

Milnor fillable contact structures are universally tight

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We show that the canonical contact structure on the link of a normal complex singularity is universally tight. As a corollary we show the existence of closed, oriented, atoroidal 3-manifolds with infinite fundamental groups which carry universally tight contact structures that are not deformations of taut (or Reebless) foliations. This answers two questions of Etnyre in [12].

1 Introduction

Let (X, x) be a normal complex surface singularity. Fix a local embedding of (X, x) in $(\mathbb{C}^N, 0)$. Then a small sphere $S_\epsilon^{2N-1} \subset \mathbb{C}^N$ centered at the origin intersects X transversely, and the complex hyperplane distribution ξ_{can} on $M = X \cap S_\epsilon^{2N-1}$ induced by the complex structure on X is called the *canonical* contact structure. For sufficiently small radius ϵ , the contact manifold is independent of ϵ and the embedding, up to isomorphism. The 3-manifold M is called the link of the singularity, and (M, ξ_{can}) is called the *contact boundary* of (X, x) .

A contact manifold (Y, ξ) is said to be *Milnor fillable* if it is isomorphic to the contact boundary (M, ξ_{can}) of some isolated complex surface singularity (X, x) . In addition, we say that a closed and oriented 3-manifold Y is Milnor fillable if it carries a contact structure ξ so that (Y, ξ) is Milnor fillable. It is known that a closed and oriented 3-manifold is Milnor fillable if and only if it can be obtained by plumbing according to a weighted graph with negative definite intersection matrix (cf. [23] and [16]). Moreover any 3-manifold has at most one Milnor fillable contact structure up to *isomorphism* (cf. [5]). Note that Milnor fillable contact structures are Stein fillable (see [4]) and hence tight [10]. Here we prove that every Milnor fillable contact structure is in fact universally tight, i.e., the pullback to the universal cover is tight. We would like to point out that universal tightness of a contact structure is not implied by any other type of fillability.

In [12], Etnyre settled a question of Eliashberg and Thurston [11] by proving that every contact structure on a closed oriented 3-manifold is obtained by a deformation of a foliation and raised two other related questions:

(Question 4 in [12]) *Is every universally tight contact structure on a closed 3-manifold with infinite fundamental group the deformation of a Reebless foliation?*

(Question 5 in [12]) *Is every universally tight contact structure on an atoroidal closed 3-manifold with infinite fundamental group the deformation of a taut foliation?*

In this note we answer both questions negatively as a consequence of our main result. The assumption on the fundamental group is necessary since every foliation on a closed 3-manifold with finite fundamental group has a Reeb component (and hence is not taut) by a theorem of Novikov. Moreover Ghiggini [14] gave examples of toroidal 3-manifolds which carry universally tight contact structures that are not weakly fillable (and therefore can not be perturbations of taut foliations by [11]).

We contrast our result with the result of Honda, Kazez and Matic in [19], where they show that for a sutured manifold with annular sutures, the existence of a (universally) tight contact structure is equivalent to the existence of a taut foliation.

We assume that all the 3-manifolds are compact and oriented, all the contact structures are co-oriented and positive and all the surface singularities are isolated and normal.

2 Milnor fillable implies universally tight

A *graph manifold* is a 3-manifold $M(\Gamma)$ obtained by plumbing circle bundles according to a connected weighted plumbing graph Γ . More precisely, let A_1, \dots, A_r denote vertices of a connected graph Γ . Each vertex is decorated with a pair (g_i, e_i) of integral weights, where $g_i \geq 0$. Here the i th vertex represents an oriented circle bundle of Euler number e_i over a closed Riemann surface of genus g_i . Then $M(\Gamma)$ is the 3-manifold obtained by plumbing these circle bundles according to Γ . A *horizontal* open book in $M(\Gamma)$ is an open book whose binding consists of some fibers in the circle bundles and whose (open) pages are transverse to the fibers. We also require that the orientation induced on the binding by the pages coincides with the orientation of the fibers induced by the fibration.

