ON SECTIONS OF ELLIPTIC FIBRATIONS

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ABSTRACT. We find a new relation among right-handed Dehn twists in the mapping class group of a k-holed torus for $4 \leq k \leq 9$. This relation induces an elliptic Lefschetz fibration on the complex elliptic surface $E(1) \rightarrow S^2$ with twelve singular fibers and k disjoint sections. More importantly we can locate these k sections in a Kirby diagram of the induced elliptic Lefschetz fibration. The n-th power of our relation gives an explicit description for k disjoint sections of the induced elliptic Lefschetz fibration on the complex elliptic surface $E(n) \rightarrow S^2$ for $n \geq 2$.

1. INTRODUCTION

It is well-known that two generic cubics P and Q in $\mathbb{C}P^2$ intersect each other in nine points $z_1, \ldots, z_9$. By constructing the corresponding *pencil* of curves

$$\{sP + tQ \mid [s:t] \in \mathbb{C}P^1\}$$

one can define a map $f : \mathbb{C}P^2 - \{z_1, \ldots, z_9\} \rightarrow \mathbb{C}P^1$. Blowing up $\mathbb{C}P^2$ at $\{z_1, \ldots, z_9\}$ one can extend f to a Lefschetz fibration $\pi : E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2} \rightarrow \mathbb{C}P^1$ with nine distinguished sections, whose generic fiber is an elliptic curve. Our aim in this paper is to describe an analogous construction in the smooth category although unfortunately we do not know whether our construction arises from an *algebraic* pencil of curves or not. Nevertheless, many 4–manifold topologists were curious about such a differential topological construction. (For instance this was posed explicitly as a question in [4]).

Let $\Gamma^s_{g,k}$ denote the mapping class group of a compact connected orientable genus g surface with k boundary components and s marked points, so that diffeomorphisms and isotopies of the surface are assumed to fix the marked points and the points on the boundary. (We will drop k if the surface is closed and drop s if there are no fixed points.) A product $\Pi^m_{i=1} t_i$ of right-handed Dehn twists in Γ_g provides a genus-g Lefschetz fibration $X \rightarrow D^2$ over the disk with closed fibers. If $\Pi^m_{i=1} t_i = 1$ in Γ_g then the fibration closes up to a fibration over the sphere S^2. A lift of the relation $\Pi^m_{i=1} t_i = 1$ to Γ^k_g shows the existence of k disjoint sections of the induced Lefschetz fibration. The self-intersection of the j-th section is $-n_j$ if $\Pi^m_{i=1} t_i = t^{n_1}_{\delta_1} \cdots t^{n_k}_{\delta_k}$ in $\Gamma_{g,k}$, for some positive integers $n_1, \ldots, n_k$, where

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t_{δ_i}'s are right-handed Dehn twists along circles parallel to the boundary components of the surface at hand (cf. [3]).

On the other hand, an expression $\Pi_{i=1}^{m} t_i = t_{\delta_1} \cdots t_{\delta_k}$ in $\Gamma_{g,k}$ naturally describes a Lefschetz pencil: The relation determines a Lefschetz fibration with k disjoint sections, where each section has self-intersection -1, and after blowing these sections down we get a Lefschetz pencil (cf. [4]). Conversely, by blowing up the base locus of a Lefschetz pencil we arrive to a Lefschetz fibration which can be captured (together with the exceptional divisors of the blow-ups, which are all sections now) by a relation of the above type.

In this paper we find relations of the form $\Pi_{i=1}^{12} t_i = t_{\delta_1} \cdots t_{\delta_k}$ in $\Gamma_{1,k}$ for $4 \leq k \leq 9$, generalizing the well-known cases $k = 1, 2, 3$. A relation of this type naturally induces a Lefschetz pencil and by blowing up we get an elliptic Lefschetz fibration with k disjoint sections. Moreover by taking the n-th power of our relation (for $n \geq 2$) we get

$$(\Pi_{i=1}^{12} t_i)^n = t_{\delta_1}^n \cdots t_{\delta_k}^n \in \Gamma_{1,k}$$

for $4 \leq k \leq 9$. Once again this relation induces an elliptic Lefschetz fibration $E(n) \to S^2$ with $12n$ singular fibers and k disjoint sections, where the self-intersection of each section is equal to $-n$.

