EMBEDDING FILLINGS OF CONTACT 3-MANIFOLDS

BURAK OZBAGCI

ABSTRACT. In this survey article we describe different ways of embedding fillings of contact 3-manifolds into closed symplectic 4-manifolds.

0. INTRODUCTION

One of the most exciting advances regarding the topology of 3-manifolds in 2004 was the solution of the "Property P" conjecture by Kronheimer and Mrowka [KrMr-04]. Namely, they proved that no surgery on a knot in S^3 can produce a counter-example to the Poincaré conjecture. The last ingredient in their proof was supplied by a recent theorem of Eliashberg [El-04]: *Any weak filling of a contact 3-manifold can be embedded symplectically into a closed symplectic 4-manifold*. This particular way of embedding a weak filling into a closed symplectic 4-manifold was also used by Ozsváth and Szabó [OzSz-04] to show that their (appropriately twisted) contact Heegaard Floer invariant of a fillable contact structure does not vanish.

In order to prove his theorem Eliashberg attaches a symplectic 2-handle along the binding of an open book compatible with the given weakly fillable contact structure such that the other end of the cobordism given by this symplectic 2-handle attachment symplectically fibres over S^1. Then he fills in this symplectic fibration by a symplectic Lefschetz fibration over D^2 to obtain a symplectic embedding of a weak filling into a closed symplectic 4-manifold. Note that the method of construction in [El-04] takes its roots from the one considered in [AO-02].

Eliashberg's theorem was obtained independently by Etnyre [Et-04] using different methods. The first step in Etnyre's construction is to embed a weak filling into a weak filling of an integral homology sphere. Note that, from the *surgery* point of view, this step also fairly easily follows from Stipsicz's results in [St-03]. Then one can modify the symplectic form near the boundary so that it becomes a strong filling (cf. [El-91], [OO-99]). This is just a homological argument. Now the problem is reduced to finding an embedding of a strong

2000 *Mathematics Subject Classification.* 57R17, 57R40, 57R65.

Key words and phrases. Symplectic filling, Lefschetz fibration, Contact structure, Open book, Property P.

The author was partially supported by the Turkish Academy of Sciences.

filling. The strategy at this point is to find a concave filling to cap off the convex boundary of this strong filling from the "other side". One way of finding this concave filling is to further reduce the problem (cf. [EH-02]) to the existence of a symplectic embedding of a Stein filling into a closed symplectic 4-manifold, which was already provided by Lisca and Matic [LM-97]. Alternatively, one can proceed with constructing an explicit concave filling (cf. [Ga-02b]) obtained by a careful investigation of the monodromies of the open books compatible with different types of symplectic and contact surgeries.

The purpose of this survey article is to describe and compare embeddings due to Eliashberg and Etnyre and discuss some previous work on the subject. We note that there are now many ways of embedding a weak filling symplectically into a closed symplectic 4-manifold. In Section 7 we construct an embedding which is obtained by combining the various ideas developed in the article. We would like to point out that these embeddings are constructed by making use of a recent theory developed by Giroux [Gi-02] which establishes a (one-to-one) correspondence between open book decompositions of 3-manifolds and contact structures.

We would also like to point out that in [Et-05] Etnyre gives quite a bit of details of the arguments in [Et-04] including the necessary background. In addition, there is another recent survey article by Geiges [Ge-05], where he emphasizes the role that contact geometry has played in the proof of "Property P" for knots.

1. Open book decompositions and contact structures

We will assume throughout this paper that a contact structure $\xi = \ker \alpha$ is coorientable (i.e., α is a global 1-form) and positive (i.e., $\alpha \wedge d\alpha > 0$) unless otherwise stated. In the following we describe the compatibility of an open book decomposition with a given contact structure on a 3-manifold.

Suppose that for a link L in a 3-manifold Y the complement $Y \setminus L$ fibers as $\pi \colon Y \setminus L \to S^1$ such that the fibers are interiors of Seifert surfaces of L. Then (L, π) is an *open book decomposition* (or just an *open book*) of Y. For each $t \in S^1$, the Seifert surface $F = \overline{\pi^{-1}(t)}$ is called a *page*, while L the *binding* of the open book. The monodromy of the fibration π is called the *monodromy* of the open book decomposition.

