SURGERY DIAGRAMS FOR HORIZONTAL CONTACT STRUCTURES

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ABSTRACT. We describe Legendrian surgery diagrams for some horizontal contact structures on non-positive plumbing trees of oriented circle bundles over spheres with negative Euler numbers. As an application we determine Milnor fillable contact structures on some Milnor fillable 3-manifolds.

1. INTRODUCTION

Suppose that for an oriented link B in a closed and oriented 3-manifold Y the complement $Y \setminus B$ fibers over the circle as $\pi \colon Y \setminus B \to S^1$ such that $\pi^{-1}(\theta) = \Sigma_\theta$ is the interior of a compact surface with $\partial \Sigma_\theta = B$, for all $\theta \in S^1$. Then (B, π) is called an *open book decomposition* (or just an *open book*) of Y. For each $\theta \in S^1$, the surface Σ_θ is called a *page*, while B is called the *binding* of the open book. The monodromy of the fibration π is defined as the diffeomorphism of a fixed page which is given by the first return map of a flow that is transverse to the pages and meridional near the binding. The isotopy class (relative to the boundary) of this diffeomorphism is independent of the chosen flow and we will refer to that as the *monodromy* of the open book decomposition. An open book (B, π) on a 3-manifold Y is said to be *isomorphic* to an open book (B', π') on a 3-manifold Y', if there is a diffeomorphism $f : (Y, B) \to (Y', B')$ such that $\pi' \circ f = \pi$ on $Y \setminus B$.

An open book on a plumbing of oriented circle bundles over spheres according to a tree is called *horizontal* if its binding is a collection of some fibers in the circle bundles, its pages (excluding the binding) are positively transverse to the fibers of the circle bundles and the orientation induced on the binding by the pages coincides with the orientation of the fibers induced by the fibration. Similarly, a contact structure on such a plumbing is called *horizontal* if the contact planes (which are oriented by the differential of the contact form) are positively transverse to the fibers of the bundles involved in the plumbing. We will call a plumbing tree *non-positive* if for every vertex of the tree the sum of the degree of the vertex and the Euler number of the bundle corresponding to that vertex is non-positive. Note that a non-positive plumbing is called a "plumbing with no bad vertex" in [11]. We will also assume that the Euler number of a circle bundle in a non-positive plumbing tree is less than or equal to -2.

In [2], Etgü and the author constructed horizontal planar open books on non-positive plumbing trees of oriented circle bundles over spheres. (See also Gay's paper [6]). Our construction of the open books were very explicit as we clearly described the bindings, the pages and vector fields whose flows induced the monodromy maps of the open books. We also showed that the contact structures compatible with these open books are horizontal as well. Note that for the existence of compatible contact structures we had to rely on a result of Thurston and Winkelnkemper [14].

Combining with a supplementary result of Etnyre in [4] we obtained a somewhat satisfactory description of these horizontal contact structures. Our goal in this paper is to produce yet another (perhaps more explicit) description of these contact structures, up to isomorphism, by finding Legendrian surgery diagrams for them. These diagrams can be used to evaluate some invariants of these contact structures.

In [13], Schönenberger also gives a construction of (not necessarily horizontal) planar open books compatible with some Stein fillable contact structures on non-positive plumbing trees. He starts with the usual surgery diagram of a given plumbing tree in S^3 and then modifies this diagram by some simple handle slides to put the surgery link in a certain form so that the components of the new surgery link can be viewed as embedded curves in distinct pages of some open book in S^3. By Legendrian realizing these curves on the pages and applying Legendrian surgery, Schönenberger obtains Stein fillable contact structures along with the open books compatible with these contact structures. The advantage of his method is that one can identify the contact structures by their surgery diagrams. It does not, however, seem possible to tell which one of these contact structures (for a fixed plumbing tree) is horizontal applying his method. Our strategy here is to compare the open books obtained by these two different approaches to determine the Legendrian surgery diagrams for the horizontal contact structures given in [2] on non-positive plumbing trees.

A contact 3-manifold is said to be *Milnor fillable* if it is isomorphic to the contact boundary of an isolated complex surface singularity. It is a well-known result of Grauert [9] that an oriented 3-manifold has a Milnor fillable contact structure if and only if it can be obtained by plumbing oriented circle bundles over surfaces according to a graph with negative definite intersection matrix. On the other hand, a recent discovery of Caubel, Némethi and Popescu-Pampu (cf. [1]) is that any 3-manifold admits at most one Milnor fillable contact structure, up to isomorphism. As an application of our constructions we will be able to determine this unique Milnor fillable contact structure for some Milnor fillable 3-manifolds.