The reader is advised to turn to [2], [6] and [8] for background material on Lefschetz fibrations and pencils. To simplify the notation in the rest of the paper we will denote a right-handed Dehn twist along a curve along α also by α. A left-handed Dehn twist along α will be denoted by $\overline{\alpha}$. We will multiply the Dehn twists from right to left, i.e., $\beta\alpha$ means that we first apply α then β.

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2. Lantern relation for the four-holed sphere

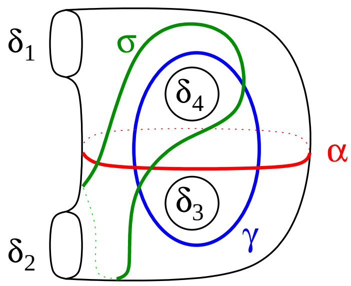

FIGURE 1. Four-holed sphere with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4\}$.

Consider the four-holed sphere depicted in Figure 1. Then there is the relation

$$\delta_1\delta_2\delta_3\delta_4 = \gamma\sigma\alpha$$

in $\Gamma_{0,4}$ which was discovered by Dehn [1]. It was rediscovered by Johnson [7] and named as the lantern relation. We will freely use this relation in any subsurface (of another surface) which is homeomorphic to a sphere with four holes. The particular depiction of the lantern relation on the four-holed sphere in Figure 1 will be convenient in the subsequent discussions. The lantern relation is classically proven by comparing the actions of both sides on a suitable system of curves whose complement is a disc.

3. Relations on torus with holes

In this section we will generalize the well known one-holed torus relation to a relation on the k-holed torus for $2 \leq k \leq 9$. We will give all the details in each case since the relation for $(k+1)$-holes is derived by using the relation for k-holes for $1 \leq k \leq 8$. The relations in the cases $k = 2, 3$ are also known, but we compute them anyway for the sake of completeness and to show our method.

We note that if two circles are disjoint, then the corresponding Dehn twists commute. Also, if two circles α and β intersect transversely at one point, then the corresponding Dehn twists satisfy the braid relation; $\alpha\beta\alpha = \beta\alpha\beta$.

3.1. **One-holed torus.** If α and β are two circles on a torus with one boundary δ_1 which intersect each other transversely at one point, then there is the relation

$$(\alpha\beta)^6 = \delta_1.$$

We call it the one-holed torus relation. It turns out that this relation was known to Dehn [1] in a slightly different form. The one-holed torus relation, just like the lantern relation, is proven by comparing the actions of both sides on a suitable system of curves whose complement is a disc.

3.2. **Two-holed torus.** Consider the two-holed torus depicted in Figure 2. By the lantern relation, we have

$$\alpha_2^2\delta_1\delta_2 = \gamma_1\sigma_1\alpha_1.$$

One-holed torus relation is

$$\gamma_1 = (\alpha_2\beta)^6.$$

Note that $\sigma_1 = \overline{\beta}\overline{\alpha}_2\overline{\alpha}_2\alpha_1\beta\overline{\alpha}_1\alpha_2\alpha_2\beta$. Then we have

$$\begin{aligned}
\delta_1\delta_2 &= \overline{\alpha}_2\,\overline{\alpha}_2\gamma_1\sigma_1\alpha_1 \\
&= \overline{\alpha}_2\,\overline{\alpha}_2(\alpha_2\beta\alpha_2\beta\alpha_2\beta\alpha_2\beta\alpha_2\beta\alpha_2\beta)(\overline{\beta}\overline{\alpha}_2\overline{\alpha}_2\alpha_1\beta\overline{\alpha}_1\alpha_2\alpha_2\beta)\alpha_1 \\
&= \overline{\alpha}_2\,\overline{\alpha}_2\alpha_2\alpha_2\beta\alpha_2\alpha_2\beta\alpha_2\beta\alpha_2\beta\overline{\alpha}_2\alpha_1\beta\overline{\alpha}_1\alpha_2\alpha_2\beta\alpha_1 \\
&= \beta\alpha_2\alpha_2\beta\alpha_2\alpha_2\beta\alpha_1\beta\overline{\alpha}_1\alpha_2\alpha_2\beta\alpha_1 \\
&= \beta\alpha_2\alpha_2\beta\alpha_2\alpha_2\alpha_1\beta\alpha_2\alpha_2\beta\alpha_1 \\
&= \beta\alpha_2\beta\alpha_2\beta\alpha_2\alpha_1\beta\alpha_2\alpha_2\beta\alpha_1 \\
&= \alpha_2\beta\alpha_2\alpha_2\beta\alpha_2\alpha_1\beta\alpha_2\alpha_2\beta\alpha_1 \\
&= \alpha_2\alpha_2\beta\alpha_2\alpha_1\beta\alpha_2\alpha_2\beta(\alpha_1\alpha_2\beta) \\
&= \alpha_2\beta\alpha_2\beta\alpha_1\beta\alpha_2\alpha_2\beta(\alpha_1\alpha_2\beta) \\
&= \alpha_2\beta\alpha_2\alpha_1\beta\alpha_1\alpha_2\alpha_2\beta(\alpha_1\alpha_2\beta) \\
&= \alpha_2\beta\alpha_1\alpha_2\beta\alpha_2(\alpha_1\alpha_2\beta)(\alpha_1\alpha_2\beta) \\
&= \alpha_2\beta\alpha_1\beta\alpha_2\beta(\alpha_1\alpha_2\beta)^2 \\
&= \alpha_2\alpha_1\beta(\alpha_1\alpha_2\beta)(\alpha_1\alpha_2\beta)^2 \\
&= (\alpha_1\alpha_2\beta)^4.
\end{aligned}$$