Any locally trivial bundle with fiber F (a compact oriented surface) over an oriented circle is canonically isomorphic to the fibration

$$\frac{I \times F}{(1,x) \sim \big(0, h(x)\big)} \to \frac{I}{\partial I} \approx S^1$$

for some orientation preserving self-diffeomorphism h of F. In fact, h is determined by the fibration up to isotopy and conjugation by an orientation preserving self-diffeomorphism

of F. The isotopy class represented by h is called the (topological) monodromy of the fibration.

The *mapping class group* Γ_F of F is defined as the quotient of the group of orientation preserving self-diffeomorphisms of F fixing ∂F pointwise modulo isotopies fixing ∂F pointwise. Given a compact oriented surface F with nonempty boundary and $h \in \Gamma_F$, then we can consider $F(h) = I \times F/(1,x) \sim \big(0, h(x)\big)$ which is called a mapping torus. Note that since h is the identity on ∂F, the boundary $\partial F(h)$ can be canonically identified with r copies of $T^2 = S^1 \times S^1$, where the first S^1 factor is identified with $I/\partial I$ and the second one is identified with a component of ∂F. Hence $F(h)$ can be completed to a closed 3-manifold Y equipped with an open book decomposition by gluing in r copies of $D^2 \times S^1$ to $F(h)$ so that ∂D^2 is identified with $S^1 = I/\partial I$ and the S^1 factor in $D^2 \times S^1$ is identified with a boundary component of ∂F. In conclusion, an element $h \in \Gamma_F$ determines a 3-manifold together with an open book decomposition on it.

Theorem 1 (Alexander [Al-23])**.** *Every closed and oriented 3-manifold admits an open book decomposition.*

The contact condition $\alpha \wedge d\alpha > 0$ can be strengthened in the presence of an open book decomposition on Y by requiring that $\alpha > 0$ on the binding and $d\alpha > 0$ on the pages.

Definition 2. *An open book decomposition of a 3-manifold Y and a contact structure ξ on Y are called* compatible *if ξ can be represented by a contact form α such that the binding is a transverse link, $d\alpha$ is a symplectic form on every page and the orientation of the transverse binding induced by α agrees with the boundary orientation of the pages.*

Theorem 3 (Giroux [Gi-02])**.** *Every contact 3-manifold admits a compatible open book (with a connected binding).*

We refer the reader to [Et-05] and [OzSt-04] for more on the correspondence between open books and contact structures.

2. Lefschetz fibrations

Suppose that X and Σ are given compact, oriented, connected 4- and 2-dimensional manifolds. A smooth map $f\colon X \to \Sigma$ is called a ***Lefschetz fibration*** if df is onto with finitely many exceptions $\{p_1, \dots, p_k\} = C \subset int\ X$ (called the set of critical points), the map f is a locally trivial surface bundle over $\Sigma - f(C)$ and around $p_i \in C$ and $q_i = f(p_i) \in f(C)$ there are orientation preserving complex charts U_i and V_i, respectively, on which f is of the form $z_1^2 + z_2^2$.

Notice that the manifolds X and Σ might have boundaries. If the typical fiber $f^{-1}(t)$ is a closed surface then $f^{-1}(\partial\Sigma) = \partial X$, but the definition also allows $f^{-1}(t)$ to have boundary,

in which case $f^{-1}(\partial\Sigma)$ forms only part of ∂X. We call the fibers $f^{-1}(q_i)$ ($q_i \in f(C)$) *singular*, while the other fibers are called *regular*. Two Lefschetz fibrations $f\colon X \to \Sigma$ and $f'\colon X' \to \Sigma'$ are called *equivalent* if there are diffeomorphisms $\Phi\colon X \to X'$ and $\phi\colon \Sigma \to \Sigma'$ such that $f' \circ \Phi = \phi \circ f$.

By definition removing the singular fibers turns a Lefschetz fibration into a fiber bundle with a connected base space. Consequently all but finitely many fibers of a Lefschetz fibration are smooth, compact and oriented surfaces, all of which have the same diffeomorphism type. We will assume that there is at most one critical point on each fiber and no fiber contains an embedded 2-sphere of self-intersection number -1. Each critical point of a Lefschetz fibration corresponds to an embedded circle called a *vanishing cycle* in a nearby regular fiber, and the singular fiber is obtained by collapsing the vanishing cycle to a point.

The boundary of a regular neighborhood of a singular fiber is a surface bundle over circle. In fact, a singular fiber can be described by the monodromy of this surface bundle which turns out to be a right-handed Dehn twist along the corresponding vanishing cycle. Once we fix an identification of a regular fiber with a compact, connected, oriented surface F, the topology of the Lefschetz fibration is determined by its *monodromy representation* $\Psi\colon \pi_1(\Sigma - \{\text{critical values}\}) \to \Gamma_F$. In case $\Sigma = D^2$ the monodromy along $\partial D^2 = S^1$ is called the *total monodromy* of the fibration; according to the above said it is the product of right-handed Dehn twists corresponding to the singular fibers.

