ from the system of equations

$$\mathbf{r}^+ + \mathbf{r}^- = s - b \quad \text{and} \quad \mathbf{r}^+ - \mathbf{r}^- = w(D) - w(\beta).$$

By the work of Rudolph [32] and Nakamura [24, Lemma 4.1], positive diagrams represent strongly quasipositive links. Conversely, Baader [2] showed that homogeneous strongly quasipositive knots are positive. Note that the class of homogeneous links, introduced by Cromwell [6], includes all alternating links. Moreover, alternating strongly quasipositive links have positive alternating diagrams by Boileau, Boyer and Gordon [3, Corollary 7.3]. Therefore, the following question appears naturally (see [2], for example):

Question 4.1. *Is it true that alternating quasipositive links have positive diagrams?*

A partial answer to this question was provided as follows:

Theorem 4.2. ([25, Theorem 4]). *If D is a reduced alternating diagram of a quasipositive link L, which satisfies the inequality*

$$2\mathbf{r}^-(D) \leq d^-(D) \tag{4.1}$$

then D is positive, and hence L is strongly quasipositive.

Proposition 4.3. ([25, Remark 2]). *The inequality (4.1) is satisfied by a standard alternating diagram of any two-bridge link or any alternating Montesinos link.*

Remark 4.4. As pointed out in [25, Remark 2], inequality (4.1) is actually proven in [7] for the diagrams from [7, Thm. 4.10, 4.12, 4.14] (in particular, for those in Proposition 4.3) even though it is not formulated in [7] explicitly. Since it is not so easy to recognize the proof of this fact without carefully reading the whole paper [7], one of our goals here is to help the reader to extract this proof from [7].

Remark 4.5. Note that by Proposition 4.3, if a two-bridge link or an alternating Montesinos link is quasipositive, then it is positive.

The statement in Proposition 4.6 was claimed in [25, Remark 2] and as mentioned there, it follows from [7]. For rational links, however, it can also be easily derived from Murasugi's work [21, Section 3].

Proposition 4.6. *Every oriented two-bridge (aka rational) link admits an alternating diagram D satisfying the inequalities $2\mathbf{r}^-(D) \leq d^-(D)$ and $2\mathbf{r}^+(D) \leq d^+(D)$.*

Proof. For rational links we follow the orientation convention used in Murasugi's book [22]. Let L be a rational oriented link of type (p, q). For the mirror image D^* of D, we have $d^\pm(D^*) = d^\mp(D)$ and $\mathbf{r}^\pm(D^*) = \mathbf{r}^\mp(D)$. Therefore, without loss of generality we may assume that q is odd and $0 < q < p$. Using the notation introduced in [21, Section 3], let

$$\frac{p}{p-q} = [2n_{1,1}, \ldots, 2n_{1,k_1},\ -2n_{2,1}, \ldots, -2n_{2,n_2},\ \ldots\ , (-1)^{t-1}2n_{t,1}, \ldots, (-1)^{t-1}2n_{t,k_t}]^-,$$

where we assume $n_{i,j} > 0$.

Let b be the braid index of L and $e = w(\beta)$ be the exponent sum of a b-braid β representing L. By [21, Prop. 4.2 and Thm. 4.3] we have

$$b = t + 1 + \sum_{i=1}^{t}\sum_{j=1}^{k_i}(n_{i,j} - 1), \qquad e = \frac{1-(-1)^t}{2} + \sum_{i=1}^{t}(-1)^{i-1}\sum_{j=1}^{k_i} n_{i,j}. \tag{4.2}$$

Using the standard properties of rational links and continued fractions, one easily checks that L admits an alternating diagram D shown in Figure 1 where T_i are the tangles defined by the braids

$$T_i = \begin{cases} \sigma_1^{1-2n_{i,1}} \Big(\prod_{j=2}^{k_i} \sigma_2 \sigma_1^{2-2n_{i,j}} \Big) \sigma_1^{-1}, & i \text{ is odd,} \\ \sigma_2^{2n_{i,1}-1} \Big(\prod_{j=2}^{k_i} \sigma_1^{-1} \sigma_2^{2n_{i,j}-2} \Big) \sigma_2, & i \text{ is even,} \end{cases}$$

as depicted in Figure 2 for odd i. Note that, for odd (resp. even) i, all crossings of T_i are positive (resp. negative).