In this paper, we will consider horizontal open books on graph manifolds coming from isolated normal complex singularities. Given an analytic function $f: (X, x) \rightarrow (\mathbb{C}, 0)$ vanishing at x , with an isolated singularity at x , the open book decomposition \mathcal{OB}_f of the boundary M of (X, x) with binding $L = M \cap f^{-1}(0)$ and projection $\pi = \frac{f}{|f|}: M \setminus L \rightarrow S^1 \subset \mathbb{C}$ is called the *Milnor open book* induced by f .

Theorem 2.1 *A Milnor fillable contact structure is universally tight.*

Proof Given a Milnor fillable contact 3-manifold (Y, ξ) . By definition (Y, ξ) is isomorphic to the link (M, ξ_{can}) of some surface singularity. Hence it suffices to show that (M, ξ_{can}) is universally tight. It is known that M is an irreducible graph manifold $M(\Gamma)$ where Γ is a negative definite plumbing graph [25]. Moreover, such a manifold is characterized by the property that there exists a unique minimal set \mathcal{T} (possibly empty) consisting of pairwise disjoint *incompressible* tori in M such that each component of $M - \mathcal{T}$ is an orientable Seifert fibered manifold with an orientable base [25]. In terms of the plumbing description \mathcal{T} is a subset of the tori that are used to glue the circle bundles in the definition of $M(\Gamma)$. The set \mathcal{T} is minimal if in plumbing of two circle bundles the homotopy class of circle fiber in one boundary torus is not identified with the homotopy class of the fiber in the other boundary torus.

Recall that an arbitrary Milnor open book \mathcal{OB} on M has the following essential features [5]: It is compatible with the canonical contact structure ξ_{can} , horizontal when restricted to each Seifert fibered piece in $M - \mathcal{T}$ which means that the Seifert fibres intersect the pages of the open book transversely, and the binding of the open book consists of some number (which we can take to be non-zero) of regular fibres of the Seifert fibration in each Seifert fibered piece.

In the rest of the proof, we will construct a universally tight contact structure ξ on M which is compatible with the Milnor open book \mathcal{OB} . This implies that the canonical contact structure ξ_{can} is isotopic to ξ by Giroux's correspondence (since they are both compatible with \mathcal{OB}) and thus we conclude that ξ_{can} on the singularity link M is universally tight.

Let V_i denote a Seifert fibered 3-manifold with boundary, which is a component of $M - N(\mathcal{T})$, where $N(\mathcal{T})$ denotes a regular neighborhood of \mathcal{T} . Consider the 3-manifold V'_i obtained by removing a regular neighborhood of the binding of \mathcal{OB} from V_i . Note that V'_i is also a Seifert fibered manifold since the binding consists of regular fibers of the Seifert fibration on V_i . Then the restriction of a page of \mathcal{OB} to V'_i is a connected horizontal surface (see the proof of Proposition 4.6 in [5]) which we denote by Σ'_i . It follows that V'_i is a surface bundle over S^1 whose fibers are precisely the restriction

of the pages of \mathcal{OB} to V'_i , since Σ'_i does not separate V'_i . Note that Σ'_i is a branched cover of the base of the Seifert fibration on V'_i and the monodromy ϕ_i of this surface bundle is a periodic self-diffeomorphism of Σ'_i of some order n_i (cf. Section 1.2 in [17]).

Now we construct, as in Section 2 in [14], a contact structure ξ'_i on V'_i which is "compatible" with the surface fibration $V'_i \rightarrow S^1$. Here compatibility means that the Reeb vector field of the contact form is transverse to the fibers, keeping in mind that a fiber of this fibration is cut out from a page of the open book \mathcal{OB} . Let β_i denote a 1-form on Σ'_i such that $d\beta_i$ is a volume form on Σ'_i and $\beta|_{\partial\Sigma'_i}$ is a volume form on $\partial\Sigma'_i$. Then the 1-form

$$\beta'_i = \frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi_i^k)^* \beta_i,$$

which also satisfies the above conditions, is a ϕ_i invariant 1-form on Σ'_i . Let t denote the coordinate on S^1 . It follows that for every real number $\epsilon > 0$, the kernel of the 1-form $dt + \epsilon\beta'_i$ is a contact structure on V'_i which is compatible with the fibers. Note that the characteristic foliation on every torus in $\partial V'_i$ is linear with a slope arbitrarily close to the slope of the foliation induced by the pages when $\epsilon \rightarrow 0$. Here we point out that different choices of ϵ give isotopic contact structures, so that we denote the isotopy type of this contact structure by ξ'_i . Moreover the Reeb vector field R_i is tangent to the circle fibers in the Seifert fibration and hence transverse to the fibers of the surface bundle $V'_i \rightarrow S^1$.