A Lefschetz fibration over S^2 with closed fibers can be decomposed into two Lefschetz fibrations over D^2, one of which is trivial. Hence a Lefschetz fibration over S^2 is determined by a relator in the mapping class group. Conversely, given a product of right-handed Dehn twists in the mapping class group we can construct the corresponding Lefschetz fibration over D^2, and if the given product of right-handed Dehn twists is isotopic to identity (and $g \geq 2$) then the fibration extends uniquely over S^2. The monodromy presentation also provides a handlebody decomposition of a Lefschetz fibration over D^2: we attach 2-handles to $F \times D^2$ along the vanishing cycles with framing -1 relative to the framing the circle inherits from the fiber. (For a more detailed introduction to the theory of Lefschetz fibrations see [GS-99] and [OzSt-04].)

3. Different types of fillings of contact 3-manifolds

In this section we give definitions of different types of symplectic fillings of contact 3-manifolds. A symplectic 4-manifold (X, ω) will be assumed to be oriented by $\omega \wedge \omega$.

3.1. **Weak filling.** A contact 3-manifold (Y, ξ) is said to be *weakly fillable* if there is a compact symplectic 4-manifold (W, ω) such that $\partial W = Y$ as oriented manifolds and $\omega|_\xi > 0$. In this case we say that (W, ω) is a *weak filling* of (Y, ξ).

3.2. **Strong filling.** A contact 3-manifold (Y, ξ) is said to be *strongly fillable* if there is a compact symplectic 4-manifold (W, ω) such that $\partial W = Y$ as oriented manifolds, ω is exact near the boundary and its primitive α (i.e., a 1-form with $d\alpha = \omega$) can be chosen in such a way that $\ker(\alpha|_{\partial W}) = \xi$. In this case we say that (W, ω) is a *strong filling* of (Y, ξ). Clearly a strong filling is a weak filling by definition.

Suppose that (W, ω) is a compact symplectic 4-manifold with nonempty boundary $\partial W = Y$ and there exists a Liouville vector field v (i.e., $\mathcal{L}_v \omega = \omega$) defined in a neighborhood of and transverse to Y. Then v induces a contact structure $\xi = \ker \alpha$ on Y where $\alpha = \iota_v \omega|_Y$ is a contact 1-form. If v points out of W along Y then we say that (W, ω) is a *convex filling* of (Y, ξ), and (Y, ξ) is said to be the *convex boundary* of (W, ω). It is easy to see that the notion of a convex filling is the same as the notion of a strong filling. If v points into W along Y, on the other hand, then we say that (W, ω) is a *concave filling* of (Y, ξ) and (Y, ξ) is said to be the *concave boundary* of (W, ω). Here notice that if v points out of W then ξ is a positive contact structure on Y, while if v points into W then ξ is a positive contact structure on $-Y$.

If a compact symplectic 4-manifold W has multiple boundary components and if Y is a boundary component of W which satisfies the definition of convexity (concavity, resp.) above then we say that Y is a convex (concave, resp.) boundary component of W. It is quite possible that a symplectic 4-manifold W can have a convex (concave, resp.) boundary component Y without W being a filling of Y, since the other components of W may not be convex (concave, resp.).

3.3. **Stein filling.** A compact 4-manifold W with nonempty boundary $\partial W = Y$ is called a *Stein domain* if there is a Stein surface X with plurisubharmonic function $\varphi\colon X \to [0, \infty)$ such that $W = \varphi^{-1}\big([0, t]\big)$ for some regular value t. So a compact manifold with boundary (and a complex structure J on its interior) is a Stein domain if it admits a proper plurisubharmonic function φ which is constant on the boundary. Then the complex line distribution induced by J is a contact structure ξ on Y. In this case we say that the contact 3-manifold (Y, ξ) is *Stein fillable* and (W, J) is a called a *Stein filling* of (Y, ξ). It is easy to verify that a Stein filling is a strong filling. In fact, $dJ^*(d\varphi)$ induces a Kähler structure on (W, J). More generally, a cobordism W (with boundary $-Y_1 \cup Y_2$) is a *Stein cobordism* if W is a complex cobordism with a plurisubharmonic function $\varphi\colon W \to \mathbb{R}$ such that $\varphi^{-1}(t_i) = Y_i$, for some regular values $t_1 < t_2$.