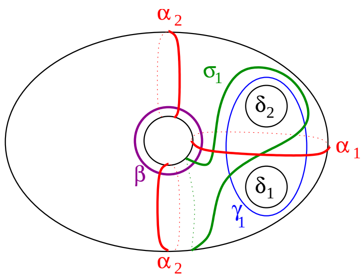

FIGURE 2. Two-holed torus with boundary $\{\delta_1, \delta_2\}$.

3.3. **Three-holed torus.** Consider the three-holed torus depicted in Figure 3. By the lantern relation,

$$\alpha_3\alpha_1\delta_2\delta_3 = \gamma_2\sigma_2\alpha_2$$

and by the two-holed torus relation,

$$\delta_1\gamma_2 = (\alpha_1\alpha_3\beta)^4.$$

Note that $\sigma_2 = \overline{\beta}\overline{\alpha}_1\overline{\alpha}_3\alpha_2\beta\overline{\alpha}_2\alpha_3\alpha_1\beta$. Then

$$\begin{aligned}
\delta_1\delta_2\delta_3 &= \overline{\alpha}_1\,\overline{\alpha}_3\delta_1\gamma_2\sigma_2\alpha_2\\
&= \overline{\alpha}_1\,\overline{\alpha}_3(\alpha_1\alpha_3\beta\alpha_1\alpha_3\beta\alpha_1\alpha_3\beta\alpha_1\alpha_3\beta)(\overline{\beta}\overline{\alpha}_1\overline{\alpha}_3\alpha_2\beta\overline{\alpha}_2\alpha_3\alpha_1\beta)\alpha_2\\
&= \beta\alpha_1\alpha_3\beta\alpha_1\alpha_3\beta\alpha_2\beta\overline{\alpha}_2\alpha_3\alpha_1\beta\alpha_2\\
&= \beta\alpha_1\alpha_3\beta\alpha_1\alpha_3\alpha_2\beta\alpha_3\alpha_1\beta\alpha_2.
\end{aligned}$$

Using the appropriate braid and commutation relations (as we did in the derivation of the two-holed torus relation) it follows that

$$\delta_1\delta_2\delta_3 = (\alpha_1\alpha_2\alpha_3\beta)^3.$$

This relation was called the star relation in [5].

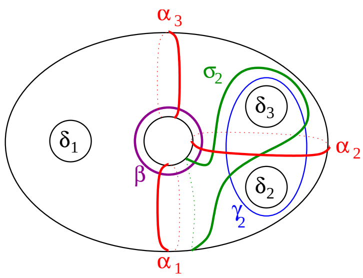

FIGURE 3. Three-holed torus with boundary $\{\delta_1, \delta_2, \delta_3\}$.