2. OPEN BOOK DECOMPOSITIONS AND CONTACT STRUCTURES

An open book can also be described as follows. First consider the mapping torus

$$\Sigma_\phi = [0,1] \times \Sigma / (1, x) \sim (0, \phi(x))$$

where Σ is a compact oriented surface with r boundary components and ϕ is an element of the mapping class group Γ_Σ of Σ. Since ϕ is the identity map on $\partial\Sigma$, the boundary $\partial\Sigma_\phi$ of the mapping torus Σ_ϕ can be canonically identified with r copies of T^2. Now we glue in r copies of $D^2 \times S^1$ to cap off Σ_ϕ so that ∂D^2 is identified with $S^1 = [0,1]/(0 \sim 1)$ and the S^1 factor in $D^2 \times S^1$ is identified with a boundary component of $\partial\Sigma$. Thus we get a closed 3-manifold $Y = \Sigma_\phi \cup_r D^2 \times S^1$ equipped with an open book decomposition whose binding is the union of the core circles of $D^2 \times S^1$'s that we glue to Σ_ϕ to obtain Y. In conclusion, an element $\phi \in \Gamma_\Sigma$ determines a 3-manifold together with an "abstract" open book decomposition on it. Notice that by conjugating the monodromy ϕ of an open book on a 3-manifold Y by an element in Γ_Σ we get an isomorphic open book on a 3-manifold Y' which is diffeomorphic to Y.

Suppose that an open book decomposition with page Σ is specified by $\phi \in \Gamma_\Sigma$. Attach a (2-dimensional) 1-handle to the surface Σ connecting two points on $\partial\Sigma$ to obtain a new surface Σ'. Let γ be a closed curve in Σ' intersecting the co-core of the new 1-handle at one point. Define a new open book decomposition with $\phi' = \phi \circ t_\gamma \in \Gamma_{\Sigma'}$, where t_γ denotes the right-handed Dehn twist along γ. The resulting open book decomposition is called a *positive stabilization* of the one defined by ϕ. Notice that although the resulting monodromy depends on the chosen curve γ, the 3-manifold specified by (Σ', ϕ') is diffeomorphic to the 3-manifold specified by (Σ, ϕ).

Our interest in finding open books on 3-manifolds arises from their connection to contact structures, which we will recall here very briefly. We will assume throughout this paper that a contact structure $\xi = \ker \alpha$ is coorientable (i.e., α is a global 1-form) and positive (i.e., $\alpha \wedge d\alpha > 0$).

Definition 2.1. *An open book decomposition (B, π) of a 3-manifold Y and a contact structure ξ on Y are called* compatible *if ξ can be represented by a contact form α such that $\alpha(B) > 0$ and $d\alpha > 0$ on every page.*

Thurston and Winkelnkemper [14] showed that every open book admits a compatible contact structure and conversely Giroux [7] proved that every contact 3-manifold admits a compatible open book. We refer the reader to [4] and [12] for more on the correspondence between open books and contact structures.
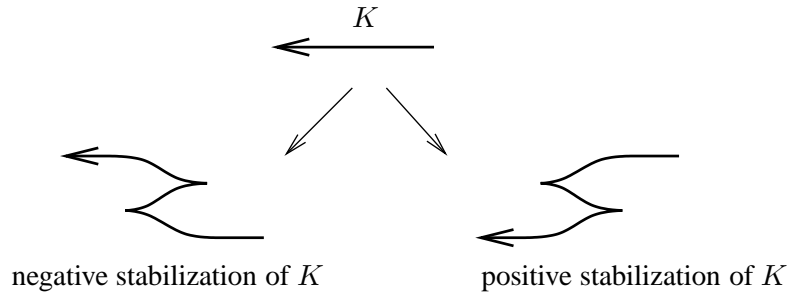

negative stabilization of K positive stabilization of K

FIGURE 1.

Let K be a Legendrian knot in the standard contact $\mathbb{R}^3$ (inside the standard contact S^3) given by its front projection. We define the positive and negative stabilization of K as follows: First we orient the knot K and then if we replace a strand of the knot by an up (down, resp.) cusp by adding a zigzag as in Figure 1 we call the resulting Legendrian knot the negative (positive, resp.) stabilization of K. Notice that stabilization is a well defined operation, i.e., it does not depend at what point the stabilization is done.

Lemma 2.2. *([3]) Let K be a Legendrian knot in a page of an open book* ob *in S^3 compatible with its standard tight contact structure. Then the stabilization of K lies in a page of another open book in S^3 obtained by stabilizing* ob.

In fact, the stabilization of K is obtained by sliding L over the 1-handle (which is attached to the page to stabilize ob). Notice that there is a positive and a negative stabilization of the oriented Legendrian knot K defined by adding a down or an up cusp, and this choice corresponds to adding a *left* (i.e., to the left-hand side of the oriented curve K) or a *right* (i.e., to the right-hand side of the oriented curve K) 1-handle to the surface respectively as shown in Figure 2.
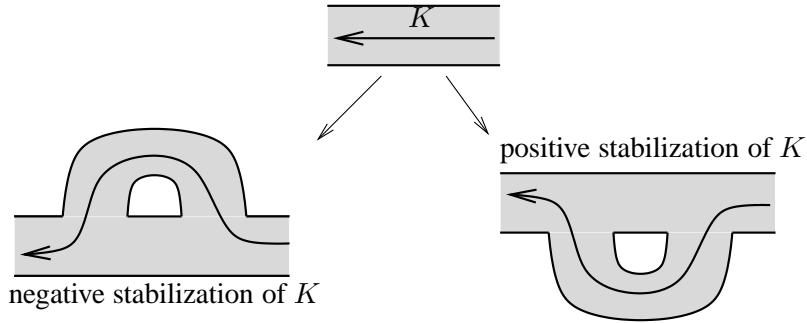
FIGURE 2. Stabilization of a page to include the stabilization of the Legendrian knot K

3. SURGERY DIAGRAMS FOR HORIZONTAL CONTACT STRUCTURES

3.1. **Circle bundles with negative Euler numbers.** Let Y_n denote the oriented circle bundle over S^2 with Euler number $e(Y_n) = n < -1$. Then the page of the horizontal open book ob_n on Y_n explicitly constructed in [2] is a planar surface with $|n|$ boundary components and the monodromy is a product of $|n|$ right-handed Dehn twists, each of which is along a curve parallel to a boundary component. Now consider the contact structure ξ_n compatible with the planar horizontal open book ob_n. In [2] it was shown that ξ_n is horizontal as well.

