CONTACT OPEN BOOKS WITH EXOTIC PAGES

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Abstract. We consider a fixed contact 3-manifold that admits infinitely many compact Stein fillings which are all homeomorphic but pairwise non-diffeomorphic. Each of these fillings gives rise to a closed contact 5-manifold described as a contact open book whose page is the filling at hand and whose monodromy is the identity symplectomorphism. We show that the resulting infinitely many contact 5-manifolds are all diffeomorphic but pairwise non-contactomorphic. Moreover, we explicitly determine these contact 5-manifolds.

1. Introduction

Recent advances in symplectic geometry and topology showed that while some closed contact 3-manifolds have only finitely many Stein fillings, others have infinitely many, up to diffeomorphism (see [11] for a recent survey). Among the 4-manifold topologists, it is common to call a Stein filling of a contact 3-manifold exotic compared to another filling, if these two fillings are homeomorphic but non-diffeomorphic.

The first examples of a closed contact 3-manifold admitting infinitely many exotic *simply-connected* Stein fillings were discovered in [3]. The Stein fillings in that article, and many others which appeared in the literature since then, were given as Lefschetz fibrations over the disk whose boundary is a fixed open book supporting the contact manifold in question.

In [1], Akbulut and Yasui gave an example of a closed 3-manifold ∂X, described as the boundary of a 4-manifold X, carrying two contact structures η and η' such that $(\partial X, \eta)$ (resp. $(\partial X, \eta')$) admits an infinite family $\{X_p \mid p \text{ odd}\}$ (resp. $\{X_p \mid p \text{ even}\}$) of pairwise exotic simply-connected Stein fillings with $b_2 = 2$. These fillings, however, are described by explicit handlebody diagrams, rather than Lefschetz fibrations.

Let $M_p := OB(X_p, id)$ denote the closed 5-manifold viewed as the open book with page X_p and monodromy the identity map. Since X_p admits a Stein structure for each p, and the identity map is a symplectomorphism, there is a contact structure ξ_p on M_p supported by this open book in the sense of Giroux [9]. Let $S^2 \tilde{\times} S^3$ denote the non-trivial S^3-bundle over S^2. Here we prove the following result.

Theorem 3.6. *If p is odd, then M_p is diffeomorphic to $S^2 \times S^3 \# S^2 \times S^3$. If p is even, then M_p is diffeomorphic to $S^2 \times S^3 \# S^2 \tilde{\times} S^3$. Moreover, (M_p, ξ_p) is contactomorphic to $(M_{p'}, \xi_{p'})$ if and only if $p = p'$.*

We give two alternative arguments to distinguish the contact 5-manifolds: one is based on the Barden's classification [4] of simply-connected closed 5-manifolds, and the other is based on the diagrammatic language developed for contact 5-manifolds in [7]. The advantage of the latter approach is that we can explicitly identify the contact 5-manifolds.

Shortly after we posted our paper in the arXiv, Akbulut and Yasui [2] showed the existence of a contact 5-manifold supported by infinitely many distinct open books with pairwise exotic Stein pages and the identity monodromy—which nicely complements our result. They also generalized the statement in our Theorem 3.6 to some contact structures on $\#_n S^2 \times S^3$ and $\#_n S^2 \tilde{\times} S^3$, for any $n \geq 2$.

2. Simply-connected exotic Stein fillings

We briefly review the infinite family of exotic simply-connected Stein fillings of a fixed contact 3-manifold due to Akbulut and Yasui [1]. Let X be the 4-manifold with boundary described by the handlebody diagram on the left in Figure 1. Note that there is an embedded $T^2 \times D^2$ in the interior of X. Throughout this section, let p denote a positive integer and let X_p be the result of p-log transform on the $T^2 \times D^2 \subset X$. A handlebody diagram of X_p is presented on the right in Figure 1.

First of all, one observes that X and X_p are simply-connected, for all p. To see this, cancel the upper 1-/2-handle pair and the lower 1-/2-handle pair to get a diagram consisting of only two 2-handles and no 1-handles for both X and X_p. This already implies that $b_2(X) = b_2(X_p) = 2$, and allows one to easily compute the intersection forms of X and X_p, which turns out to be unimodular and indefinite, for all p. Such forms are classified, up to isomorphism, by their rank, signature and parity. The signature of all the forms are zero, the form of X is even, and the form of X_p is even if and only if p is odd. Thus, by Boyer's generalization [5] of Freedman's celebrated theorem [8], one concludes that X_p is homeomorphic to $X_{p'}$ if and only if p and p' have the same parity and X_p is homeomorphic to X if and only if p is odd. Moreover, for all p, $\partial X = \partial X_p$ is a homology 3-sphere.