3.4. **Four-holed torus.** The lantern relation for the sphere with boundary $\{\alpha_4, \alpha_2, \delta_3, \delta_4\}$ in Figure 4 is

$$\alpha_4\alpha_2\delta_3\delta_4 = \gamma_3\sigma_3\alpha_3.$$

The relation on the three-holed torus with boundary $\{\delta_1, \delta_2, \gamma_3\}$ given in Section 3.3 is

$$\delta_1\delta_2\gamma_3 = (\alpha_1\alpha_2\alpha_4\beta)^3.$$

Here we identify the curves $(\alpha_1, \alpha_2, \alpha_3)$ in Figure 3 with the curves $(\alpha_1, \alpha_2, \alpha_4)$ in Figure 4. Combining we get

$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4 &= \delta_1\delta_2\,\overline{\alpha}_2\,\overline{\alpha}_4\gamma_3\sigma_3\alpha_3\\
&= \overline{\alpha}_2\,\overline{\alpha}_4\delta_1\delta_2\gamma_3\sigma_3\alpha_3\\
&= \overline{\alpha}_2\,\overline{\alpha}_4(\alpha_1\alpha_2\alpha_4\beta)^3\sigma_3\alpha_3\\
&= \alpha_1\beta(\alpha_1\alpha_2\alpha_4\beta)^2\sigma_3\alpha_3\\
&= (\alpha_1\alpha_2\alpha_4\beta)^2\sigma_3\alpha_3\alpha_1\beta.
\end{aligned}$$

Remark. Although we will not need it in the rest of the paper, by plugging in

$$\sigma_3 = \overline{\beta}\overline{\alpha}_4\overline{\alpha}_2\alpha_3\beta\overline{\alpha}_3\alpha_2\alpha_4\beta,$$

it is easy to see that this relation may also be written in a more symmetric form as

$$\delta_1\delta_2\delta_3\delta_4 = (\alpha_1\alpha_3\beta\alpha_2\alpha_4\beta)^2.$$

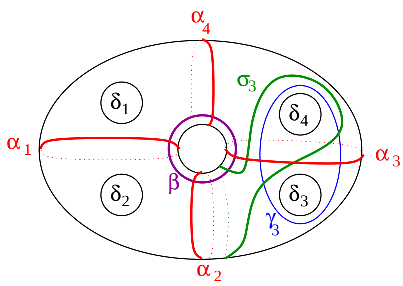

FIGURE 4. Four-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4\}$.

3.5. **Five-holed torus.** The lantern relation for the sphere with boundary $\{\alpha_5, \alpha_3, \delta_4, \delta_5\}$ in Figure 5 is

$$\alpha_5\alpha_3\delta_4\delta_5 = \gamma_4\sigma_4\alpha_4.$$

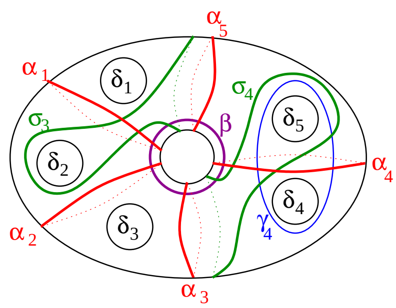

FIGURE 5. Five-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$.

The relation on the four-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \gamma_4\}$ given in Section 3.4 is

$$\delta_1\delta_2\delta_3\gamma_4 = (\alpha_3\alpha_5\alpha_2\beta)^2\sigma_3\alpha_1\alpha_3\beta.$$

Here we identify the curves $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in Figure 4 with the curves $(\alpha_3, \alpha_5, \alpha_1, \alpha_2)$ in Figure 5. Combining we get

$$\begin{aligned}\delta_1\delta_2\delta_3\delta_4\delta_5 &= \overline{\alpha}_3\,\overline{\alpha}_5\delta_1\delta_2\delta_3\gamma_4\sigma_4\alpha_4\\ &= \overline{\alpha}_3\,\overline{\alpha}_5(\alpha_3\alpha_5\alpha_2\beta)^2\sigma_3\alpha_1\alpha_3\beta\sigma_4\alpha_4\\ &= \alpha_2\beta\alpha_3\alpha_5\alpha_2\beta\sigma_3\alpha_1\alpha_3\beta\sigma_4\alpha_4\\ &= \alpha_2\alpha_3\alpha_5\beta\sigma_3\alpha_1\alpha_3\beta\sigma_4\alpha_4\alpha_2\beta.\end{aligned}$$