Furthermore we observe that ξ'_i is transverse to the Seifert fibration on V'_i and can be extended over to V_i along the neighborhood of the binding so that it remains transverse to the Seifert fibration. Consequently the resulting contact structure ξ_i on V_i is universally tight by Corollary 2.2 in [20], and also by Proposition 4.4 in [22]. (Strictly speaking, the cited results prove universal tightness for transverse contact structures on closed Seifert fibered 3-manifolds, but the proofs, especially the one in [22], can easily be extended to cover the case of Seifert fibered 3-manifolds with boundary a disjoint union of tori.)

Let V_1, \dots, V_n denote the Seifert fibered manifolds in the decomposition of $M - N(\mathcal{T})$. Our goal is to glue together ξ_i 's on V_i 's to get a universally tight contact structure ξ on M which is *compatible* with \mathcal{OB} . We should point out that if one ignores the compatibility with \mathcal{OB} , then ξ_i 's can be glued along the incompressible pre-Lagrangian tori on ∂V_i 's to yield a universally tight contact structure on M , by Colin's gluing theorem [6]. This was already described in Theorem 1.4 in [7], although the contact structures on Seifert fibered pieces were obtained by perturbing Gabai's taut foliations [13].

By construction ξ_i on V_i is compatible with the restriction of \mathcal{OB} to V_i . We first modify ξ_i near each component of ∂V_i to put it in a certain standard form. To this end, let $N(T_{ij})$ denote the normal neighborhood of a torus $T_{ij} \in \mathcal{T}$ along which plumbing is performed between V_i and V_j . By picking a section s_i of each V_i we can identify the boundary of $N(T_{ij})$ in V_i with T^2 so that the basis (r_i, s_i) is sent to the standard basis $\{\partial_x, \partial_y\}$ of T^2 , where r_i denotes a fiber of the Seifert fibration on V_i . Then we can identify $N(T_{ij}) = T^2 \times [a_i, b_i] \cup_{\rho_{ij}} -T^2 \times [a_j, b_j]$ where $\rho_{ij} : T^2 \times \{b_i\} \rightarrow -T^2 \times \{b_j\}$ is the gluing map sending $(r_i, s_i) \rightarrow (s_j, r_j)$.

Let \mathcal{F}_i denote the foliation by circles with a certain rational slope m_i/m_j on $T^2 \times \{a_i\}$ induced by the pages of \mathcal{OB} . We extend the pages into $T^2 \times [a_i, b_i]$ as $\mathcal{F}_i \times [a, b]$. Similarly, \mathcal{F}_j denote the foliation by circles given by the intersection of the pages of \mathcal{OB} with $T^2 \times \{a_j\}$ which necessarily has rational slope m_j/m_i so that the gluing map ρ_{ij} glues the pages in each piece together to form \mathcal{OB} .

For convenience, we will choose a_i so that $-\cot a_i = m_i/(m_j - \epsilon)$ is the slope of the characteristic foliation of the contact structure ξ_i and b_i so that $-\cot b_i = m_i/m_j$ is the slope of the pages of \mathcal{OB} . Note that $b_i \in [a_i, a_i + \pi_2)$ and by choosing ϵ small enough, we can arrange b_i to be as close to a_i as we want.

We now need to glue together the contact forms that we constructed on V_i by extending them to $N(T_{ij})$. For our purposes, we need to pay special attention to compatibility with \mathcal{OB} on $N(T_{ij})$.