Next, one shows that X and X_p admit Stein structures, for all p, by turning the smooth handlebody diagrams in Figure 1 into Legendrian handlebody diagrams (see Figure 2), after cancelling the upper 1-/2-handle pairs for convenience.

In particular, $\partial X = \partial X_p$ admits a Stein fillable contact structure. Finally, one uses the adjunction inequality coupled with the genus function to distinguish the smooth structures on X_p's:

Proposition 2.1 (Akbulut and Yasui). *(i) There exists a contact structure η on ∂X such that for infinitely many values of $q \geq 1$, X_{2q-1} is a Stein filling of $(\partial X, \eta)$ with the property that all of these fillings are homeomorphic but pairwise non-diffeomorphic.*

(ii) There exists a contact structure η' on ∂X such that for infinitely many values of $q \geq 1$, X_{2q} is a Stein filling of $(\partial X, \eta')$ with the property that all of these fillings are homeomorphic but pairwise non-diffeomorphic.

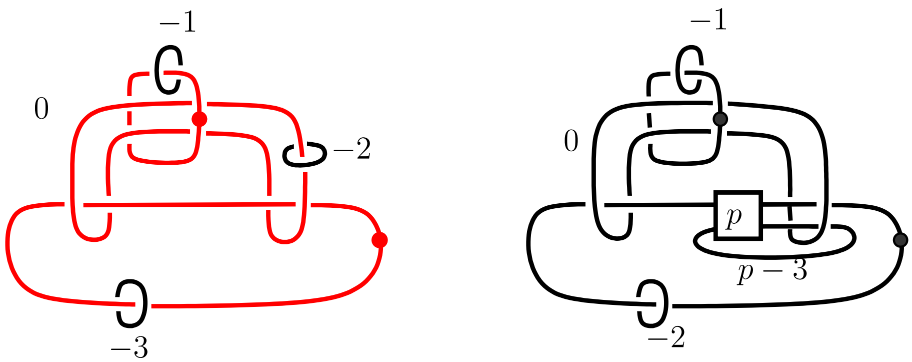

FIGURE 1. On the left: the handlebody diagram of X, where $T^2 \times D^2 \subset X$ is indicated with a different color. On the right: the handlebody diagram of X_p.

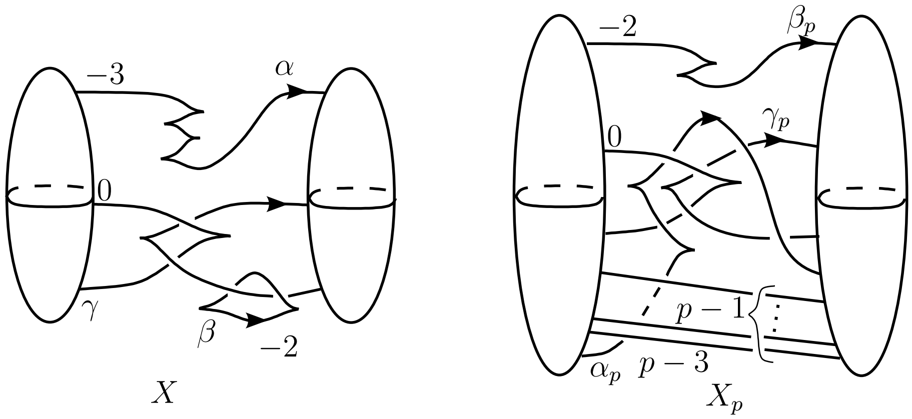

FIGURE 2. Stein handlebody diagrams for X and X_p

3. Contact open books with exotic pages

Let $M_p := OB(X_p, id)$ (resp. $M := OB(X, id)$) denote the closed contact 5-manifold viewed as the contact open book with page X_p (resp. X) and monodromy the identity map. Let ξ_p (resp. ξ) denote the contact structure supported by this open book in the sense of Giroux [9].