3.6. **Six-holed torus.** The lantern relation for the sphere with boundary $\{\alpha_6, \alpha_4, \delta_5, \delta_6\}$ in Figure 6 is

$$\alpha_6\alpha_4\delta_5\delta_6 = \gamma_5\sigma_5\alpha_5.$$

The relation for the five-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4, \gamma_5\}$ given in Section 3.5 is

$$\delta_1\delta_2\delta_3\delta_4\gamma_5 = \alpha_4\alpha_6\alpha_2\beta\sigma_3\alpha_3\alpha_6\beta\sigma_4\alpha_1\alpha_4\beta.$$

We identify the curves $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ in Figure 5 with the curves $(\alpha_3, \alpha_4, \alpha_6, \alpha_1, \alpha_2)$ in Figure 6. Combining we get

$$\begin{aligned}\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 &= \overline{\alpha}_4\,\overline{\alpha}_6\delta_1\delta_2\delta_3\delta_4\gamma_5\sigma_5\alpha_5\\ &= \overline{\alpha}_4\,\overline{\alpha}_6\alpha_4\alpha_6\alpha_2\beta\sigma_3\alpha_3\alpha_6\beta\sigma_4\alpha_1\alpha_4\beta\sigma_5\alpha_5\\ &= \alpha_2\beta\sigma_3\alpha_3\alpha_6\beta\sigma_4\alpha_1\alpha_4\beta\sigma_5\alpha_5\\ &= \beta_2\alpha_2\sigma_3\alpha_3\alpha_6\beta\sigma_4\alpha_1\alpha_4\beta\sigma_5\alpha_5\\ &= \alpha_2\alpha_3\alpha_6\beta\sigma_4\alpha_1\alpha_4\beta\sigma_5\alpha_5\beta_2\sigma_3,\end{aligned}$$

where $\beta_2 = \alpha_2\beta\overline{\alpha}_2$.

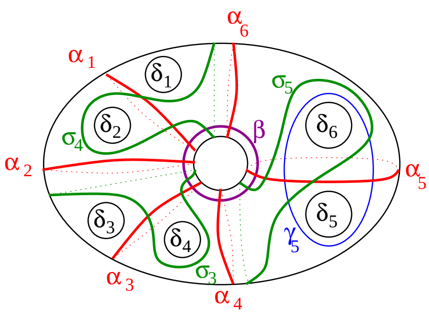

FIGURE 6. Six-holed torus with boundary $\{\delta_1, \delta_2, \ldots, \delta_6\}$.

3.7. **Seven-holed torus.** The lantern relation for the sphere with boundary $\{\alpha_7, \alpha_5, \delta_6, \delta_7\}$ in Figure 7 is

$$\alpha_7\alpha_5\delta_6\delta_7 = \gamma_6\sigma_6\alpha_6.$$

The relation on the six-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \gamma_6\}$ given in Section 3.6 is

$$\delta_1\delta_2\delta_3\delta_4\delta_5\gamma_6 = \alpha_5\alpha_7\alpha_3\beta\sigma_4\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3,$$

where we use the identification $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \to (\alpha_4, \alpha_5, \alpha_7, \alpha_1, \alpha_2, \alpha_3)$ to go from Figure 6 to Figure 7. Combining we get

$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 &= \overline{\alpha}_5\,\overline{\alpha}_7\delta_1\delta_2\delta_3\delta_4\delta_5\gamma_6\sigma_6\alpha_6 \\
&= \overline{\alpha}_5\,\overline{\alpha}_7\alpha_5\alpha_7\alpha_3\beta\sigma_4\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3\sigma_6\alpha_6 \\
&= \alpha_3\beta\sigma_4\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3\sigma_6\alpha_6 \\
&= \beta_3\alpha_3\sigma_4\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3\sigma_6\alpha_6 \\
&= \beta_3\sigma_4\alpha_3\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3\sigma_6\alpha_6 \\
&= \alpha_3\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3\sigma_6\alpha_6\beta_3\sigma_4,
\end{aligned}$$

where $\beta_3 = \alpha_3\beta\overline{\alpha}_3$ and $\beta_5 = \alpha_5\beta\overline{\alpha}_5$.