We refer the reader to [Et-98] and [OzSt-04] for a further detailed discussion of different types of fillings of contact 3-manifolds.

4. Embedding a Stein filling

The first result in the literature about embedding a filling of a contact 3-manifold into a closed symplectic 4-manifold was obtained by Lisca and Matic. Recall that a Stein filling (i.e., a Stein domain) admits a Kähler form $dJ^*(d\varphi)$ which is an exact symplectic form, where φ is the plurisubharmonic function defining the Stein filling.

Theorem 4 (Lisca-Matic [LM-97])**.** *A Stein filling admits a Kähler embedding into a (minimal) compact Kähler surface X (of general type), such that the pull-back of the Kähler form on X is the exact symplectic form on the Stein filling.*

Apparently what motivated Lisca and Matic to construct such an embedding was their search for a method to distinguish tight contact structures. Using Seiberg-Witten theory coupled with their embedding result, Lisca and Matic were able to show that for any positive integer n, there exists a homology 3-sphere with at least n homotopic but non-isomorphic tight contact structures. Lisca and Matic use analytical tools in the construction of their embedding and the starting point of their embedding is given by a holomorphic embedding of a Stein domain into an affine algebraic manifold with trivial normal bundle (cf. [DLS-94]). Roughly speaking, the idea here is to approximate analytical maps by algebraic ones, namely by polynomials.

A very different approach to embed a Stein filling *smoothly* into a closed symplectic 4-manifold was presented in [AO-02]. The construction in [AO-02] is topologically more explicit than the method of Lisca and Matic although the result is weaker since only the smoothness of the embedding is clear from the presentation.

The simple construction in [AO-02] is based on a theorem of Loi and Piergallini ([LP-01], cf. also [AO-01]) which says that every Stein domain admits a Lefschetz fibration over D^2, whose vanishing cycles are homologically non-trivial on the respective nearby regular fibers. Notice that the fibers of such a Lefschetz fibration will necessarily have non-empty boundaries. It is easy to see that the boundary a Lefschetz fibration (whose fibers have non-empty boundaries) admits a canonical open book decomposition and we can assume that the binding of this open book is connected. To embed a Stein filling (which has a Lefschetz fibration structure) into a closed symplectic 4-manifold, we first attach a 2-handle to the binding of the open book in the boundary of this Lefschetz fibration over D^2 to get a Lefschetz fibration over D^2 with closed fibers. Then we extend this fibration to a Lefschetz fibration over S^2. The resulting 4-manifold is known to be symplectic by a result of Gompf ([GS-99]). This construction gives a smooth embedding of a Stein filling into a closed symplectic 4-manifold.

5. EMBEDDING A STRONG FILLING

In [EH-02], Etnyre and Honda proved that every contact 3-manifold has (infinitely many distinct) concave fillings. Their proof was based on the embedding result of Lisca and Matic we discussed in the previous section. In [Ga-02b], Gay proved the same existence result (independent of the Lisca-Matic embedding) by presenting a method to explicitly construct, handle by handle, a concave filling of a given contact 3-manifold. A symplectic embedding of a strong filling of a contact 3-manifold into a closed symplectic 4-manifold trivially follows from Proposition 5.

Proposition 5 (Etnyre-Honda [EH-02], Gay [Ga-02b])**.** *Any contact 3-manifold admits a concave filling.*

Proof. We will describe a proof (cf. [OzSt-04]) which is very similar to the one given in [EH-02]. The difference here is that we rather do not translate contact (± 1)-surgeries along Legendrian knots into the monodromy language of open books.

Given an arbitrary contact 3-manifold (Y, ξ). Let α be a contact 1-form for ξ. Consider a compact piece $(W = Y \times I, \omega = d(e^t \alpha))$ of the symplectization of (Y, ξ). It is easy to see that $Y \times \{1\}$ is a convex boundary component of (W, ω) while $Y \times \{0\}$ is a concave boundary component. Our strategy here will be to cap off the convex end of (W, ω) obtaining a concave filling of $Y = Y \times \{0\}$.