We start with some basic observations about the contact 5-manifolds (M, ξ) and (M_p, ξ_p). First of all, (M, ξ) and (M_p, ξ_p) are subcritically Stein fillable, and in fact $W := X \times D^2$ and $W_p := X_p \times D^2$ are their subcritical Stein fillings, respectively [7, Prop. 3.1].

Since X and X_p are simply-connected for all p, so are W and W_p. Moreover, M and M_p are simply-connected for all p, as a result of the following simple but useful lemma which generalizes Proposition 1.10 in [12]. Note that a compact Stein filling (a.k.a. a Stein domain) is a Weinstein domain (cf. [6]).

Lemma 3.1. *If V^{2n} is a Weinstein domain, then the inclusion map $i : \partial V \to V$ induces an isomorphism on π_k for $k < n-1$ and a surjection on π_{n-1}.*

Proof. As V is a Weinstein domain, there is an almost complex structure J and a J-convex Morse function f on V. By [6, Corollary 3.4], any critical point of f has index at most n, so the Morse function $-f$ has only critical points of index at least n. Now we reconstruct V from a collar neighborhood of ∂V by attaching the handles corresponding to the critical points of $-f$. Therefore, we see that $i : \partial V \to V$ induces an isomorphism on π_k for $k < n-1$ and a surjection on π_{n-1}. □

Remark 3.2. The inclusion of the boundary does not always induce an isomorphism on π_{n-1} as, for instance, $T^2 \times D^2$ is a Stein (hence Weinstein) filling of the standard contact T^3.

Lemma 3.3. *M_p is diffeomorphic to either $S^2 \times S^3 \# S^2 \times S^3$ or $S^2 \times S^3 \# S^2 \tilde{\times} S^3$.*

Proof. For the subcritical Stein filling W_p, the homology sequence

$$\underbrace{H_3(W_p, \partial W_p)}_{\cong H^3(W_p)=0} \longrightarrow H_2(\partial W_p) \xrightarrow{i_*} H_2(W_p) \longrightarrow \underbrace{H_2(W_p, \partial W_p)}_{\cong H^4(W_p)=0},$$

of the pair $(W_p, \partial W_p)$ implies that i_* is an isomorphism, since the homology of subcritical Weinstein manifolds vanishes in degree at least half the dimension. As a consequence we have

$$H_2(M_p) \cong H_2(W_p) \cong H_2(X_p) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The statement in the lemma follows from Barden's classification [4] of diffeomorphism classes of simply-connected closed 5-manifolds. Note that $S^2 \tilde{\times} S^3 \# S^2 \tilde{\times} S^3$ is diffeomorphic to $S^2 \times S^3 \# S^2 \tilde{\times} S^3$. □

Next, in order to identify the contact 5-manifold (M_p, ξ_p), for all p, we would like to determine the first Chern class $c_1(\xi_p)$ using the following general results. Let c denote the total Chern class.

Lemma 3.4. *Suppose that (V, ω) is a strong symplectic filling of some closed contact manifold (Y, ξ). If $i : Y \to V$ denotes the inclusion map, then $i^* c(TV) = c(\xi)$.*

Proof. There is a Liouville vector field Z defined near ∂V with the Liouville form $\lambda := i_Z \omega$. Its restriction $\alpha := \lambda|_{\partial V}$ defines a contact form. We get a map $((-\epsilon, 0] \times Y, d(e^t \alpha)) \to (V, \omega)$ by sending (t, p) to $Fl^Z_t(i(p))$, where Fl^Z denotes the flow induced by Z. This map preserves the symplectic structure. We conclude that $TV|_{\partial V = Y}$ symplectically splits as

$$(span_{\mathbb{R}}(Z, R) \oplus \xi, \omega_0 \oplus \omega|_\xi).$$

Here R is the Reeb vector field on the contact manifold (Y, α), and ω_0 defines the standard symplectic form. Note that Z, R forms a symplectic frame as $\omega(Z, R) = 1$. The rank 2 symplectic vector bundle $\epsilon = span_{\mathbb{R}}(Z, R)$ is clearly trivial, so we see that

$$c(i^* TV) = c(\epsilon \oplus \xi) = c(\epsilon) c(\xi) = c(\xi).$$

□

Lemma 3.5. *Suppose V^{2n} is a Weinstein domain with $n \geq 3$. Assume in addition that W is subcritical for $n = 3$. Then the inclusion map $i : \partial V \to V$ induces an isomorphism on H^2. In particular, if (Y, ξ) is the contact boundary of V, then $c_1(\xi)$ determines and is determined by $c_1(TV)$.*