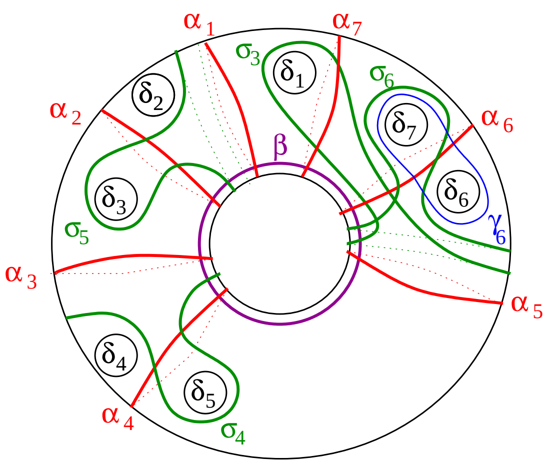

FIGURE 7. Seven-holed torus with boundary $\{\delta_1, \delta_2, \dots, \delta_7\}$.

3.8. **Eight-holed torus.** The lantern relation for the sphere with boundary $\{\alpha_8, \alpha_6, \delta_7, \delta_8\}$ in Figure 8 is

$$\alpha_8\alpha_6\delta_7\delta_8 = \gamma_7\sigma_7\alpha_7.$$

The relation on the seven-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \gamma_7\}$ given in Section 3.7 is

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\gamma_7 = \alpha_6\alpha_8\alpha_4\beta\sigma_5\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4,$$

where we use the identification $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \to (\alpha_4, \alpha_5, \alpha_6, \alpha_8, \alpha_1, \alpha_2, \alpha_3)$ to go from Figure 7 to Figure 8. Combining we get

$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 &= \overline{\alpha}_6\,\overline{\alpha}_8\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\gamma_7\sigma_7\alpha_7 \\
&= \overline{\alpha}_6\,\overline{\alpha}_8\alpha_6\alpha_8\alpha_4\beta\sigma_5\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7 \\
&= \alpha_4\beta\sigma_5\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7 \\
&= \beta_4\alpha_4\sigma_5\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7 \\
&= \beta_4\sigma_5\alpha_4\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7 \\
&= \alpha_4\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7\beta_4\sigma_5,
\end{aligned}$$

where $\beta_1 = \alpha_1\beta\overline{\alpha}_1$, $\beta_4 = \alpha_4\beta\overline{\alpha}_4$ and $\beta_6 = \alpha_6\beta\overline{\alpha}_6$.

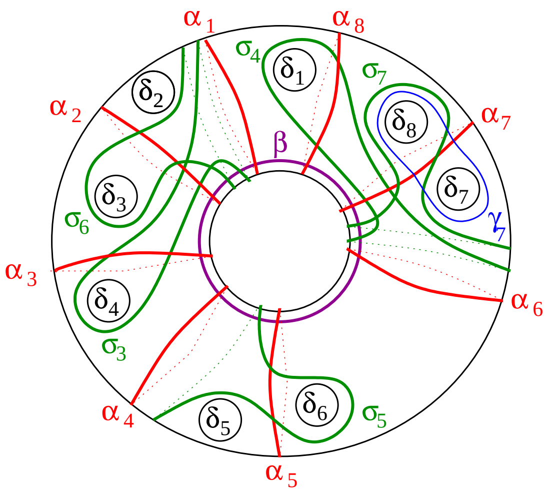

FIGURE 8. Eight-holed torus with boundary $\{\delta_1, \delta_2, \ldots, \delta_8\}$.

3.9. **Nine-holed torus.** The lantern relation for the sphere with boundary $\{\alpha_9, \alpha_7, \delta_8, \delta_9\}$ in Figure 9 is

$$\alpha_9\alpha_7\delta_8\delta_9 = \gamma_8\sigma_8\alpha_8.$$

The relation on the eight-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\gamma_8\}$ given in Section 3.8 is

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\gamma_8 = \alpha_7\alpha_9\beta_4\sigma_3\sigma_6\alpha_5\beta_1\sigma_4\sigma_7\alpha_2\beta_7\sigma_5,$$

where we identify $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$ with $(\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_9, \alpha_1, \alpha_2, \alpha_3)$ to go from Figure 8 to Figure 9. Combining we get

$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9 &= \overline{\alpha}_7\,\overline{\alpha}_9\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\gamma_8\sigma_8\alpha_8 \\
&= \overline{\alpha}_7\,\overline{\alpha}_9\alpha_7\alpha_9\beta_4\sigma_3\sigma_6\alpha_5\beta_1\sigma_4\sigma_7\alpha_2\beta_7\sigma_5\sigma_8\alpha_8 \\
&= \beta_4\sigma_3\sigma_6\alpha_5\beta_1\sigma_4\sigma_7\alpha_2\beta_7\sigma_5\sigma_8\alpha_8,
\end{aligned}$$

where $\beta_1 = \alpha_1\beta\overline{\alpha}_1$, $\beta_4 = \alpha_4\beta\overline{\alpha}_4$ and $\beta_7 = \alpha_7\beta\overline{\alpha}_7$.