In [DG-04], Ding and Geiges proved that every (closed) contact 3-manifold (Y, ξ) can be given by a contact (± 1)-surgery on a Legendrian link $\mathbb{L}$ in the standard contact S^3. Here the surgery coefficients are measured with respect to the contact framing. Let $\mathbb{L}^{\pm} \subset \mathbb{L}$ denote the set of the components of the link $\mathbb{L}$ with (± 1)-surgery coefficients, respectively. Let $\mathbb{K}$ denote the Legendrian link we get by considering Legendrian push-offs of the components of $\mathbb{L}^+$.

Proposition 6 (Weinstein [We-91])**.** *Let (W, ω) be a compact symplectic 4-manifold with a convex boundary component (Y, ξ). A 2-handle can be attached symplectically to (W, ω) along a Legendrian knot $L \subset (Y, \xi)$ in such a way that the symplectic structure extends to the 2-handle and the new symplectic 4-manifold $(\widetilde{W}, \widetilde{\omega})$ has a convex boundary component $(\widetilde{Y}, \widetilde{\xi})$, where $(\widetilde{Y}, \widetilde{\xi})$ is given by contact (-1)-surgery (i.e., Legendrian surgery) along $L \subset (Y, \xi)$.*

Thus when we attach symplectic 2-handles to (W, ω) along the knots of $\mathbb{K} \subset Y = Y \times \{1\}$ we get a symplectic 4-manifold (W', ω') with a convex boundary component (Y', ξ') by Proposition 6. We observe that the contact manifold (Y', ξ') can be given by a Legendrian surgery along $\mathbb{L}^-$, since a combination of a contact $(+1)$-surgery on a Legendrian knot in $\mathbb{L}^+$ and a contact (-1)-surgery on its push-off in $\mathbb{K}$ cancels out (cf. [DG-04]). We note that

the cancellation of these contact (± 1)-surgeries just corresponds to the cancellation of a right-handed Dehn twist along a curve with a left-handed Dehn twist along a curve parallel to it in the monodromy of an open book in the proof of Etnyre and Honda [EH-02].

Consequently (Y', ξ') is Stein fillable by a result of Eliashberg [El-90] since it is obtained from the standard contact S^3 via Legendrian surgeries only. Consider a Stein filling (W'', J) of (Y', ξ') and embed this filling into a closed symplectic 4-manifold (Z, ω_Z) using Theorem 4. Then since a Stein filling is a convex filling by definition, $(Z \setminus \text{int } W'')$ will be a concave filling of (Y', ξ'). Hence we conclude that

$$(W', \omega') \bigcup_{(Y', \xi')} (Z \setminus \text{int } W'', \omega_Z)$$

is a concave filling of (Y, ξ), which is illustrated in Figure 1. Here we use Lemma 11 to glue these symplectic 4-manifolds symplectically.

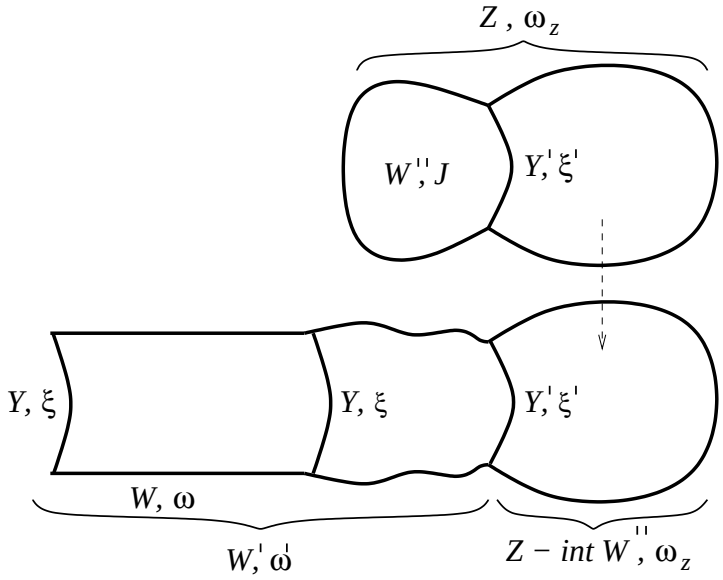

FIGURE 1. A concave filling of (Y, ξ)

□

Next we will discuss another proof of Theorem 5 given in [Et-05] which is not based on the embedding of Lisca and Matic. This method of proof is essentially due to Gay ([Ga-02b]) except for a slight short-cut at the end. We first collect below a few results that we will need. We denote by t_β a right-handed Dehn twist about a curve β on a surface F.