Consider the contact form $\alpha_i = \cos t dx + \sin t dy$ on $T^2 \times [a_i, b_i]$. By [8] Lemma 9.1 we can isotope ξ_i on V_i relative to the boundary so that it is defined by a contact form that glue to α_i . Moreover, after this isotopy the Reeb vector field of ξ_i still remains transverse to the pages of \mathcal{OB} on V_i . Furthermore, the Reeb vector field of α_i is perpendicular to the slope $-\cot a_i$ at $T^2 \times \{a_i\}$ which we know to be arbitrarily close the slope of the foliation $\mathcal{F}_i \times \{a_i\}$ induced by the page of \mathcal{OB} . Since the slope of the Reeb vector field changes by strictly less than $\pi/2$ as we go from a_i to b_i , the Reeb vector field still remains transverse to $\mathcal{F}_i \times [a_i, b_i]$. Therefore, the form α_i is compatible with \mathcal{OB} in $T^2 \times [a_i, b_i]$. Finally, to finish the construction of the contact structure ξ on M , we observe that the gluing map ρ_{ij} sends α_i to α_j , since we arranged that the slope of α_i and the slope of the characteristic foliation induced by the page are the same at $T^2 \times \{b_i\}$.

We constructed a contact structure ξ which is compatible with a Milnor open book (hence is isomorphic to ξ_{can}) such that ξ is isotopic to ξ_i on V_i , a universally tight contact structure, furthermore for each incompressible torus $T \in \mathcal{T}$, the characteristic

foliation of ξ is a linear foliation (with slope m_i/m_j). Therefore, we are in a position to apply the gluing result of Colin [6] which states that universally tight contact structures can be glued along pre-Lagrangian tori to a universally tight contact structure. This shows that ξ_{can} is a universally tight contact structure. \square

Remark 2.2 The above construction shows that when the fibres of each Seifert fibered piece is not contractible, then ξ_{can} is *hypertight*, that is, it can be defined by a contact form whose associated Reeb vector field has no contractible orbits. Thus, for example when $\mathcal{T} \neq \emptyset$, ξ_{can} is hypertight. Note that hypertight contact structures are tight [18] and any finite cover of a hypertight contact manifold is hypertight [14]. These results together with the fact that graph manifolds have residually finite fundamental groups give another proof of universal tightness (avoiding Colin's gluing result). Since M is irreducible, its universal cover is diffeomorphic to either S^3 or \mathbb{R}^3 depending on whether $\pi_1(M)$ is finite or infinite. The universal cover can be S^3 only if M is atoroidal, then M is either a small Seifert fibered space or a lens space and these have no hypertight contact structures. Therefore, M is hypertight if and only if $\pi_1(M)$ is infinite (or equivalently its universal cover is \mathbb{R}^3).

Remark 2.3 Since any Milnor fillable contact 3-manifold (Y, ξ) is Stein fillable (see [4]), it follows from Theorem 1.5 in [26] that the contact invariant $c(\xi) \in \widehat{HF}(-Y)/(\pm 1)$ is non-trivial. Therefore, by [15], the Giroux torsion of Y is zero. In particular, the incompressible tori in \mathcal{T} have zero torsion. This was predicted in [24] and was raised as a question there.

3 Universally tight but no taut

A rational homology sphere is called an L -space if $\text{rk } \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$. Lens spaces are basic examples of L -spaces which explains the name. A characterization of L -spaces among Seifert fibered 3-manifolds is given by

Theorem 3.1 [21] *A rational homology sphere which is Seifert fibered over S^2 is an L -space if and only if it does not carry a taut foliation.*

A huge class of examples of L -spaces come from complex surface singularities. Recall that an isolated normal surface singularity (X, x) is rational if the geometric genus $p_g := \dim_{\mathbb{C}} H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is equal to zero, where $\tilde{X} \rightarrow X$ is a resolution of the singular point $x \in X$. This definition does not depend on the resolution.

Theorem 3.2 [24] *The link of a rational surface singularity is an L -space.*

Corollary 3.3 *If Y is the link of a rational surface singularity which is Seifert fibered over S^2 , then Y carries a universally tight contact structure that can not be obtained by a deformation of a taut foliation.*

Proof The link of a rational surface singularity is an L -space by Theorem 3.2 and hence it does not carry any taut foliations by Theorem 3.1. Moreover, Theorem 2.1 implies that the canonical contact structure on this link is universally tight. \square

Remark 3.4 Note that Seifert fibered 3-manifolds as above carry transverse contact structures (by Theorem 1.3 in [20]) and such contact structures are known to be universally tight (cf. Corollary 2.2 in [20] and also Proposition 4.4 in [22]).

