

ON THE RELATIVE GIROUX CORRESPONDENCE

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ABSTRACT. Recently, Honda, Kazez and Matic [9] described an adapted partial open book decomposition of a compact contact 3-manifold with convex boundary by generalizing the work of Giroux in the closed case. They also implicitly established a one-to-one correspondence between isomorphism classes of partial open book decompositions modulo positive stabilization and isomorphism classes of compact contact 3-manifolds with convex boundary. In this expository article we explicate the relative version of Giroux correspondence.

0. INTRODUCTION

A sutured manifold (M, Γ) is a compact oriented 3-manifold with nonempty boundary, together with a compact subsurface $\Gamma = A(\Gamma) \cup T(\Gamma) \subset \partial M$, where $A(\Gamma)$ is a union of pairwise disjoint annuli and $T(\Gamma)$ is a union of tori. Moreover each component of $\partial M \setminus \Gamma$ is oriented, subject to the condition that the orientation changes every time we nontrivially cross $A(\Gamma)$. Let $R_+(\Gamma)$ (resp. $R_-(\Gamma)$) be the open subsurface of $\partial M \setminus \Gamma$ on which the orientation agrees with (resp. is the opposite of) the boundary orientation on ∂M . A sutured manifold (M, Γ) is balanced if M has no closed components, $\pi_0(A(\Gamma)) \rightarrow \pi_0(\partial M)$ is surjective, and $\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))$ on every component of M . It turns out that if (M, Γ) is balanced, then $\Gamma = A(\Gamma)$ and every component of ∂M nontrivially intersects Γ . Since all the sutured manifolds that we will deal with in this paper are balanced, we will think of Γ as a set of *oriented curves* on ∂M by identifying each annulus in Γ with its core circle. Here we orient Γ as the boundary of $R_+(\Gamma)$.

Let ξ be a contact structure on a compact oriented 3-manifold M whose dividing set on the convex boundary ∂M is denoted by Γ . Then it is fairly easy to see that (M, Γ) is a *balanced* sutured manifold (with annular sutures) via the identification we mentioned above. Conversely, given a balanced sutured manifold (M, Γ) , there exists a contact structure ξ on M which makes ∂M convex and realizes Γ as its dividing set on ∂M . One should keep in mind, however, that such a contact structure is neither tight nor unique in general.

Let (M, Γ) be a balanced sutured 3-manifold and let ξ be a contact structure on M with convex boundary whose dividing set on ∂M is isotopic to Γ . Recently, Honda, Kazez and Matic [9] introduced an invariant of the contact structure ξ which lives in the sutured Floer

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homology group defined by Juhász [6]. This invariant is a relative version of the contact class in Heegaard Floer homology in the closed case as defined by Ozsváth and Szabó [11] and reformulated in [8]. Both the original definition in [11] and the reformulation of the contact class by Honda, Kazez and Matić depend heavily on the so called Giroux correspondence [5] which is a one-to-one correspondence between open book decompositions modulo positive stabilization and isotopy classes of contact structures on closed 3-manifolds.

In order to adapt their reformulation [8] of the contact class to the case of a contact manifold (M, ξ) with convex boundary, Honda, Kazez and Matić described in [9], a partial open book decomposition of M (adapted to ξ) by generalizing the work of Giroux in the closed case. This description coupled with Theorem 1.2 (and the subsequent discussion) in [9] induces a map from isomorphism classes of compact contact 3-manifolds with convex boundary to isomorphism classes of partial open book decompositions modulo positive stabilization. The inverse of this map is implicit in [9], and here we spell out this map by describing a compact contact 3-manifold with convex boundary compatible with an *abstract* partial open book decomposition. One could possibly adapt the explicit construction of Thurston and Winkelnkemper [13] in the closed case to define a contact structure compatible with an abstract partial open book decomposition but we rather chose to mimic the analogous result of Torisu [12] which conveniently allowed us to keep track of the dividing set on the boundary. Consequently, one obtains a relative version of Giroux correspondence, namely the following theorem.

Theorem 0.1 ([9]). *There is a one-to-one correspondence between isomorphism classes of partial open book decompositions modulo positive stabilization and isomorphism classes of compact contact 3-manifolds with convex boundary.*

This paper grew from our attempt to understand, in simpler terms, the description of a partial open book decomposition of Honda, Kazez and Matić and it maybe considered as a reinterpretation of their work with an “abstract” perspective. The paper is organized as follows: In Section 1 we give an *abstract* definition of a partial open book decomposition (S, P, h) , construct a balanced sutured manifold (M, Γ) associated to (S, P, h) , construct a (unique) *compatible* contact structure ξ on M which makes ∂M convex with a dividing set isotopic to Γ . In Section 2 we prove Theorem 0.1 after reviewing the related results due to Honda, Kazez and Matić [9]. The reader is advised to turn to Etnyre’s notes [3] for the related material on contact topology of 3-manifolds.