Lemma 3.1. *The contact structures ξ_n is given, up to isomorphism, by the Legendrian surgery diagram depicted in Figure 3.*

Proof. We will show that the open book compatible with the contact structure depicted in Figure 3 is isomorphic to ob_n.
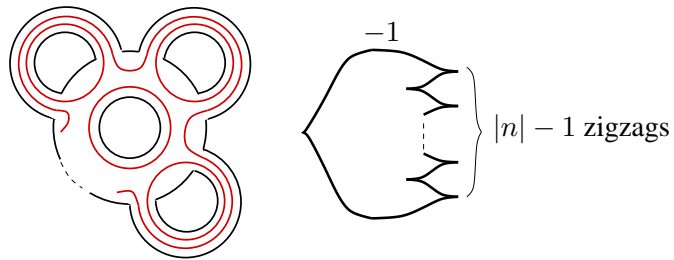

FIGURE 3. A page of a planar open book on the left compatible with the contact structure given by the Legendrian surgery diagram on the right. The monodromy is given by the product of right-handed Dehn twists along the thicker curves.

Consider the core circle C of the open book ob_H in S^3 given by the positive Hopf link H. The page of ob_H is an annulus and its monodromy is a right-handed Dehn twist along C. First we Legendrian realize C on a page of ob_H. Then we use Lemma 2.2, to stabilize C so that the stabilized knot will still be embedded on a page of another open book in S^3. Note that there are two distinct ways of stabilizing an open book corresponding to two distinct ways of stabilizing a Legendrian knot in the standard contact S^3. We apply this trick to stabilize ob_H, $(|n|-1)$-times, by successively attaching 1-handles while keeping the genus of the page to be zero. Here we choose

to stabilize ob_H always on the same side of the core circle. Then we slide C over all the attached 1-handles, Legendrian realize the resulting knot on the page of the stabilized open book and perform Legendrian surgery on this knot to get an open book compatible with the contact structure induced by the Legendrian surgery diagram in Figure 3. We observe that the resulting compatible open book is isomorphic to ob_n since its page is a sphere with $|n|$ holes and its monodromy is a product of one right-handed Dehn twist along each boundary component.

$\square$

Note that by reversing the orientations of the pages (and hence the orientation of the binding) of ob_n we get another planar open book $\overline{\mathrm{ob}}_n$ on Y_n. The open book $\overline{\mathrm{ob}}_n$ is in fact isomorphic to ob_n, since ob_n and $\overline{\mathrm{ob}}_n$ have identical pages and the same monodromy map measured with the respective orientations. To see that $\overline{\mathrm{ob}}_n$ is also horizontal we simply reverse the orientation of the fiber (to agree with the orientation of the binding) as well as the orientation of base S^2 of the circle bundle Y_n, so that we do not change the orientation of Y_n. In addition we observe that the contact structure $\overline{\xi}_n$ compatible with $\overline{\mathrm{ob}}_n$ can be obtained from ξ_n by changing the orientations of the contact planes since $\overline{\mathrm{ob}}_n$ is obtained from ob_n by changing the orientations of the pages. This immediately implies that $c_1(\overline{\xi}_n) = -c_1(\xi_n)$. We conclude that the contact structure induced by the mirror image of the Legendrian surgery diagram in Figure 3 is isomorphic to $\overline{\xi}_n$. Finally note that ξ_n is isomorphic but not homotopic to $\overline{\xi}_n$.

3.2. Non-positive plumbing trees of circle bundles.

In [2], an explicit planar horizontal open book ob_Γ on a given non-positive plumbing tree Γ of oriented circle bundles over spheres was constructed. Moreover it was shown that the compatible contact structure ξ_Γ is horizontal as well. In fact as in the previous section by changing the orientations of the pages of this horizontal open book and the orientations of the corresponding contact planes we get another horizontal contact structure $\overline{\xi}_\Gamma$. In the following, we will find Legendrian surgery diagrams for these horizontal contact structures. We will first consider the easier case of lens spaces and then give a construction for the general case.

3.2.1. Lens spaces.

The lens space $L(p,q)$ for $p > q \geq 1$ is defined by a $(-\frac{p}{q})$–surgery on an unknot in S^3. Equivalently $L(p,q)$ can be obtained by a linear plumbing Γ of oriented circle bundles over spheres with Euler numbers $n_1, n_2, \ldots, n_k$ (see Figure 6) where $[n_1, n_2, \ldots, n_k]$ is the continued fraction expansion for $(-\frac{p}{q})$ with $n_i \leq -2$, for all $i = 1, \cdots, k$.