Lemma 7 (Wajnryb [Wa-99])**.** *The relation* $t_c = (t_{a_1} \circ t_{a_2} \circ \cdots \circ t_{a_{2g-1}} \circ t_{a_{2g}})^{4g+2}$ *holds in the mapping class group* Γ_F*, where* a_i*'s are the curves on a genus* g *surface* F *with one boundary component depicted in Figure 2 and* c *is a curve parallel to* ∂F*.*

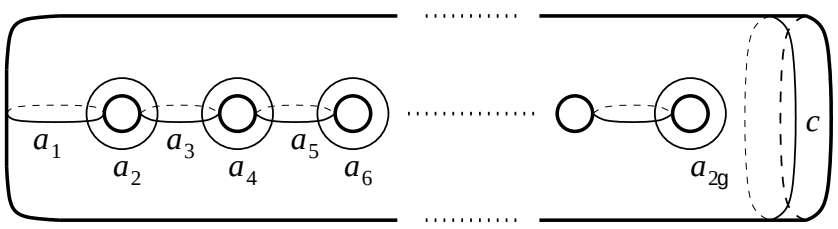

FIGURE 2. Genus g surface F with boundary

Lemma 8. *Any element* ϕ *of the mapping class group of a surface* F *with one boundary component can be expressed as* $\phi = t_c^m \circ t_{\gamma_1}^{-1} \circ \cdots \circ t_{\gamma_n}^{-1}$ *for some* $m \in \mathbb{Z}$ *and some non-separating curves* $\gamma_i \subset F$*, where* c *is a curve parallel to* ∂F*.*

Proof. We can express t_{a_1} as a product of non-separating left-handed Dehn twists and t_c by Lemma 7. Therefore any non-separating right-handed Dehn twist — being conjugate to t_{a_1} — is a product of non-separating left-handed Dehn twists and t_c. This finishes the proof since it is well-known that the mapping class group of a surface with one boundary component is generated by non-separating Dehn twists.

□

Lemma 9 (Kanda [Ka-97])**.** *A non-separating curve* γ *on a convex surface in a contact 3-manifold can be made Legendrian by isotoping the surface through convex surfaces such that the contact framing of* γ *agrees with its surface framing.*

Lemma 10 (Gay [Ga-02b])**.** *Given a Legendrian knot* L *on a page of an open book* $\mathfrak{obb}_\xi$ *compatible with* (Y, ξ)*. Let* $h \in \Gamma_F$ *denote the monodromy of* $\mathfrak{obb}_\xi$*. Then a contact* (-1)*-surgery on* L *induces a contact structure* ξ' *compatible with the open book* $\mathfrak{obb}_{\xi'}$ *whose monodromy is given by* $h' = h \circ t_L \in \Gamma_F$*.*

Lemma 11. *If* (Y, ξ) *is a convex boundary component of* (W_1, ω_1) *and is a concave boundary component of* (W_2, ω_2) *then we can glue* (W_1, ω_1) *and* (W_2, ω_2) *symplectically along their common boundary component* (Y, ξ)*.*

The result above was first explicitly stated in [Et-98] although it was implicit in Eliashberg's work in [El-96]. Note that after gluing one of the symplectic forms needs to be scaled appropriately.

We are now ready to describe a second proof of Theorem 5. Consider the compact piece (W,ω) of the symplectization of (Y,ξ) as in the proof above. Let $\mathfrak{ob}_\xi$ be an open book decomposition of Y with a connected binding which is compatible with ξ . Let ϕ be the monodromy of this open book. Now use Lemma 8 to write $\phi = t_c^m \circ t_{\gamma_1}^{-1} \circ \cdots \circ t_{\gamma_n}^{-1}$. We can assume that the curve γ_n is a Legendrian curve which lies on a convex page of $\mathfrak{ob}_\xi$ by Lemma 9. Then contact (-1)-surgery along γ_n yields a contact structure which has a compatible open book whose monodromy is given by $\phi \circ t_{\gamma_n} = t_c^m \circ t_{\gamma_1}^{-1} \circ \cdots \circ t_{\gamma_{n-1}}^{-1}$ by Lemma 10.

We repeat this process for all the curves γ_i (for $i = n-1, \cdots, 1$) to obtain a contact 3-manifold (Y',ξ') whose compatible open book $\mathfrak{ob}_{\xi'}$ has monodromy t_c^m. Moreover we can assume that m is odd and $m \geq 1$, otherwise we can just perform some more contact (-1)-surgeries along a_i's (depicted in Figure 2), after making them Legendrian on distinct convex pages using Lemma 9.