1. PARTIAL OPEN BOOK DECOMPOSITIONS AND COMPATIBLE CONTACT STRUCTURES

The first description of a partial open book decomposition has appeared in [9]. In this paper we give an abstract version of this description.

Definition 1.1. A partial open book decomposition is a triple (S, P, h) satisfying the following conditions:

- (1) S is a compact oriented connected surface with $\partial S \neq \emptyset$,
- (2) $P = P_1 \cup P_2 \cup \dots \cup P_r$ is a proper (not necessarily connected) subsurface of S such that S is obtained from $\overline{S \setminus P}$ by successively attaching 1-handles P_1, P_2, \dots, P_r ,
- (3) $h : P \rightarrow S$ is an embedding such that $h|_A = \text{identity}$, where $A = \partial P \cap \partial S$.

Remark 1.2. It follows from the above definition that A is a 1-manifold with nonempty boundary, and $\partial P \setminus \partial S$ is a nonempty set consisting of some arcs (but no closed components). The connectedness condition on S is not essential, but simplifies the discussion.

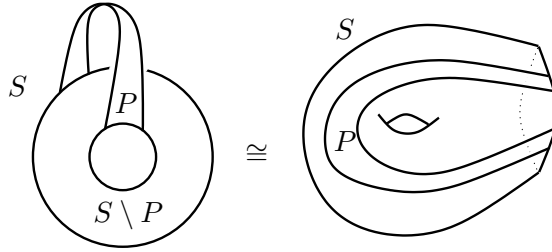


FIGURE 1. An example of S and P satisfying the conditions in Definition 1.1: $\overline{S \setminus P}$ is an annulus and S is a once punctured torus.

A sutured manifold (M, Γ) is a compact oriented 3-manifold with nonempty boundary, together with a compact subsurface $\Gamma = A(\Gamma) \cup T(\Gamma) \subset \partial M$, where $A(\Gamma)$ is a union of pairwise disjoint annuli and $T(\Gamma)$ is a union of tori. Moreover we orient each component of $\partial M \setminus \Gamma$, subject to the condition that the orientation changes every time we nontrivially cross $A(\Gamma)$. Let $R_+(\Gamma)$ (resp. $R_-(\Gamma)$) be the open subsurface of $\partial M \setminus \Gamma$ on which the orientation agrees with (resp. is the opposite of) the boundary orientation on ∂M .

Given a partial open book decomposition (S, P, h) , we construct a sutured manifold (M, Γ) as follows: Let

$$H = (S \times [-1, 0]) / \sim$$

where $(x, t) \sim (x, t')$ for $x \in \partial S$ and $t, t' \in [-1, 0]$. It is easy to see that H is a solid handlebody whose oriented boundary is the surface $S \times \{0\} \cup -S \times \{-1\}$ (modulo the relation $(x, 0) \sim (x, -1)$ for every $x \in \partial S$). Similarly let

$$N = (P \times [0, 1]) / \sim$$

where $(x, t) \sim (x, t')$ for $x \in A$ and $t, t' \in [0, 1]$. Since P is not necessarily connected N is not necessarily connected. Observe that each component of N is also a solid handlebody. The oriented boundary of N can be described as follows: Let the arcs c_1, c_2, \dots, c_n denote the connected components of $\partial P \setminus \partial S$. Then, for $1 \leq i \leq n$, the disk $D_i = (c_i \times [0, 1]) / \sim$

belongs to ∂N . Thus part of ∂N is given by the disjoint union of D_i 's. The rest of ∂N is the surface $P \times \{1\} \cup -P \times \{0\}$ (modulo the relation $(x, 0) \sim (x, 1)$ for every $x \in A$).

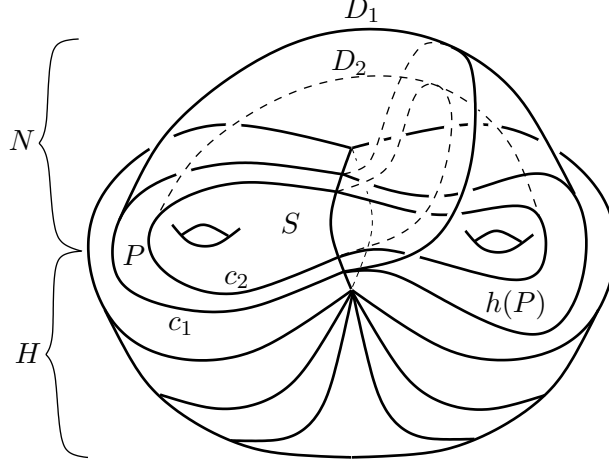


FIGURE 2. A partial open book decomposition: M as the union of N and H

Let $M = N \cup H$ where we glue these manifolds by identifying $P \times \{0\} \subset \partial N$ with $P \times \{0\} \subset \partial H$ and $P \times \{1\} \subset \partial N$ with $h(P) \times \{-1\} \subset \partial H$. Since the gluing identification is orientation reversing M is a compact oriented 3-manifold with oriented boundary

$$\partial M = (S \setminus P) \times \{0\} \cup -(S \setminus h(P)) \times \{-1\} \cup (\overline{\partial P \setminus \partial S}) \times [0, 1]$$