Proof. We just consider the long exact sequence of cohomology groups

$$\underbrace{H^2(V, \partial V)}_{\cong H^{2n-2}(V)=0} \longrightarrow H^2(V) \xrightarrow{i^*} H^2(\partial V) \longrightarrow \underbrace{H^3(V, \partial V)}_{\cong H^{2n-3}(V)=0}$$

of the pair $(V, \partial V)$ to conclude that i^* is an isomorphism, assuming in addition that V is subcritical for the case $n = 3$. Lemma 3.4 then shows the last claim. □

Finally, we are ready to prove the main result of the paper.

Theorem 3.6. *If p is odd, then M_p is diffeomorphic to $S^2 \times S^3 \# S^2 \times S^3$. If p is even, then M_p is diffeomorphic to $S^2 \times S^3 \# S^2 \tilde{\times} S^3$. Moreover, (M_p, ξ_p) is contactomorphic to $(M_{p'}, \xi_{p'})$ if and only if $p = p'$.*

Proof. We know that the first Chern class of any Stein surface can be calculated using an explicit Legendrian handlebody diagram representing the surface [10, Prop. 2.3]. Let α_p, β_p, and γ_p be the Legendrian curves in the handlebody diagram for X_p depicted in Figure 2 and let $T_p = [\gamma_p]$ and $R_p = [\alpha_p] - p[\beta_p] \in H_2(X_p)$. As it was observed in [1], $H_2(X_p)$ has a basis consisting of T_p and S_p, where

$$S_p = \begin{cases} R_{q-1} + \big((2q-1)^2 - q + 1\big) T_{2q-1} & : p = 2q-1 \\ R_{2q} + \big((2q)^2 - q + 1\big) T_{2q} & : p = 2q \end{cases}$$

It is easy to see that $c_1(TX_p)$ evaluates on these homology classes as:

$$\langle c_1(TX_p), S_p \rangle = -1 - p, \quad \langle c_1(TX_p), T_p \rangle = 0.$$

The first Chern class $c_1(\xi_p)$ can be viewed as a linear map from $H_2(M_p) \cong \mathbb{Z} \oplus \mathbb{Z}$ to $\mathbb{Z}$. The cycles $S_p \times \{1\}$, and $T_p \times \{1\}$ in $M_p = \partial(X_p \times D^2)$, where we think of $1 \in \partial D^2$, generate $H_2(M_p)$. Using Lemma 3.5, we conclude that $c_1(\xi_p)$ evaluates as $(-1-p, 0)$ with respect to the chosen basis of $H_2(M_p)$, which is sufficient to mutually distinguish ξ_p's.

The calculation above shows that M_p is spin for odd p and non-spin otherwise. We conclude, by Lemma 3.3, that M_p is diffeomorphic to $S^2 \times S^3 \# S^2 \times S^3$ for odd p, and to $S^2 \times S^3 \# S^2 \tilde{\times} S^3$, otherwise. □

We can say more explicitly what contact manifold we get with the following lemma.

Lemma 3.7. [7, Prop. 4.5] *Suppose that V is a Stein surface obtained by attaching a single 2-handle to the standard Stein domain D^4 along a Legendrian knot δ in the standard tight S^3. Then $OB(V, id)$ is diffeomorphic to*

- *$S^2 \times S^3$ if $rot(\delta)$ is even*
- *$S^2 \tilde{\times} S^3$ if $rot(\delta)$ is odd.*

Moreover, if V' is another Stein surface obtained as above using a Legendrian knot δ', then $OB(V, id)$ and $OB(V', id)$ are contactomorphic if and only if $|rot(\delta)| = |rot(\delta')|$.

For each integer k, choose a Legendrian unknot δ_k in the tight S^3 with rotation number equal to k. We view this S^3 as the boundary of the Stein domain D^4. Define V_k as the handlebody obtained by attaching a Weinstein 2-handle to D^4 along δ_k.

Then $\partial(V_k \times D^2)$ is an S^3-bundle over S^2, and it is diffeomorphic to $S^2 \times S^3$ if k is even and to $S^2 \tilde{\times} S^3$ if k is odd. Up to contactomorphism, the contact structure only depends on $|k|$ by Lemma 3.7. We denote the resulting contact structure by ζ_k.