On the other hand, by Proposition 6, a contact (-1)-surgery along a Legendrian knot L in a convex boundary component of a symplectic 4-manifold can be obtained by a symplectic 2-handle attachment along L. Hence there exists a symplectic 4-manifold (W',ξ') with a convex boundary component (Y',ξ') which is obtained from (W,ω) by attaching symplectic 2-handles along γ_i's in the convex end of (W,ω). Next we will prove that we can actually assume that $m = 1$.

We note that Proposition 6 is also true for attaching symplectic 1-handles. Namely, one can attach a symplectic 1-handle to a symplectic 4-manifold along two points on the binding of a compatible open book decomposition of a convex boundary component in such a way that the symplectic structure extends over the 1-handle. In addition the induced surgery on the convex boundary component corresponds to taking a connected sum with a copy of standard contact $S^1 \times S^2$. At the level of compatible open books, attaching a (4-dimensional) symplectic 1-handle to a convex boundary component along two points in the binding of a compatible open book corresponds to attaching a (2-dimensional) 1-handle to the page of that open book. Note that we extend the old monodromy by identity over the new 1-handles.

Let $g' = mg + \frac{1}{2}(m-1)$. Now we attach symplectic 1-handles to (W',ξ') so that the resulting compatible open book on the boundary has a page F' of genus g' with one boundary component. Let c' be a curve parallel to $\partial F'$ as shown in Figure 3.

Then by Lemma 7 we have

$$t_{c'} = (t_{a_1} \circ t_{a_2} \circ \cdots \circ t_{a_{2g'-1}} \circ t_{a_{2g'}})^{4g'+2}$$

$$= (t_{a_1} \circ t_{a_2} \circ \cdots \circ t_{a_{2g}} \circ t_{a_{2g+1}} \circ \cdots \circ t_{a_{2g'-1}} \circ t_{a_{2g'}})^{m(4g+2)} \in \Gamma_{F'}.$$

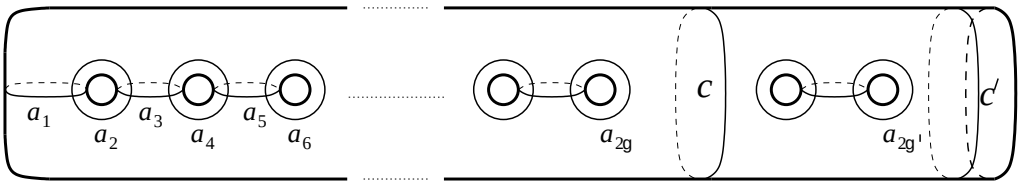

FIGURE 3. Genus g' surface F' with boundary

To simplify the notation we will denote the result of attaching symplectic 1-handles to (W', ω') again as (W', ω'). Now attach more symplectic 2-handles to (W', ω') along the Legendrian curves $a_{2g+1}, a_{2g+2}, \cdots, a_{2g'}$ sufficiently many times so that the resulting convex boundary has a compatible open book with monodromy $t_{c'}$. Here note that we are inserting (rather than appending as in Lemma 10) some right-handed Dehn twists, but nevertheless Lemma 10 holds true in this case (cf. [Ga-02b]). We will still denote the resulting symplectic 4-manifold by (W', ω'), to simplify the notation.

Summarizing the above discussion, by attaching symplectic 1- and 2-handles to (W, ω) we end up with a symplectic 4-manifold (W', ω') with a convex boundary component (Y', ξ') whose compatible open book $\mathfrak{ob}_{\xi'}$ has the following description: The page F' is a genus g' surface with one boundary component and the monodromy is a single right-handed Dehn twist along a curve c' parallel to $\partial F'$. Let $\widehat{F}$ denote the surface obtained by capping off the surface F' by gluing a 2-disk along $\partial F'$.

Lemma 12. *The 3-manifold Y' is a circle bundle over the surface $\widehat{F}$ with Euler number -1.*

Proof. This is a well-known result; we repeat the proof described in [AO-02]. Recall the relation

$$(t_{a_1} t_{a_2} \cdots t_{a_{2g'}})^{4g'+2} = t_{c'}$$

in the mapping class group $\Gamma_{F'}$. It induces a relation

$$(t_{a_1} t_{a_2} \cdots t_{a_{2g'}})^{4g'+2} = 1.$$

in the mapping class group $\Gamma_{\widehat{F}}$. This later relation induces a Lefschetz fibration $f : X \to S^2$ admitting a section of square -1. Consider a neighborhood U of a regular fiber union this section. We observe that $\partial U = -Y$. This is because $X \setminus \operatorname{int} U$ is a Lefschetz fibration (with bounded fibers) with monodromy

$$(t_{a_1} \circ t_{a_2} \circ \cdots \circ t_{a_{2g'-1}} \circ t_{a_{2g'}})^{4g'+2} = t_{c'}.$$

Moreover U is obtained by plumbing a $D^2 \times \widehat{F}$ (a regular neighborhood of the fiber) and a disk bundle over S^2 with Euler number -1 (a regular neighborhood of the section). In Figure 4 we illustrated a handlebody diagram of the 4-manifold U.