(modulo the identifications given above).

Definition 1.3. *If a compact 3-manifold M with boundary is obtained from (S, P, h) as discussed above, then we call the triple (S, P, h) a partial open book decomposition of M .*

We define the suture Γ on ∂M as the set of closed curves (see Remark 1.4) obtained by gluing the arcs $c_i \times \{1/2\} \subset \partial N$, for $1 \leq i \leq n$, with the arcs in $(\overline{\partial S \setminus \partial P}) \times \{0\} \subset \partial H$, hence as an oriented simple closed curve and modulo identifications

$$\Gamma = (\overline{\partial S \setminus \partial P}) \times \{0\} \cup -(\overline{\partial P \setminus \partial S}) \times \{1/2\}.$$

Remark 1.4. *If a sutured manifold (M, Γ) has only annular sutures, then it is convenient to refer to the set of core circles of these annuli as Γ .*

Definition 1.5. *The sutured manifold (M, Γ) obtained from a partial open book decomposition (S, P, h) as described above is called the sutured manifold associated to (S, P, h) .*

Definition 1.6 ([6]). *A sutured manifold (M, Γ) is balanced if M has no closed components, $\pi_0(A(\Gamma)) \rightarrow \pi_0(\partial M)$ is surjective, and $\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))$ on every component of M .*

Remark 1.7. *It follows that if (M, Γ) is balanced, then $\Gamma = A(\Gamma)$ and every component of ∂M nontrivially intersects the suture Γ .*

Lemma 1.8. *The sutured manifold (M, Γ) associated to a partial open book decomposition (S, P, h) is balanced.*

Proof. It is clear that M is connected since we assumed that S is connected. We observe that $\partial M \neq \emptyset$ since P is a proper subset of S by our definition and thus M has no closed components. By our construction every component of ∂M contains a disk $D_i = (c_i \times [0, 1]) / \sim$ for some $1 \leq i \leq n$. Hence every component of ∂M contains a $c_i \times \{1/2\} \subset \Gamma$ and therefore $\pi_0(A(\Gamma)) \rightarrow \pi_0(\partial M)$ is surjective. Now let $R_+(\Gamma)$ be the open subsurface in ∂M obtained by gluing

$$((S \setminus \partial S) \setminus P) \times \{0\} \subset \partial H \quad \text{and} \quad \cup_{i=1}^n (c_i \times [0, 1/2]) / \sim \subset \partial N$$

and $R_-(\Gamma)$ be the open subsurface in ∂M obtained by gluing

$$((S \setminus \partial S) \setminus h(P)) \times \{-1\} \subset \partial H \quad \text{and} \quad \cup_{i=1}^n (c_i \times (1/2, 1]) / \sim \subset \partial N$$

under the gluing map that is used to construct M . Since $h : P \rightarrow S$ is an embedding we have $\chi(P) = \chi(h(P))$ and it follows that $\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))$. \square

The following result is inspired by Torisu's work [12] in the closed case.

Proposition 1.9. *Let (M, Γ) be the balanced sutured manifold associated to a partial open book decomposition (S, P, h) . Then there exists a contact structure ξ on M satisfying the following conditions:*

- (1) ξ is tight when restricted to H and N ,
- (2) ∂H is a convex surface in (M, ξ) whose dividing set is $\partial S \times \{0\}$,
- (3) ∂N is a convex surface in (M, ξ) whose dividing set is $\partial P \times \{1/2\}$.

Moreover such ξ is unique up to isotopy.

Proof. We will prove that there is a unique tight contact structure (up to isotopy) on each piece H and N with the given boundary conditions. Then one can conclude that there is a unique contact structure (up to isotopy) on M satisfying the above conditions, since the dividing sets on ∂H and ∂N agree on the subsurface along which we glue H and N .

The existence of a unique tight contact structure on the handlebody H with the assumed boundary conditions was already shown by Torisu [12]. We include here a proof (see also page 97 in [10]) which is different from Torisu's original proof.