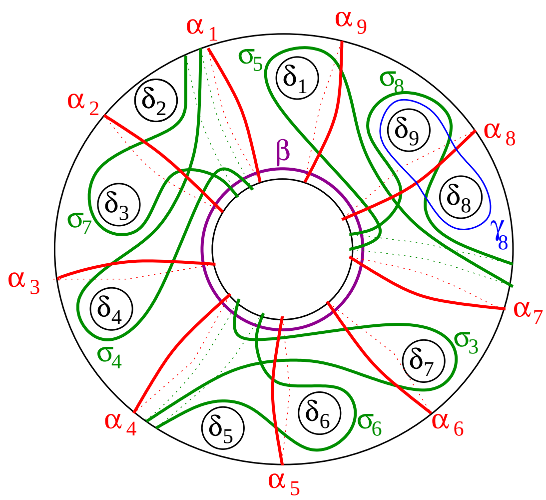

FIGURE 9. Nine-holed torus with boundary $\{\delta_1, \delta_2, \dots, \delta_9\}$.

Remark. The curious reader might wonder why we stopped at $k = 9$. First of all our process will not allow us to go any further because we will not have the cancelling right-handed Dehn twists to kill off the left-handed Dehn twists which appear in the appropriate lantern relations. In fact, there is a good reason for that: An elliptic fibration $E(1) \to S^2$ admits at most nine disjoint sections (all with negative self-intersections). So we conclude that there is no such relation for k-holed torus with $k \geq 10$.

4. SECTIONS OF THE ELLIPTIC FIBRATIONS

First we consider the case $k = 4$. The relation

$$\delta_1\delta_2\delta_3\delta_4 = (\alpha_1\alpha_2\alpha_4\beta)^2\sigma_3\alpha_3\alpha_1\beta$$

in $\Gamma_{1,4}$ we derived in Section 3.4 induces the word $(\alpha^3\beta)^3 = 1$ in Γ_1 which gives us an elliptic Lefschetz fibration on the elliptic surface $E(1) = \mathbb{C}P^2 \# 9\, \overline{\mathbb{C}P^2}$. To draw a Kirby diagram (cf. [6]) of this elliptic fibration we start with a 0-handle, attach two 1-handles (see Figure 10) and attach a 2-handle which yields $D^2 \times T^2$. A torus fiber of the trivial fibration $D^2 \times T^2 \to D^2$ can be viewed in Figure 10 as follows: Take the obvious disk on the page, attach two 2-dimensional 1-handles (going through two 4-dimensional 1-handles) and cap off by a 2-dimensional disk. Then we draw the curves which appear in the monodromy of the elliptic fibration on parallel copies of this fiber. Notice that these curves are the attaching curves of some 2-handles. Once we attach all twelve of these 2-handles with framing one less than the page framing we get an elliptic Lefschetz fibration over D^2 with twelve singular fibers, which then can be closed off to an elliptic Lefschetz fibration over S^2. We depicted the four disjoint sections s_1, s_2, s_3, s_4 of the induced fibration in Figure 10. (Imagine replacing s_i's in Figure 10 by holes where they intersect the page and embed the curves in Figure 4 into distinct fibers.) For each $i = 1, 2, 3, 4$, the curve s_i bounds two disks—one in the neighborhood of a regular fiber, one outside of that neighborhood—which in turn gives a section of the elliptic Lefschetz fibration when we glue these two disks along their common boundary s_i.

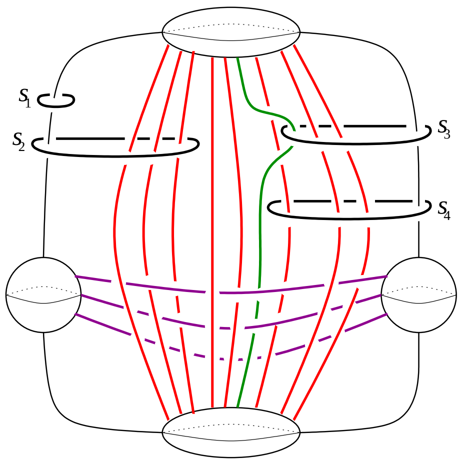

FIGURE 10. An elliptic Lefschetz fibration $E(1) \to S^2$ with four disjoint sections.