Proposition 3.2. *The horizontal contact structure ξ_Γ is given, up to isomorphism, by the Legendrian surgery diagram depicted in Figure 4.*

Proof. We apply the recipe in [2] to construct a horizontal open book ob_Γ on $L(p,q)$, with respect to its plumbing description. First we consider the horizontal open book ob_n (see Figure 3) we constructed on a circle bundle with a negative Euler number n. Then we glue the open books ob_{n_i} corresponding to the circle bundles represented in the linear plumbing presentation of $L(p,q)$. The key point is that when we glue two boundary components with boundary parallel right-handed Dehn twists we end up with only one right-handed Dehn twist on the connecting neck, as illustrated abstractly in Figure 5. The monodromy of the open book ob_Γ in Figure 5 is given by the product of right-handed Dehn twists along the thicker curves.
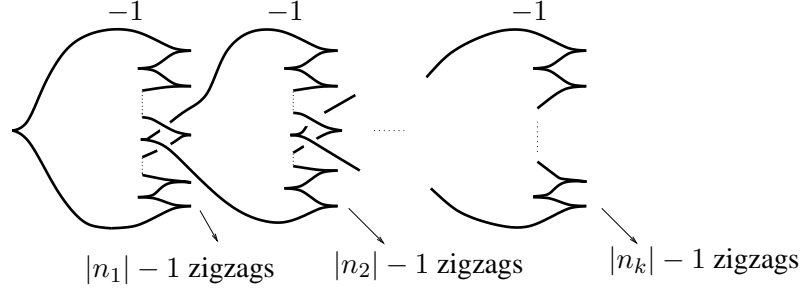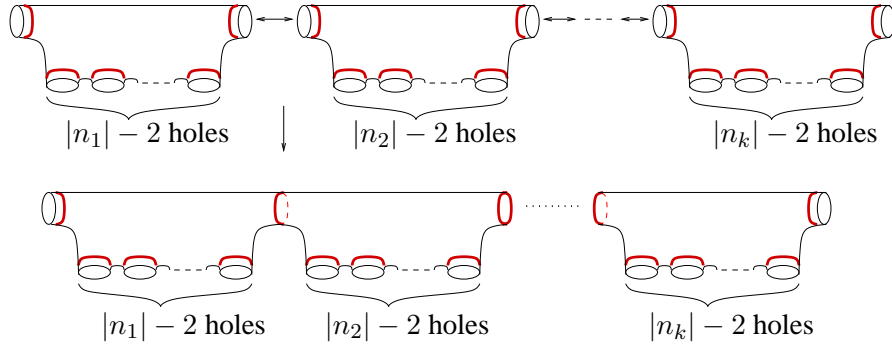
$|n_1| - 1$ zigzags $|n_2| - 1$ zigzags $|n_k| - 1$ zigzags

FIGURE 4.

$|n_1| - 2$ holes $|n_2| - 2$ holes $|n_k| - 2$ holes

$|n_1| - 2$ holes $|n_2| - 2$ holes $|n_k| - 2$ holes

FIGURE 5. We illustrate how to plumb open books on circle bundles to get the horizontal open book ob_Γ in $L(p, q)$.

Now from [13] (see also [5]) we recall how to "roll up" a linear plumbing tree Γ. Let Γ be the linear plumbing tree with vertices $Y_{n_1}, \ldots, Y_{n_k}$ where each Y_{n_i} is plumbed only to $Y_{n_{i-1}}$ and $Y_{n_{i+1}}$, for $i = 2, \ldots, k-1$, as shown on the left-hand side of Figure 6. The standard surgery diagram for Γ is a chain of unknots $U_1, \ldots, U_k$ with each U_i simply linking U_{i-1} and $U_{i+1}, i = 2, \ldots, k-1$ such that U_i has framing n_i. We think of this chain as horizontal with components labelled from left to right. Let $U_1' = U_1$. Slide U_2 over U_1 to get a new link with U_2 replaced by an unknot U_2' that now links U_1, $n_1 + 1$ times. Now slide U_3 over U_2'. Continue in this way until U_k is slid over U_{k-1}'. The new link L is called the "rolled up" surgery diagram as depicted on the right-hand side of Figure 6. Note that a 2-handle slide corresponds to a change of basis in the second homology of the 4-manifold bounded by the 3-manifold at hand. We observe a few simple features of this construction:

• For fixed i, U_j' links U_i' the same number of times for any $j > i$. If we denote this linking number by l_i, then we have $l_i = n_1 + \ldots + n_i + 2i - 1$.

• Since $m_{i+1} - m_i = n_{i+1} + 2$, we have $m_i = 2(i-1) + \Sigma_{j=1}^{i} n_j$. Note that the framings m_i on the U_i''s are non-increasing and decrease only when $n_i < -2$.

• The meridian μ_i for U_i simply links $U_i' \cup \ldots \cup U_k'$.

• L sits in an unknotted solid torus neighborhood of U_1.