In order to prove the uniqueness we take a set $\{d_1, d_2, \dots, d_p\}$ of properly embedded pairwise disjoint arcs in S whose complement is a single disk. (It follows that the set $\{d_1, d_2, \dots, d_p\}$ represents a basis of $H_1(S, \partial S)$.) For $1 \leq k \leq p$, let δ_k denote the closed curve on ∂H which is obtained by gluing the arc d_k on $S \times \{0\}$ with the arc d_k on $S \times \{-1\}$. Then we observe that $\{\delta_1, \delta_2, \dots, \delta_p\}$ is a set of homologically linearly independent closed

curves on ∂H so that δ_k bounds a compressing disk $D_k^\delta = (d_k \times [0, -1]) / \sim$ in H . It is clear that when we cut H along D_k^δ 's (and smooth the corners) we get a 3-ball B^3 . Moreover δ_k intersects the dividing set twice by our construction. Now we put each δ_k into Legendrian position (by the Legendrian realization principle [7]) and make the compressing disk D_k^δ convex [4]. The dividing set on D_k^δ will be an arc connecting two points on $\partial D_k^\delta = \delta_k$. Then we cut along these disks and round the edges (see [7]) to get a connected dividing set on the remaining B^3 . Consequently, the following fundamental theorem of Eliashberg implies the uniqueness of a tight contact structure on H with the assumed boundary conditions.

Theorem 1.10 ([1]). *There is a unique tight contact structure on a 3-ball B^3 with a connected dividing set on ∂B^3 up to isotopy fixing the dividing set.*

The existence of such a tight contact structure on H essentially follows from the explicit construction of Thurston and Winkelnkemper [13]. We just embed H into an open book decomposition (in the usual sense) with page S and trivial monodromy whose compatible contact structure is Stein fillable by [5] (and hence tight by [2]). To be more precise, we embed H into

$$Y = (S \times [-2, 0]) / \sim$$

where $(x, 0) \sim (x, -2)$ for $x \in S$ and $(x, t) \sim (x, t')$ for $x \in \partial S$ and $t, t' \in [-2, 0]$. Let ξ' be the tight structure on Y which is compatible with the above open book decomposition. Then $\partial H = S \times \{0\} \cup -S \times \{-1\}$ which is obtained by gluing two pages along the binding can be made convex with respect to ξ' so that the dividing set on ∂H is exactly the binding (see [3] for example).

By a similar argument we will prove the existence of a unique tight contact structure on N (each of whose components is a handlebody) with the assumed boundary conditions. By the definition of a partial open book decomposition (S, P, h) , P is a proper subsurface of S such that S is obtained from $\overline{S \setminus P}$ by successively attaching 1-handles P_1, P_2, \dots, P_r . Then it is easy to see that there are properly embedded pairwise disjoint arcs a_1, a_2, \dots, a_r in P with endpoints on A so that $S \setminus \cup_j a_j$ deformation retracts onto $\overline{S \setminus P}$: just take a suitable cocore a_j of each 1-handle P_j in P (see Figure 3 for an example). It follows that $P \setminus \cup_j a_j$ is a disjoint union of some disks. (In fact $\{a_1, a_2, \dots, a_r\}$ represents a basis of $H_1(P, A)$.)

For $1 \leq j \leq r$, let α_j denote the closed curve on ∂N which is obtained by gluing the arc a_j on $P \times \{0\}$ with the arc a_j on $P \times \{1\}$. Then we observe that α_j is a closed curve on ∂N which bounds the compressing disk $D_j^\alpha = (a_j \times [0, 1]) / \sim$ in N . Thus we conclude that we can find pairwise disjoint compressing disks in N each of whose boundary intersects the dividing set twice in such a way that when we cut along these disks we get a disjoint union of B^3 's with connected dividing sets after rounding the edges. The uniqueness of a tight contact structure on N with the assumed boundary conditions again follows from Eliashberg's Theorem 1.10.

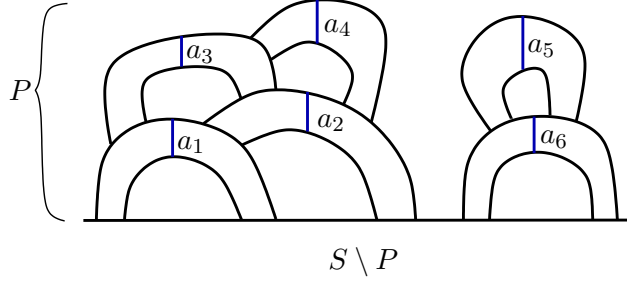


FIGURE 3. A basis of $H_1(P, A)$: cocores a_1, a_2, \dots, a_6 of the 1-handles in P