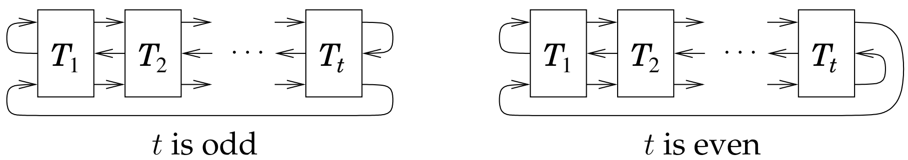

FIGURE 2. Alternating diagram D for the rational link L.

Let $w = w(D)$ and $s = s(D)$ be the writhe and the number of Seifert circles of D. Each tangle T_i contributes $(-1)^{i-1} w_i$ to w, where $w_i = 1 + \sum_j (2n_{i,j} - 1)$. If i is odd (resp. even), then T_i contributes $w_i - k_i$ to d^+ (resp. to d^-). We have $s - (d^+ + d^-) = 1$ (the Euler characteristic of a spanning tree).

Hence

$$s = t + 1 + \sum_{i=1}^{t} \sum_{j=1}^{k_i} 2(n_{i,j} - 1), \quad w = \frac{1 - (-1)^t}{2} + \sum_{i=1}^{t} (-1)^{i-1} \sum_{j=1}^{k_i} (2n_{i,j} - 1), \tag{4.3}$$

$$d^+ = \lceil t/2 \rceil + \sum_{i \text{ odd}} \sum_{j=1}^{k_i} 2(n_{i,j} - 1), \qquad d^- = \lfloor t/2 \rfloor + \sum_{i \text{ even}} \sum_{j=1}^{k_i} 2(n_{i,j} - 1). \tag{4.4}$$

By combining (4.2) and (4.3) we obtain

$$s - b = \sum_{i=1}^{t} \sum_{j=1}^{k_i} (n_{i,j} - 1), \qquad w - e = \sum_{i=1}^{t} (-1)^{i-1} \sum_{j=1}^{k_i} (n_{i,j} - 1). \tag{4.5}$$

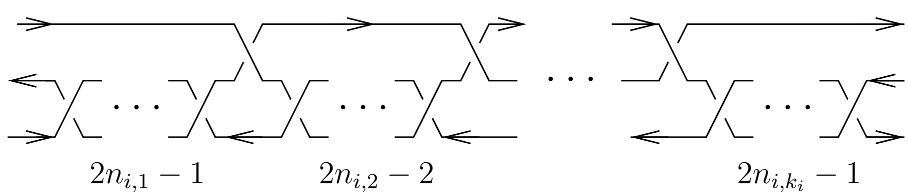

FIGURE 3. The tangle T_i for an odd i.

Recall that $\mathbf{r}^{\pm} = \mathbf{r}^{\pm}(D)$ are defined by

$$\mathbf{r}^{+} + \mathbf{r}^{-} = s - b, \qquad \mathbf{r}^{+} - \mathbf{r}^{-} = w - e. \tag{4.6}$$

By combining (4.4), (4.5), and (4.6) we obtain

$$2\mathbf{r}^{+} = d^{+} - \lceil t/2 \rceil \le d^{+}, \qquad 2\mathbf{r}^{-} = d^{-} - \lfloor t/2 \rfloor \le d^{-}.$$