Note that we can realize L as a Legendrian link in the standard contact S^3 such that U_i' is the Legendrian push off of U_{i-1}' with $|n_i + 2|$ stabilizations. Using this observation we can find an open
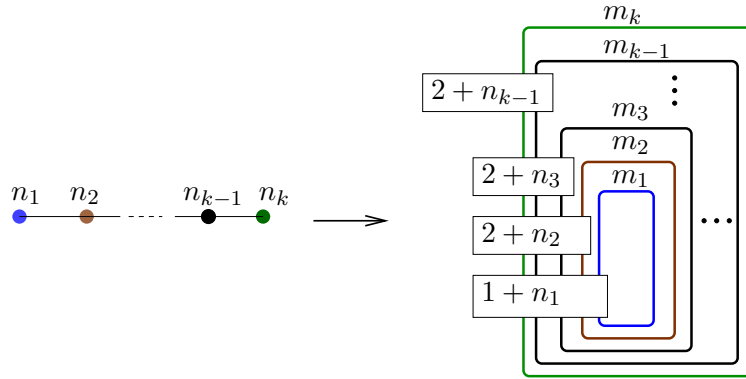
FIGURE 6. Linear diagram for $L(p, q)$ on the left, rolled up version on the right. The number inside a box indicates the number of full-twists that should be applied to the strands entering into that box.

book in S^3 such that $U'_1 \cup \ldots \cup U'_k$ are embedded in distinct pages of this open book as follows: First find an open book in S^3 as in Section 3.1 for which the innermost knot U'_1 in L is embedded on a page. Then stabilize this open book appropriately so that U'_2 is also embedded on a page. We will get the desired open book by continuing this way. Note that there are two different ways of stabilizing our open book at every step of this construction. For our purposes in this section we will always choose to stabilize in the same direction. Consequently when we Legendrian realize these knots on distinct pages of our open book in S^3, they will all be stabilized in the same direction in S^3. By performing Legendrian surgeries on them, we get a planar open book on $L(p, q)$ compatible with the resulting contact structure shown in Figure 7.
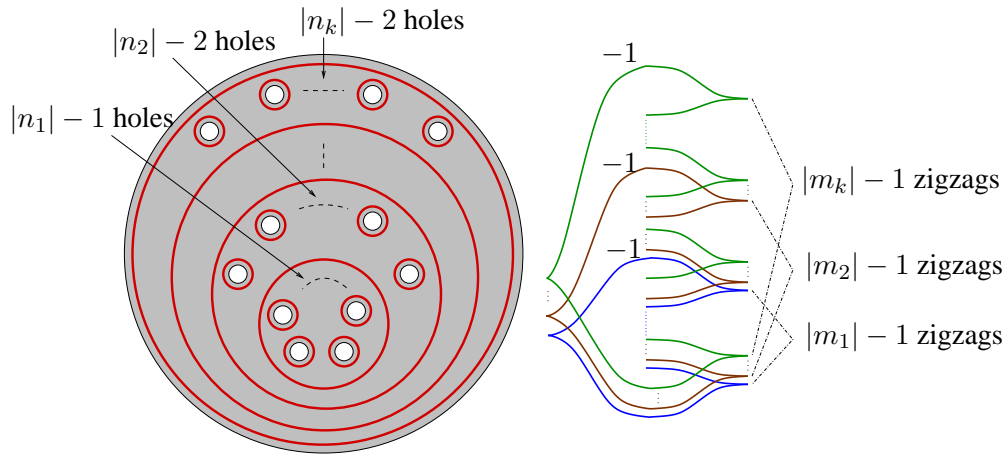

FIGURE 7. A page of a planar open book on the left compatible with the contact structure on the right. The monodromy is given by the product of right-handed Dehn twists along the thicker curves.

It is not too hard to see that the open book in Figure 7 is isomorphic to the horizontal open book ob_Γ shown in Figure 5. Hence we conclude that the contact structure in Figure 7 is isomorphic to the horizontal contact structure ξ_Γ. Next we claim that the surgery diagrams in Figure 4 and

in Figure 7 induce isomorphic contact structures. Let us denote these contact structures by ξ and ξ', respectively. To prove our claim we first note that these diagrams induce *diffeomorphic* Stein fillings (X, J) and (X', J') of the contact 3-manifolds $(\partial X, \xi)$ and $(\partial X', \xi')$, where $\partial X \cong \partial X' \cong L(p, q)$. Now consider U_i's and U_i''s as second homology classes in these Stein fillings. Then we can identify U_i' as $U_1 + \ldots + U_i$, by our construction. It is well-known (cf. [8]) that

$$< c_1(X, J), U_i >= rot(U_i) \text{ and } < c_1(X', J'), U_i' >= rot(U_i')$$

where rotation numbers $rot(U_i)$ and $rot(U_i')$ are computed using the diagrams in Figure 4 and in Figure 7, respectively, with appropriate orientations. It follows that we have

$$< c_1(X', J'), U_i >=< c_1(X, J), U_i >$$

for $1 \leq i \leq k$. Thus the classification of tight contact structures on lens spaces (cf. [10]) implies that ξ is isomorphic to ξ'.