To prove the existence of such a tight contact structure on N we first observe that $\partial P \times \{1/2\}$ is the union of $A \times \{0\}$ and the arcs $c_i \times \{1/2\}$, for $1 \leq i \leq n$. Note that we can trivially embed N into H . Then we claim that the restriction to N of the above tight contact structure on H will have a convex boundary with the required dividing set. In order to prove our claim we observe that the dividing set on $P \times \{1\} \cup -P \times \{0\} = \partial N \cap \partial H$ is the set $A \times \{0\} = \partial N \cap (\partial S \times \{0\})$. The rest of ∂N consists of the disks $D_i = (c_i \times [0, 1]) / \sim$. Each one of these disks can be made convex so that the dividing set is a single arc since its boundary intersects the dividing set twice. It follows that the dividing set on ∂N is as required after rounding the edges. \square

Proposition 1.9 leads to the following definition of compatibility of a contact structure and a partial open book decomposition.

Definition 1.11. Let (M, Γ) be the balanced sutured manifold associated to a partial open book decomposition (S, P, h) . A contact structure ξ on (M, Γ) is said to be compatible with (S, P, h) if it satisfies conditions (1), (2) and (3) stated in Proposition 1.9.

Definition 1.12. Two partial open book decompositions (S, P, h) and $(\tilde{S}, \tilde{P}, \tilde{h})$ are isomorphic if there is a diffeomorphism $f : S \rightarrow \tilde{S}$ such that $f(P) = \tilde{P}$ and $\tilde{h} = f \circ h \circ (f^{-1})|_{\tilde{P}}$.

Remark 1.13. It follows from Proposition 1.9 that every partial open book decomposition has a unique compatible contact structure, up to isotopy, on the balanced suture manifold associated to it, such that the dividing set of the convex boundary is isotopic to the suture. Moreover if (S, P, h) and $(\tilde{S}, \tilde{P}, \tilde{h})$ are isomorphic partial open book decompositions, then the associated compatible contact 3-manifolds (M, Γ, ξ) and $(\tilde{M}, \tilde{\Gamma}, \tilde{\xi})$ are also isomorphic.

The definition of a positive stabilization of a partial open book decomposition in page 9 of [9] can be interpreted as follows.

Definition 1.14. *Let (S, P, h) be a partial open book decomposition. A partial open book decomposition (S', P', h') is called a positive stabilization of (S, P, h) if there is a properly embedded arc s in S such that*

- S' is obtained by attaching a 1-handle to S along ∂s ,
- P' is defined as the union of P and the attached 1-handle,
- $h' = R_\sigma \circ h$, where the extension of h to S' by identity is also denoted by h and R_σ denotes the right-handed Dehn twist along the closed curve σ which is the union of s and the core of the attached 1-handle.

The effect of positively stabilizing a partial open book decomposition on the associated sutured manifold and the compatible contact structure is taking a connected sum with (S^3, ξ_{std}) away from the boundary.

We now digress to review basic definitions and properties of Heegaard diagrams of sutured manifolds (cf. [6]). A sutured Heegaard diagram is given by (Σ, α, β) , where the Heegaard surface Σ is a compact oriented surface with nonempty boundary and $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ and $\beta = \{\beta_1, \beta_2, \dots, \beta_n\}$ are two sets of pairwise disjoint simple closed curves in $\Sigma \setminus \partial\Sigma$. Every sutured Heegaard diagram (Σ, α, β) , uniquely defines a sutured manifold (M, Γ) as follows: Let M be the 3-manifold obtained from $\Sigma \times [0, 1]$ by attaching 3-dimensional 2-handles along the curves $\alpha_i \times \{0\}$ and $\beta_j \times \{1\}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. The suture Γ on ∂M is defined by the set of curves $\partial\Sigma \times \{1/2\}$ (see Remark 1.4).

In [6], Juhász proved that if (M, Γ) is defined by (Σ, α, β) , then (M, Γ) is balanced if and only if $|\alpha| = |\beta|$, the surface Σ has no closed components and both α and β consist of curves linearly independent in $H_1(\Sigma, \mathbb{Q})$. Hence a sutured Heegaard diagram (Σ, α, β) is called balanced if it satisfies the conditions listed above. We will abbreviate balanced sutured Heegaard diagram as balanced diagram from now on.