Similarly we can draw the Kirby diagrams corresponding to the relations we derived for $k = 5, 6, \dots, 9$, and explicitly indicate the locations of the k disjoint sections of $E(1) \to S^2$ in these diagrams. We skip the cases $k = 5, \dots, 8$ and jump to the case $k = 9$. The relation

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9 = \sigma_4\sigma_7\alpha_2\beta_7\sigma_5\sigma_8\alpha_8\beta_4\sigma_3\sigma_6\alpha_5\beta_1$$

on the nine-holed torus induces the word $(\alpha^3\beta_\alpha)^3 = 1$ in the mapping class group Γ_1 which gives us an elliptic Lefschetz fibration on the elliptic surface $E(1) = \mathbb{C}P^2 \# 9\, \overline{\mathbb{C}P^2}$, where

$\beta_\alpha = \alpha\beta\overline{\alpha}$ (which is indeed a right-handed Dehn twist). Note that we cyclicly permuted the curves in the equation we derived in Section 3.9 to obtain the relation above.

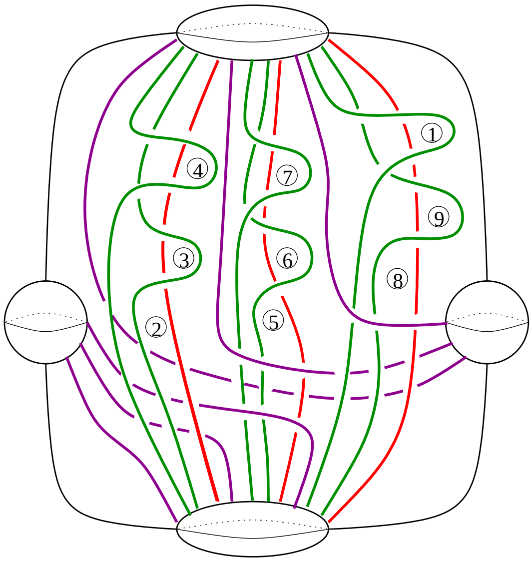

FIGURE 11. An elliptic Lefschetz fibration $E(1) \to S^2$ with nine disjoint sections. Note that we just indicated the intersection points of sections with the regular fiber by the encircled numbers corresponding to the boundary components $\delta_1, \ldots, \delta_9$ of the nine-holed torus.

Finally, for $4 \leq k \leq 9$, by taking the n-th power of our relation for the k-holed torus we can find k disjoint sections of the corresponding elliptic fibration on the elliptic surface $E(n)$ for any $n \geq 1$.

5. FINAL COMMENTS

Suppose that the product $\delta_1\delta_2 \ldots \delta_k$, where δ_i denotes a right-handed Dehn twist along a curve parallel to the ith boundary component of a surface with k boundary components, can be expressed as a product of right-handed Dehn twists along interior (i.e., non-boundary parallel) curves on the surface. We will call such a relation in the corresponding mapping class group as a *boundary-interior relation*. The technique we applied to derive a boundary-interior relation in $\Gamma_{1,k}$ (for $2 \leq k \leq 9$) can be easily generalized to derive a boundary-interior relation in $\Gamma_{g,k}$ for $g \geq 2$: As we elaborated in this paper, one can start with a certain boundary-interior relation in $\Gamma_{g,1}$ and derive a boundary-interior relation in $\Gamma_{g,2}$ and then a boundary-interior relation in $\Gamma_{g,3}$ and so on and so forth. In fact, applying our trick, it might be possible to derive a boundary-interior relation in $\Gamma_{g,k+1}$ if we are given a boundary-interior relation in $\Gamma_{g,k}$. Consequently our method can be applied to construct additional sections of a given Lefschetz fibration in certain situations.

It is intriguing to note that once we fix a boundary-interior relation in $\Gamma_{g,1}$ for some $g \geq 2$ then there is a maximum "k" (for simple homological reasons) for which we get a boundary-interior relation in $\Gamma_{g,k}$ applying our method. It appears that this number depends not only on g but also on the initial boundary-interior relation in $\Gamma_{g,1}$.

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