Similarly, by putting all the zigzags on the left-hand side in Figure 4 we get a Legendrian surgery diagram which induces a contact structure isomorphic to $\overline{\xi}_\Gamma$. Note that ξ_Γ is isomorphic but not homotopic to $\overline{\xi}_\Gamma$. $\square$

3.2.2. *General case.* We would like to explain the main idea in this section before we get into the details. Given a non-positive plumbing tree of circle bundles over spheres. We first modify (cf. [13]) the usual surgery description of this plumbing tree as in the previous section to end up with a rolled up version of the original diagram such that each component of the new link is embedded in a page of some open book in S^3. To find this open book in S^3 we need to appropriately stabilize ob_H several times paying attention to the fact that there are two possible ways of stabilizing at each time. Note that this choice will determine how to stabilize a Legendrian push-off of a Legendrian knot K in S^3 by adding either a right or a left zigzag (cf. Lemma 2.2) when we slide K over the attached 1-handle. Moreover this will also determine the monodromy curves of the open book. The key observation is that the monodromy of the horizontal open book ob_Γ constructed in [2] for a non-positive plumbing tree Γ is given by a product of right-handed Dehn twists along *disjoint* curves. Thus we will apply the same strategy as in the previous section so that our choices in the stabilizations are dictated by this "disjointness" condition. Below we give the details of this discussion.

Consider a (connected) tree Γ which has one degree three vertex and all the other vertices have degree less than or equal to two. Thus we can decompose Γ into two linear trees Γ_1 and Γ_2, where the first sphere bundle of Γ_2 is plumbed into the ith sphere bundle of Γ_1 as shown in Figure 8.

Let $L_1 = U_1' \cup \ldots \cup U_k'$ be the rolled up surgery diagram of $\Gamma_1 = U_1 \cup \ldots \cup U_k$. Note that a neighborhood of the meridian μ_i for U_i is wrapped once around $U_i', U_{i+1}', \ldots, U_k'$. Now we identify V_1 with μ_i and roll up Γ_2 to obtain $L_2 = V_1' \cup \ldots \cup V_{k'}'$, where $V_1' = V_1$. Here since V_1 is identified with μ_i, V_1' is wrapped once around $U_i', U_{i+1}', \ldots, U_k'$ as shown in Figure 9. We will call the resulting surgery link $L = L_1 \cup L_2$ as the rolled up diagram of Γ.

Now we claim that we can realize $L = L_1 \cup L_2$ as a Legendrian link using the Legendrian realizations of L_1 and L_2 we discussed in the previous section. By the construction there will be a zigzag in the stabilization of U_i' (and in its subsequent Legendrian push-offs $U_{i+1}', \ldots, U_k'$ with additional zigzags) so that we may link V_1' (and hence all of L_2) into $U_i', U_{i+1}', \ldots, U_k'$ using this zigzag as shown in Figure 10.
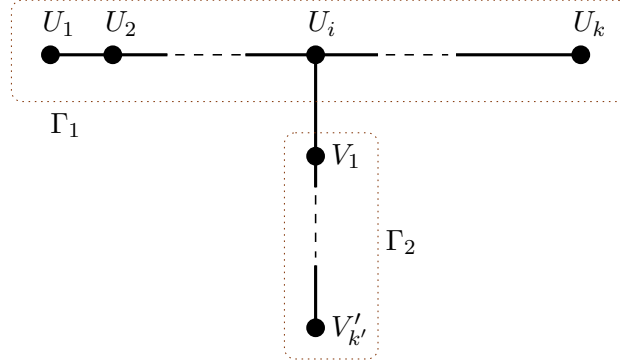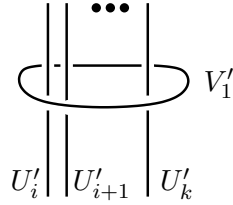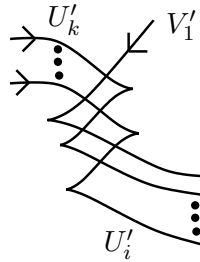
FIGURE 8.

FIGURE 9.

FIGURE 10.

Hence we can Legendrian realize L_1 and L_2 as in the previous section to get a Legendrian realization of L. On the other hand, once we have the rolled up diagram of Γ then we can find an open book in S^3 which includes the surgery curves in L on its pages as follows: Consider the open book corresponding to Γ_1 constructed as in Section 3.2.1. Let us call the annuli cut out by the large concentric circles in Figure 7 as *levels* of this open book. Observe that the ith sphere bundle of Γ_1 corresponds to the ith level in the open book corresponding to Γ_1. Since V_1 is linked once to $U_i', U_{i+1}', \ldots, U_k'$ we can take one of the annulus (a neighborhood of one of the punctures) in this level as the starting annulus while building up the open book which includes the surgery curves in L_2. We stabilize this annulus in the ith level *towards the inside direction* (as many times as necessary) so that we can embed the curves of Γ_2 (starting with V_1) into the resulting open book. Note that we need to stabilize towards the inside direction to keep the curves in L_1 disjoint from the curves in L_2. We illustrate this in Figure 11. On the left a knot K is stabilized once negatively and then the core circle C of the 1-handle we use for this stabilization is stabilized once positively. The two resulting stabilized knots are clearly disjoint. On the right a knot K is stabilized once