A partial open book decomposition of (M, Γ) gives a sutured Heegaard diagram (Σ, α, β) of $(M, -\Gamma)$ as follows: Let

$$\Sigma = P \times \{0\} \cup -S \times \{-1\} / \sim \subset \partial H$$

be the Heegaard surface. Observe that, modulo identifications,

$$\partial\Sigma = (\overline{\partial P \setminus \partial S}) \times \{0\} \cup -(\overline{\partial S \setminus \partial P}) \times \{-1\} \simeq -\Gamma.$$

As in the proof of Proposition 1.9, let a_1, a_2, \dots, a_r be properly embedded pairwise disjoint arcs in P with endpoints on A such that $S \setminus \cup_j a_j$ deformation retracts onto $\overline{S \setminus P}$. Then define two families $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ and $\beta = \{\beta_1, \beta_2, \dots, \beta_r\}$ of simple closed curves in the Heegaard surface Σ by $\alpha_j = a_j \times \{0\} \cup a_j \times \{-1\} / \sim$ and $\beta_j = b_j \times \{0\} \cup h(b_j) \times \{-1\} / \sim$, where b_j is an arc isotopic to a_j by a small isotopy such that

- the endpoints of a_j are isotoped along ∂S , in the direction given by the boundary orientation of S ,

- a_j and b_j intersect transversely in one point x_j in the interior of S ,
- if we orient a_j , and b_j is given the induced orientation from the isotopy, then the sign of the intersection of a_j and b_j at x_j is $+1$.

(Σ, α, β) is a sutured Heegaard diagram of $(M, -\Gamma)$. Here the suture is $-\Gamma$ since $\partial\Sigma$ is isotopic to $-\Gamma$.

Lemma 1.15. *The balanced sutured manifold associated to a partial open book decomposition and the compatible contact structure are invariant under positive stabilization.*

Proof. Let (S, P, h) be a partial open book decomposition of (M, Γ) , s be a properly embedded arc in S , and (S', P', h') be the corresponding positive stabilization of (S, P, h) . Consider the sutured Heegaard diagram (Σ, α, β) of $(M, -\Gamma)$ given by (S, P, h) using properly embedded disjoint arcs a_1, a_2, \dots, a_r in P .

Let a_0 be the cocore of the 1-handle attached to S during stabilization. The endpoints of a_0 are on $A' = \partial P' \cap \partial S'$ and $S' \setminus \cup_{j=0}^r a_j$ deformation retracts onto $S' \setminus P' = S \setminus P$. Using the properly embedded disjoint arcs $a_0, a_1, a_2, \dots, a_r$ in P' we get a sutured Heegaard diagram $(\Sigma', \alpha', \beta')$ of $(M', -\Gamma')$, where (M', Γ') is the sutured manifold associated to (S', P', h') . Observe that $\alpha' = \{\alpha_0\} \cup \alpha$, $\beta' = \{\beta_0\} \cup \beta$, and

$$\Sigma' = P' \times \{0\} \cup -S' \times \{-1\} / \sim \cong T^2 \# \Sigma .$$

Since h' is a right-handed Dehn twist along σ composed with the extension of h which is identity on $P' \setminus P$, α_0 intersects β_0 at one point and is disjoint from every other β_j . Therefore $(\Sigma', \alpha', \beta')$ is a stabilization of the Heegaard diagram (Σ, α, β) , and consequently $(M', \Gamma') \cong (M, \Gamma)$. The contact structure ξ' compatible with (S', P', h') is contactomorphic to ξ since ξ' is obtained from ξ by taking a connected sum with (S^3, ξ_{std}) away from the boundary. \square

2. RELATIVE GIROUX CORRESPONDENCE

The following theorem is the key to obtaining a description of a partial open book decomposition of (M, Γ, ξ) in the sense of Honda, Kazez and Matić.

Theorem 2.1 ([9], Theorem 1.1). *Let (M, Γ) be a balanced sutured manifold and let ξ be a contact structure on M with convex boundary whose dividing set $\Gamma_{\partial M}$ on ∂M is isotopic to Γ . Then there exist a Legendrian graph $K \subset M$ whose endpoints lie on $\Gamma \subset \partial M$ and a regular neighborhood $N(K) \subset M$ of K which satisfy the following:*

- (A) (i) $T = \overline{\partial N(K)} \setminus \partial M$ is a convex surface with Legendrian boundary.
(ii) For each component γ_i of ∂T , $\gamma_i \cap \Gamma_{\partial M}$ has two connected components.

(iii) There is a system of pairwise disjoint compressing disks D_j^α for $N(K)$ so that ∂D_j^α is a curve on T intersecting the dividing set Γ_T of T at two points and each component of $N(K) \setminus \cup_j D_j^\alpha$ is a standard contact 3-ball, after rounding the edges.

- (B) (i) Each component H of $\overline{M \setminus N(K)}$ is a handlebody (with convex boundary).
(ii) There is a system of pairwise disjoint compressing disks D_k^δ for H so that each ∂D_k^δ intersects the dividing set $\Gamma_{\partial H}$ of ∂H at two points and $H \setminus \cup_k D_k^\delta$ is a standard contact 3-ball, after rounding the edges.