Proposition 3.8. *The contact* 5*-manifold* (M_p, ξ_p) *is contactomorphic to*

- $(S^2 \times S^3, \zeta_0)\#(S^2 \times S^3, \zeta_{p+1})$ *if* p *is odd*
- $(S^2 \times S^3, \zeta_0)\#(S^2 \tilde{\times} S^3, \zeta_{p+1})$ *if* p *is even.*

Proof. A complete argument is given in the proof of Proposition 4.3. □

4. An alternative argument

As in [7] we encode a closed contact 5-manifold (described as a contact open book) by a handlebody diagram for its page—which can be assumed to be a compact Stein domain. We will only consider situations where the symplectic monodromy is a product of Dehn twists along Lagrangian spheres.

Definition 4.1. Two Stein surfaces X and X' are called **contact stably equivalent** if there are handlebody diagrams for X and X', and a third handlebody diagram for some Stein surface X'' with the property that the handlebody diagrams for X and X' can be transformed into the one for X'' by a finite sequence of the following moves:

- usual handlebody moves for Stein surfaces (cf. [10])
- stabilizing the attaching circles for the 2-handles (move I)
- changing a crossing (move II)

See Figure 3 for a description of moves I and II.

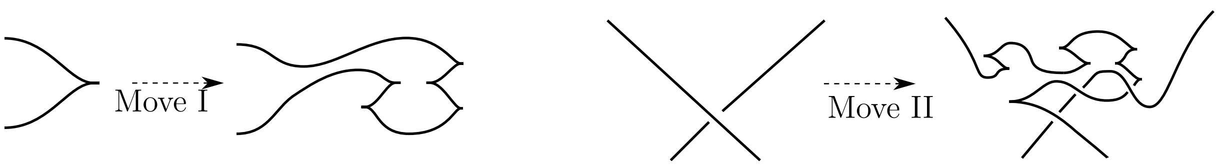

Figure 3. Moves I and II give contactomorphic 5-manifolds

With the arguments from [7, Section 4.1] we obtain the following proposition.

Proposition 4.2. *Suppose that* X *and* X' *are contact stably equivalent. Then* $X \times D^2$ *and* $X' \times D^2$ *are symplectically deformation equivalent with contactomorphic boundaries.*

We briefly summarize [7, Section 4.1] and explain why moves I and II give rise to contactomorphic manifolds. Given a contact open book $OB(W, id)$ we first positively stabilize the open book. This is done by first taking any properly embedded Lagrangian disk D. We attach a Weinstein 2-handle along ∂D to obtain a new page $\tilde{W}$, which contains a Lagrangian sphere L formed by gluing the core of the 2-handle to the Lagrangian disk D. We can then perform a right-handed Dehn twist along L. We denote this Dehn twist by τ_L. According to Giroux, the contact open book $OB(\tilde{W}, \tau_L)$ is contactomorphic to $OB(W, id)$. We use the extra 2-handle in the page to perform a handle slide. We then destabilize the open book by simply not performing the Dehn twist and removing the extra 2-handle. Since the monodromy on W was assumed to be the identity this last step can be done. See [7, Section 4.1] for a detailed description of moves I and II and a proof that these moves do not change the symplectic deformation type of the filling.

The upshot is that move I changes the Thurston-Bennequin invariant of an attaching circle of a 2-handle while preserving its rotation number. Move II changes an overcrossing into an undercrossing. This move can be used to change the knot type of the attaching circles.

Proposition 4.3. *The contact 5-manifolds (M_p, ξ_p) and $(M_{p'}, \xi_{p'})$ are diffeomorphic if and only if $p \equiv p'$ (mod 2), and contactomorphic if and only if $p = p'$.*

Proof. In the following, we refer to [10, Figures 3 and 9] for Legendrian Reidemeister moves. We consider the handlebody diagram of X_p in Figure 2 and apply a Legendrian isotopy to obtain step 1 in Figure 4. Then we apply move II simultaneously to the two overcrossings in the shaded region in step 1. Next we apply a Legendrian Reidemeister move 2 to the indicated cusp in step 2 and another Legendrian Reidemeister move 2 to the indicated cusp in step 3. We do this twice more, and then apply Legendrian Reidemeister move 4 to move the Legendrian knot γ_p off the 1-handle. Finally, by several applications of Reidemeister move 2 we obtain step 5 of Figure 4.