negatively and then the core circle C of the 1-handle we use for this stabilization is also stabilized negatively once. The two resulting stabilized knots are clearly *not* disjoint.
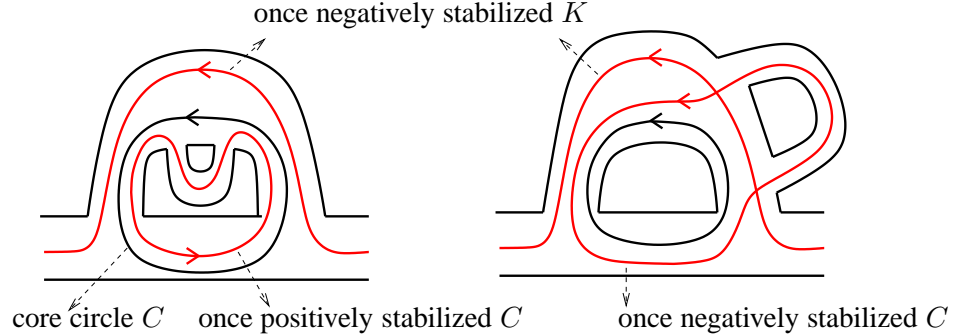

FIGURE 11.

Since we have all the surgery curves in L embedded in the pages of an open book ob in S^3 (compatible with its standard tight contact structure) now we claim that the open book obtained by performing Legendrian surgery on L is isomorphic to ob_Γ. Recall that we obtain the open book ob_Γ by gluing some pieces (see Figure 5) which can be viewed as nothing but plugging in these pieces when converted into planar diagrams. In order to prove our claim we simply observe that the result of all the necessary stabilizations of one of the punctures in the ith level of the open book of Γ_1 towards the inside direction (to embed all the curves of Γ_2) is equivalent to "plugging in" the planar open book of Γ_2 into that puncture.

Summarizing this argument what we proved so far is that the Legendrian surgery diagram for the horizontal contact structure ξ_Γ can be obtained by Legendrian realizing L_1 and L_2 such that the direction we stabilize knots in L_2 is opposite to the direction we stabilize knots in L_1. By taking orientations into account as in Figure 10, however, we realize that we need to put all the zigzags *on the same side* of all the Legendrian knots that appear in the diagram. That is because U_i' has down-cusps so that V_1' should have up-cusps. We would like to illustrate the ideas above in the following example.

Example 3.3. *Consider the non-positive plumbing tree and its rolled up diagram in Figure 12. On the right-hand side in Figure 12 we depict the Legendrian surgery diagram for ξ_Γ. In Figure 13 we depict ob_Γ.*

Remark 3.4. *We may indeed roll up the tree Γ in Example 3.3 in a different way (see Figure 14). Nevertheless, we end up with a surgery diagram whose induced contact structure is isomorphic to the one induced from the surgery diagram given in Figure 12, since the respective corresponding compatible open books are isomorphic.*

So far we analyzed our problem locally, where there is only one vertex of branching in the plumbing tree. It turns out, however, the argument can be easily generalized to arbitrary non-positive plumbing trees. By the non-positivity assumption every time there is a branching at a vertex there are enough punctures to plug in the open books corresponding to the linear branches coming out of that vertex. This argument leads to the following theorem.

Theorem 3.5. *To obtain a Legendrian surgery diagram for the horizontal contact structure ξ_Γ on a non-positive plumbing tree Γ of circle bundles over spheres we need to put the rolled up diagram*
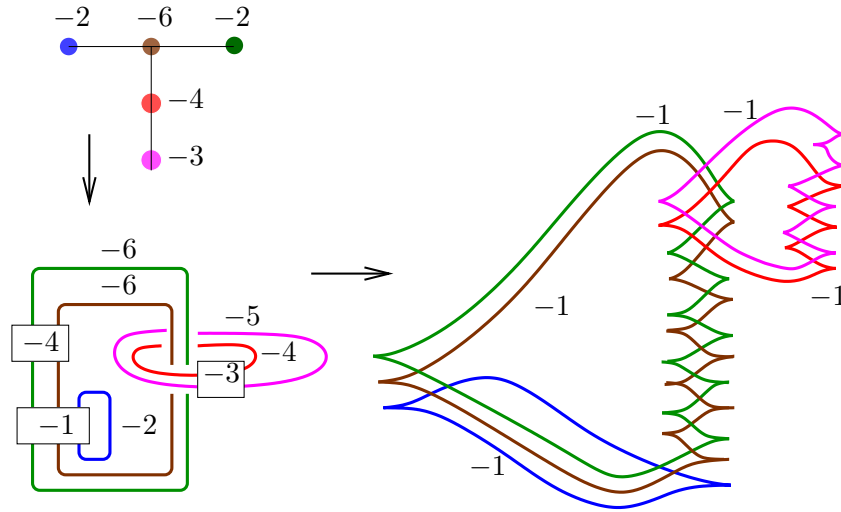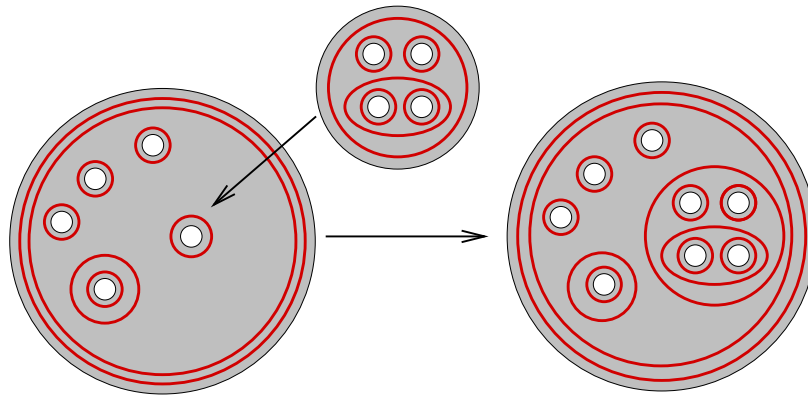
FIGURE 12.