□

4.1. **How to extract the proof of Proposition 4.3 from [7].** The paper [7] is devoted to a computation of the braid index of alternating links presented by link diagrams of some specific forms. An upper bound for the braid index is the number of Seifert circles. A lower bound is given by Morton-Franks-Williams (MFW) inequality ([12, 20]). In many cases (those indicated in Theorem 4.2) it is shown in [7] that the MFW bound is sharp. In each case, this is done in [7] as follows. Given a reduced alternating diagram D of a link L, one chooses a certain collection C of lone crossings, and applies the Murasugi-Przytycki move [23] (MP-move) to each of them (MP-moves are also depicted in [34, Fig. 2] and [8, Fig. 11]). The number of Seifert surfaces of the resulting diagram D' (in general, non-alternating) is equal to the braid index of L and, moreover, the number of the performed MP-moves at positive (negative) crossings is equal to $\mathbf{r}^{+}(D)$ (resp. $\mathbf{r}^{-}(D)$).

A *constant-sign path of length* n in G_D is a sequence $v_1, \dots, v_n$ of pairwise distinct vertices of G_D such that each pair of consecutive vertices (v_i, v_{i+1}) is connected by an edge, and all these edges are of the same sign. One can check that the vertices of G_D corresponding to C are always chosen in [7] in some pairwise disjoint constant-sign paths. Moreover, at most $\lfloor (n-1)/2 \rfloor$ crossings is chosen in each of the paths of length n. It is clear that any collection of pairwise disjoint paths is contained in some spanning tree. Thus we obtain (4.1) for all diagrams mentioned in Remark 4.4, in particular, this gives a proof of Proposition 4.3.

REFERENCES

[1] S. Baader, *Slice and Gordian numbers of track knots.* Osaka J. Math. 42 (2005), no. 1, 257-271.

[2] S. Baader, *Quasipositivity and homogeneity.* Math. Proc. Cambridge Philos. Soc. 139 (2005), no. 2, 287-290.

[3] M. Boileau, S. Boyer, and C. McA. Gordon, *Branched covers of quasi-positive links and L-spaces.* J. Topol. 12 (2019), no. 2, 536-576.

[4] M. Boileau and L. Rudolph, *Nœuds non concordants à un $\mathbb{C}$-bord.* Vietnam J. Math., vol 23 (1995), Special issue "Actes du Séminaire Franco-Vietnamien Analyse Pluricomplexe et la Topologie de Singularités", 13-28, arXiv:math/0201260.

[5] G. Burde and H. Zieschang, *Knots.* De Gruyter Stud. Math., 5 Walter de Gruyter & Co., Berlin, 2003.

[6] P. R. Cromwell, *Homogeneous links.* J. London Math. Soc. (2) 39 (1989), no. 3, 535-552.