Corollary 3.5 *There exist infinitely many atoroidal 3-manifolds with infinite fundamental groups which carry universally tight contact structures that are not deformations of taut (or Reebless) foliations.*

Proof It is known (cf. [9]) that the link of a complex surface singularity has finite fundamental group if and only if it is a quotient singularity. Thus the link of a rational but not quotient surface singularity has an infinite fundamental group. Note that the links of a quotient surface singularities (all small Seifert fibered 3-manifolds) are explicitly listed in [2] via their dual resolution graphs. It is easy to see that there are many infinite families of small Seifert fibered 3-manifolds which are links of rational but not quotient surface singularities. This finishes the proof using Corollary 3.3 since all small Seifert fibered 3-manifolds are known to be atoroidal. Note that on an atoroidal 3-manifold, a Reebless foliation is taut. \square

Consequently, Corollary 3.5 answers Questions 4 and 5 of Etnyre [12] negatively. For the sake of completeness we give an explicit example. Consider the small Seifert fibered 3-manifold

$$Y_p = Y(-2; \frac{1}{3}, \frac{2}{3}, \frac{p}{p+1})$$

for $p \geq 1$, which is depicted in Figure 1. Note that Y_p is the link of a complex surface singularity whose dual resolution graph is given in Figure 2.

A surface singularity can be identified as rational (defined by Artin in [1]) by its dual resolution graph as follows. Let (X, x) be a germ of a complex surface singularity. Fix a resolution $\pi: \tilde{X} \rightarrow X$ and denote the irreducible components of the exceptional

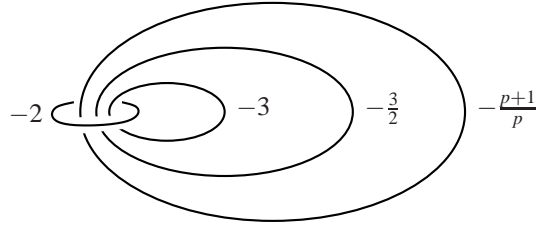
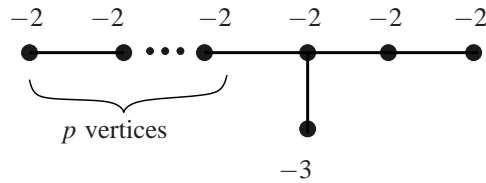
Figure 1: Rational surgery diagram for Y_p 

Figure 2: Dual resolution graph

divisor $E = \pi^{-1}(x)$ by $\bigcup_{i=1}^r E_i$. The *fundamental cycle* of E is by definition the componentwise smallest nonzero effective divisor $Z = \sum z_i E_i$ satisfying $Z \cdot E_i \leq 0$ for all i .

Definition 3.6 The singularity (X, x) is called *rational* if each irreducible component E_i of the exceptional divisor E is isomorphic to $\mathbb{C}P^1$ and

$$Z \cdot Z + \sum_{i=1}^n z_i (-E_i^2 - 2) = -2,$$

where $Z = \sum z_i E_i$ is the fundamental cycle of E .

Enumerate the vertices of the plumbing graph for Y_p from left to right along the top row with the bottom vertex coming last. It is then easy to check (cf.[3]) that the coefficients of the fundamental cycle corresponding to the dual resolution graph of Y_p is given by $(1, 2, 3, 3, \dots, 3, 3, 2, 1, 1)$, which implies that Y_p is the link of a rational surface singularity by Definition 3.6. On the other hand, Y_p is not a quotient singularity for $p \geq 2$ (cf. [2]). Thus the canonical contact structure ξ_{can} on Y_p is universally tight but it can not be obtained by perturbing a taut foliation. We would like to point out that Y_1 is the link of a quotient singularity (see [2]).

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