FIGURE 13. On the left: the open book for the linear plumbing of bundles with Euler numbers $(-2, -6, -2)$; in the middle: the open book for the linear plumbing of bundles with Euler numbers $(-4, -3)$. The horizontal open book on the right-hand side is obtained by the indicated "plugging in" operation.

of Γ into Legendrian position (as described above) so that all the zigzags are on the right-hand side for all the knots in the diagram.

4. MILNOR FILLABLE CONTACT STRUCTURES

Let (X, x) be an isolated complex-analytic singularity. Given a local embedding of (X, x) in $(\mathbb{C}^N, 0)$, then a small sphere $S_\epsilon^{2N-1} \subset \mathbb{C}^N$ centered at the origin intersects X transversely, and the complex hyperplane distribution ξ on $Y = X \cap S_\epsilon^{2N-1}$ induced by the complex structure on X is a contact structure. For sufficiently small radius ϵ the contact manifold (Y, ξ) is independent of the embedding and ϵ up to isomorphism and this isomorphism type is called the *contact boundary* of (X, x). A contact manifold (Y', ξ') is said to be *Milnor fillable* and the germ (X, x) is called a *Milnor filling* of (Y', ξ') if (Y', ξ') is isomorphic to the contact boundary (Y, ξ) of (X, x).
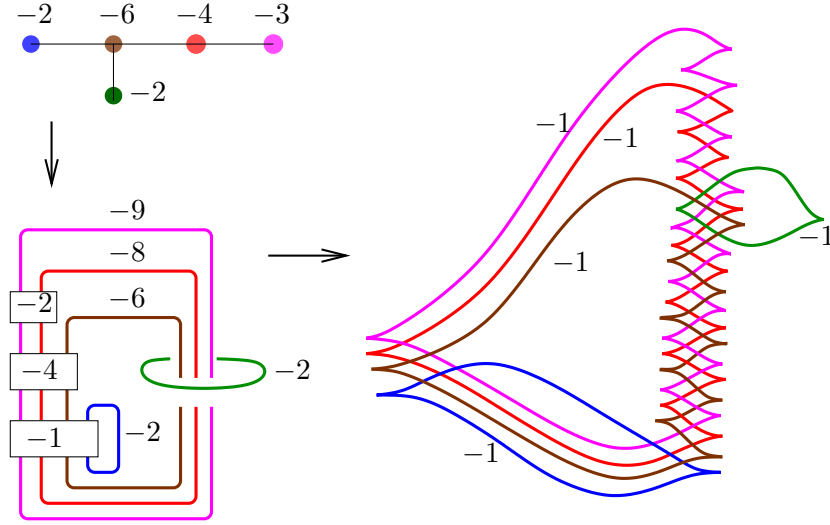
FIGURE 14.

It is a well-known result of Grauert [9] that an oriented 3–manifold has a Milnor fillable contact structure if and only if it can be obtained by plumbing oriented circle bundles over surfaces according to a weighted graph with negative definite intersection matrix. On the other hand, a recent discovery (cf. [1]) is that any 3–manifold admits at most one Milnor fillable contact structure, up to isomorphism. Note that any Milnor fillable contact structure is horizontal (cf. [2], Remark 14). The following proposition was proved in [2].

Proposition 4.1. *Let Y_Γ be a 3-manifold obtained by a plumbing of circle bundles over spheres according to a tree Γ. Suppose that the inequality*

$$e_i + 2d_i \leq 0$$

holds for every vertex of Γ, where e_i denotes the Euler number and d_i denotes the degree of the ith vertex. Then Y_Γ is Milnor fillable and the planar horizontal open book ob_Γ is compatible with the unique Milnor fillable contact structure on Y_Γ.

Corollary 4.2. *A Legendrian surgery diagram for the unique Milnor fillable contact structure on Y_Γ (where Γ satisfies the inequality in Proposition 4.1) can be explicitly determined using the algorithm presented in Theorem 3.5.*

Proof. The horizontal contact structure ξ_Γ compatible with ob_Γ is isomorphic to the unique Milnor fillable contact structure on Y_Γ by Proposition 4.1 and ξ_Γ can be explicitly determined using Theorem 3.5 since the inequality $e_i + 2d_i \leq 0$ trivially implies non-positivity for Γ.

□

Example 4.3. *The Legendrian surgery diagram in Figure 12 represents the unique Milnor fillable contact structure, up to isomorphism, on the given non-positive plumbing tree. Note that by reversing the way we stabilized the knots in Figure 12 we end up with an isomorphic but non-isotopic Milnor fillable contact structure.*

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