[7] Y. Diao, C. Ernst, G. Hetyei, and P. Liu, *A diagrammatic approach for determining the braid index of alternating links.* J. Knot Theory Ramifications 30 (2021), no. 5, Paper No. 2150035, 36 pp.
[8] Y. Diao, G. Hetyei, and P. Liu, *The braid index of reduced alternating links.* Math. Proc. Cambridge Philos. Soc. 168 (2020), no. 3, 415-434.
[9] I. A. Dynnikov and M. V. Prasolov, *Bypasses for rectangular diagrams. A proof of the Jones conjecture and related questions.* Trudy mosk. mat. obshch. 74 (2013), no. 1, 115–173 (Russian); English transl., Trans. Moscow Math. Soc. 74 (2013), 97-144.
[10] J. B. Etnyre and A. Roy, *Symplectic fillings and cobordisms of lens spaces.* Trans. Amer. Math. Soc. 374 (2021), no. 12, 8813-8867.
[11] J. B. Etnyre and B. Tosun, *Homology spheres bounding acyclic smooth manifolds and symplectic fillings.* arXiv:2004.07405.
[12] J. Franks and R. F. Williams, *Braids and the Jones polynomial.* Trans. Amer. Math. Soc. 303 (1987), no. 1, 97-108.
[13] M. Golla and L. Starkston, *The symplectic isotopy problem for rational cuspidal curves.* Compos. Math. 158 (2022), no. 7, 1595-1682.
[14] R. E. Gompf, *Handlebody construction of Stein surfaces.* Ann. of Math. (2) 148 (1998), no. 2, 619-693.
[15] K. Hayden, *Minimal braid representatives of quasipositive links.* Pacific J. Math. 295 (2018), no. 2, 421-427.
[16] K. Honda, *On the classification of tight contact structures. I.* Geom. Topol. 4 (2000), 309-368.
[17] P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces. I.* Topology 32 (1993), no. 4, 773-826.
[18] P. Lisca, *Lens spaces, rational balls and the ribbon conjecture.* Geom. Topol. 11 (2007), 429-472.
[19] A. Loi and R. Piergallini, *Compact Stein surfaces with boundary as branched covers of B^4.* Invent. Math. 143 (2001), no. 2, 325-348.
[20] H. R. Morton, *Seifert Circles and Knot Polynomials.* Math. Proc. Cambridge Philos. Soc. 99 (1986), no. 1, 107-109.
[21] K. Murasugi, *On the braid index of alternating links.* Trans. Amer. Math. Soc. 326 (1991), no. 1, 237-260.
[22] K. Murasugi, *Knot theory and its applications.* Translated from the 1993 Japanese original by Bohdan Kurpita. Birkhäuser Boston, Inc., Boston, MA, 1996.
[23] K. Murasugi, J. Przytycki, *Index of a graph with applications to knot theory.* Mem. Amer. Math. Soc. 106 (508), 1993.
[24] T. Nakamura, *Four-genus and unknotting number of positive knots and links.* Osaka J. Math. 37 (2000), no. 2, 441-451.
[25] S. Yu. Orevkov, *On alternating quasipositive links.* Doklady Akademii Nauk, 494 (2020), 53-56 (Russian); English transl., Dokl. Math. 102 (2020), no. 2, 403-405.
[26] O. Plamenevskaya, *Transverse knots, branched double covers and Heegaard Floer contact invariants.* J. Symplectic Geom. 4 (2006), no. 2, 149–170.
[27] O. Riemenschneider, *Deformationen von Quotientensingularitäten (nach zyklischen Gruppen).* Math. Ann. 209 (1974) 211-248.
[28] L. Rudolph, *Algebraic functions and closed braids.* Topology 22 (1983), no. 2, 191–202.
[29] L. Rudolph, *Quasipositivity as an obstruction to sliceness.* Bull. Amer. Math. Soc. (N.S.) 29 (1993), no. 1, 51–59.

[30] L. Rudolph, *An obstruction to sliceness via contact geometry and "classical" gauge theory.* Invent. Math. 119 (1995), no. 1, 155–163.
[31] L. Rudolph, *Quasipositive plumbing (constructions of quasipositive knots and links. V).* Proc. Amer. Math. Soc. 126 (1998), no. 1, 257-267.
[32] L. Rudolph, *Positive links are strongly quasipositive.* Proceedings of the Kirbyfest (Berkeley, CA, 1998), 555–562, Geom. Topol. Monogr., 2, Geom. Topol. Publ., Coventry, 1999.
[33] L. Rudolph, *Knot theory of complex plane curves.* Handbook of knot theory, 349-427, Elsevier B. V., Amsterdam, 2005.
[34] T. Tanaka, *Unknotting numbers of quasipositive knots.* Topology Appl. 88 (1998), no. 3, 239-246.

INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UNIVERSITÉ PAUL SABATIER, 118 ROUTE DE NARBONNE, 31062 TOULOUSE, FRANCE

STEKLOV MATHEMATICAL INSTITUTE, GUBKINA 8, MOSCOW, RUSSIA
E-mail address: stepan.orevkov@math.univ-toulouse.fr

DEPARTMENT OF MATHEMATICS, KOÇ UNIVERSITY, 34450, ISTANBUL,TURKEY
E-mail address: bozbagci@ku.edu.tr