Remark 2.2. Here note that we use a slightly different notation than the one that is used in Theorem 1.1 in [9] to denote the compressing disks. Their D_j' corresponds to our D_j^α in part (A) and their D_j^α corresponds to our D_k^δ in part (B). We hope that our notation here is consistent with our notation in the previous sections.

Remark 2.3. Given a balanced sutured manifold (M, Γ) , there exists a contact structure ξ on M which makes ∂M convex and realizes Γ as its diving set on ∂M . Conversely given a contact 3-manifold (M, ξ) (with convex boundary) whose diving set is denoted by Γ on ∂M , then (M, Γ) is a balanced sutured manifold.

Definition 2.4. A standard contact 3-ball is a tight contact 3-ball with a convex boundary whose dividing set is connected.

Based on Theorem 2.1, Honda, Kazez and Matic describe a partial open book decomposition on (M, Γ) in Section 2 of their article [9]. In this paper, for the sake of simplicity and without loss of generality, we will assume that M is connected. As a consequence $M \setminus N(K)$ in Theorem 2.1 is also connected.

We claim that their description gives a partial open book decomposition (S, P, h) , the balanced sutured manifold associated to (S, P, h) is isotopic to (M, Γ) , and ξ is compatible with (S, P, h) — all in the sense that we defined in this paper. In the rest of this section we prove these claims and Lemma 2.6 to obtain a proof of Theorem 0.1.

The tubular portion T of $-\partial N(K)$ in Theorem 2.1(A)(i) is split by its dividing set into positive and negative regions, with respect to the orientation of $\partial(M \setminus N(K))$. Let P be the positive region. Note that the negative region $T \setminus P$ is diffeomorphic to P . Since (M, Γ) is assumed to be a (balanced) sutured manifold, ∂M is divided into $R_+(\Gamma)$ and $R_-(\Gamma)$ by the suture Γ . Let $R_+ = R_+(\Gamma) \setminus \cup_i D_i$, where D_i 's are the components of $\partial N(K) \cap \partial M$ and let S be the surface which is obtained from $\overline{R_+}$ by attaching the positive region P . If we denote the dividing set of T by $A = \partial P \cap \partial S$, then it is easy to see that

$$N(K) \cong (P \times [0, 1]) / \sim$$

where $(x, t) \sim (x, t')$ for $x \in A$ and $t, t' \in [0, 1]$, such that the dividing set of $\partial N(K)$ is given by $\partial P \times \{1/2\}$.

In [9], Honda, Kazez and Matić observed that

$$\overline{M \setminus N(K)} \cong (S \times [-1, 0]) / \sim$$

where $(x, t) \sim (x, t')$ for $x \in \partial S$ and $t, t' \in [-1, 0]$, such that the dividing set of $\overline{M \setminus N(K)}$ is given by $\partial S \times \{0\}$.

Moreover the embedding $h : P \rightarrow S$ which is obtained by first pushing P across $N(K)$ to $T \setminus P \subset \partial(M \setminus N(K))$, and then following it with the identification of $\overline{M \setminus N(K)}$ with $(S \times [-1, 0]) / \sim$ is called the monodromy map in the Honda-Kazez-Matić description of a partial open book decomposition.

In conclusion, we see that the triple (S, P, h) satisfies the conditions in Definition 1.1:

(1) The compact oriented surface S is connected since we assumed that M is connected and it is clear that $\partial S \neq \emptyset$.

(2) The surface P is a proper subsurface of S such that S is obtained from $\overline{S \setminus P}$ by successively attaching 1-handles by construction.

(3) The monodromy map $h : P \rightarrow S$ is an embedding such that h fixes $A = \partial P \cap \partial S$ pointwise.

Next we observe that $N(K)$ (resp. $\overline{M \setminus N(K)}$) corresponds to N (resp. H) in our construction of the balanced sutured manifold associated to a partial open book decomposition proceeding Definition 1.1. The monodromy map h amounts to describing how $N = N(K)$ and $H = \overline{M \setminus N(K)}$ are glued together along the appropriate subsurface of their boundaries. This proves that the balanced sutured manifold associated to (S, P, h) is diffeomorphic to (M, Γ) .

Lemma 2.5. *The contact structure ξ in Theorem 2.1 is compatible with the partial open book decomposition (S, P, h) described above.*

Proof. We have to show that the contact structure ξ in Theorem 2.1 satisfies the conditions (1), (2) and (3) stated in Theorem 1.9 with respect to the partial open book decomposition (S, P, h) described above. We already observed that $N = N(K)$ and $H = \overline{M \setminus N(K)}$. Then

(1) The restrictions of the contact structure ξ onto $N(K)$ and $\overline{M \setminus N(K)}$ are tight by conditions (A)(iii) and (B)(ii) of Theorem 2.1, respectively. This is because in either case one obtains a standard contact 3-ball or a disjoint union of standard contact 3-balls by cutting the manifold along a collection of compressing disks each of whose boundary geometrically intersects the dividing set exactly twice.