We now perform some moves so that we will be able to cancel the 1-handle. Note that the dotted curve near β_p depicted in the initial diagram in Figure 5 is not part of the handlebody and it will be used for a handle-slide. First we perform an isotopy of α_p, and then in step 2, we slide the α_p handle over the β_p handle using handle subtraction. We keep calling the resulting curve the α_p-curve. Note that its rotation number has decreased by 1 after the handle subtraction.

Now we use move II to unlink the curves α_p and β_p. We get additional cusps which we remove with the inverse of move I. This move keeps the rotation number unchanged. Now we perform Legendrian Reidemeister move 4 to move one strand of the curve α_p off the 1-handle.

We repeat this procedure until there are no strands of α_p left which is going over the 1-handle. To do this, we apply Reidemeister move 2 to the indicated cusp and apply an isotopy to move this cusp in the position of step 1. With move II we can make the curve α_p into an unknot without changing the rotation number, so we end up with an unknotted curve whose rotation number is equal to $-1-p$. With Legendrian Reidemeister move 4 we make the curve α_p disjoint from the 1-handle, and then cancel the 1-handle with the 2-handle attached along β_p. We end up with a handlebody diagram containing only 2-handles attached along the curve γ_p with rotation number 0 and to the curve α_p which has rotation number $-1-p$.

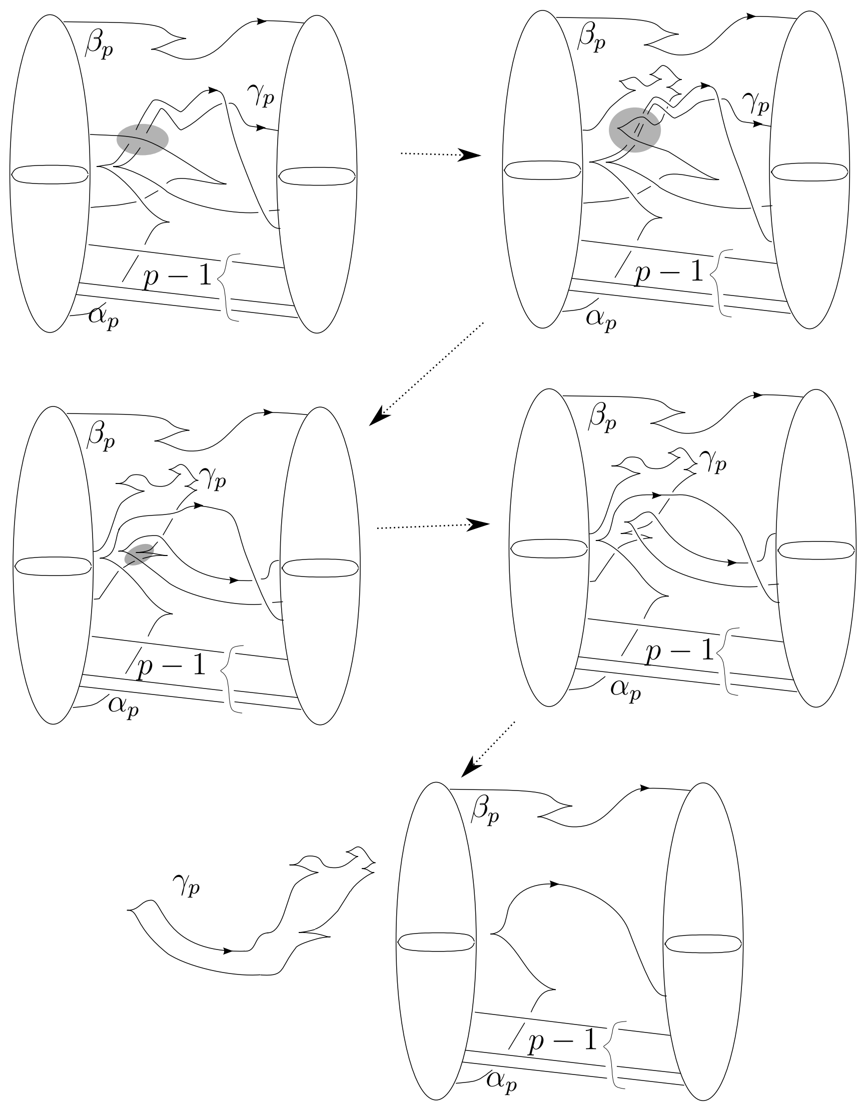

FIGURE 4. Constructing a contactomorphism between 5-manifolds

Now apply Lemma 3.7 to see that M_p is contactomorphic to

- $(S^2 \times S^3, \zeta_0)\#(S^2 \times S^3, \zeta_{p+1})$ if p is odd
- $(S^2 \times S^3, \zeta_0)\#(S^2 \tilde{\times} S^3, \zeta_{p+1})$ if p is even.

By forgetting the contact structure we get the diffeomorphism claim.

□

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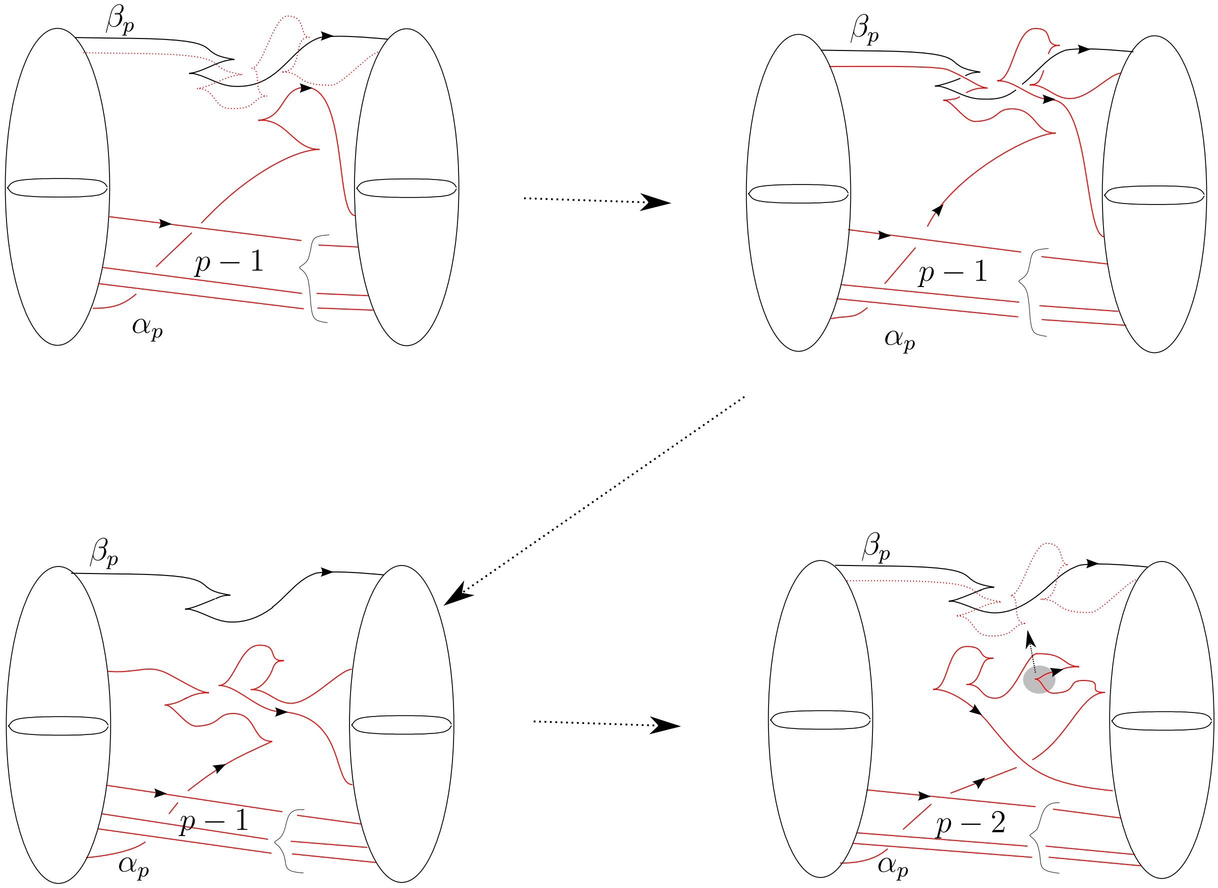

FIGURE 5. Sliding strands off the 1-handle

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