(2) $\partial H = \partial(M \setminus N(K)) = (\partial M \setminus \cup_i D_i) \cup T$ is convex by the convexity of ∂M and the convexity of T (condition (A)(i) in Theorem 2.1). Its dividing set is the union of those of $\partial M \setminus \cup_i D_i$ and T , hence it is isotopic to $(\overline{\partial S \setminus \partial P}) \times \{0\} \cup A \times \{0\} = \partial S \times \{0\}$.

(3) $\partial N = \partial N(K) = \cup_i D_i \cup T$ is convex by the convexity of $D_i \subset \partial M$ and the convexity of T . Its dividing set is the union of those of D_i 's and T , hence it is isotopic to $(\overline{\partial P \setminus \partial S}) \times \{1/2\} \cup A \times \{0\} = \partial P \times \{1/2\}$. \square

The following lemma is the only remaining ingredient in the proof of Theorem 0.1.

Lemma 2.6. *Let (S, P, h) be a partial open book decomposition, (M, Γ) be the balanced sutured manifold associated to it, and ξ be a compatible contact structure. Then (S, P, h) is given by the Honda-Kazez-Matić description.*

Proof. Consider the graph K in P that is obtained by gluing the core of each 1-handle in P (see Figure 4 for example).

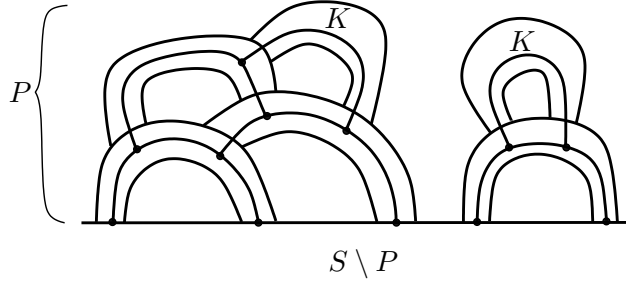


FIGURE 4. Legendrian graph K in P

It is clear that P retracts onto K . We will denote $K \times \{1/2\} \subset P \times \{1/2\}$ also by K . We can first make $P \times \{1/2\}$ convex and then Legendrian realize K with respect to the compatible contact structure ξ on $N \subset M$. This is because each component of the complement of K in P contains a boundary component (see Remark 4.30 in [3]). Hence K is a Legendrian graph in (M, ξ) with endpoints in $\partial P \times \{1/2\} \setminus \partial S \times \{0\} \subset \Gamma \subset \partial M$ such that $N = P \times [0, 1] / \sim$ is a neighborhood $N(K)$ of K in M . Then all the conditions except (A)(i) in Theorem 2.1 on $N(K) = N$ and $\overline{M \setminus N(K)} = H$ are satisfied because of the way we constructed ξ in Proposition 1.9. Since ∂N is convex T is also convex. It remains to check that the boundary of the tubular portion T of N is Legendrian. Note that each component of this boundary $\partial D_i = \partial(c_i \times [0, 1]) \subset \partial N$ is identified with $\gamma_i = c_i \times \{0\} \cup h(c_i) \times \{-1\}$ in the convex surface $\partial H = S \times \{0\} \cup -S \times \{-1\}$. Since each γ_i intersects the dividing set $\Gamma_{\partial H} = S \times \{0\}$ of ∂H transversely at two points $\partial c_i \times \{0\}$, the set $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ is non-isolating in ∂H and hence we can use the Legendrian Realization Principle to make each γ_i Legendrian. \square

Proof of Theorem 0.1. By Proposition 1.9 each partial open book decomposition is compatible with a unique compact contact 3-manifold with convex boundary up to contact isotopy.

This gives a map from the set of all partial open book decompositions to the set of all compact contact 3-manifolds with convex boundary and by Remark 1.13 this map descends to a map from the set of isomorphism classes of all partial open book decompositions to the set of isomorphism classes of all compact contact 3-manifolds with convex boundary. Moreover by Lemma 1.15 this gives a well-defined map Ψ from the isomorphism classes of all partial open book decompositions modulo positive stabilization to that of isomorphism classes of compact contact 3-manifolds with convex boundary. On the other hand, Honda-Kazez-Matić description gives a well-defined map Φ in the reverse direction by Theorems 1.1 and 1.2 in [9]. Furthermore, $\Psi \circ \Phi$ is identity by Lemma 2.5 and $\Phi \circ \Psi$ is identity by Lemma 2.6. \square

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