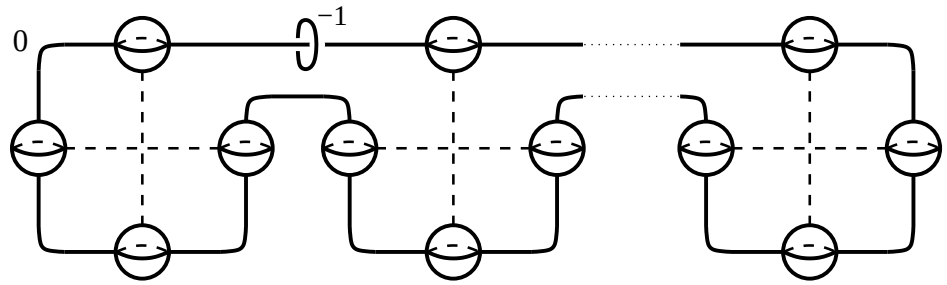

FIGURE 4. Plumbing a $D^2 \times \widehat{F}$ and a D^2-bundle over S^2 with Euler number -1

We can blow down the -1 sphere to get a disk bundle over $\widehat{F}$ with Euler number $+1$ (cf. Figure 5). Blowing down a -1 sphere changes the 4-manifold but the boundary 3-manifold remains the same (up to diffeomorphism). Note that the boundary of a disk bundle over $\widehat{F}$ with Euler number $+1$ is circle bundle over $\widehat{F}$ with Euler number $+1$. Our claim follows by reversing the orientations, since when we change the orientation of a circle bundle over $\widehat{F}$ with Euler number $+1$, we get a circle bundle over $\widehat{F}$ with Euler number -1.

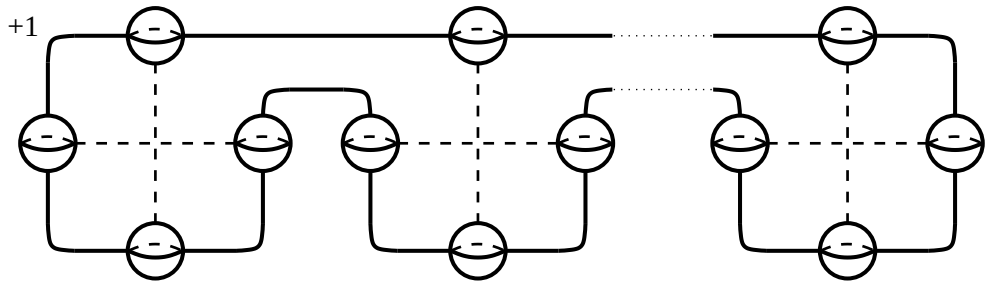

FIGURE 5. D^2-bundle over $\widehat{F}$ with Euler number $+1$

□

Now consider the disk bundle M over $\widehat{F}$ with Euler number 1. Then M admits a natural symplectic structure ω_M so that (M, ω_M) has a concave boundary (Y', ξ') (cf. [McD-91]).

Thus

$$(W', \omega') \bigcup_{(Y', \xi')} (M, \omega_M)$$

is a concave filling of (Y, ξ) by Lemma 11. This finishes the proof of Proposition 5.

Finally we would like to point out how Gay's proof in [Ga-02b] differs from the proof in [Et-05]. Consider the open book $\mathfrak{ob}_{\xi'}$ which is compatible with (Y', ξ') as above. Then Gay explains how to attach a symplectic 2-handle along the binding of $\mathfrak{ob}_{\xi'}$ with framing $+1$ relative to the page framing of the binding so that the resulting contact 3-manifold (Y'', ξ'') is also a *concave* boundary component of the symplectic cobordism given by the 2-handle attachment. Note that this operation has the affect of turning a convex boundary component of a symplectic 4-manifold to a concave boundary component. Denote the resulting symplectic 4-manifold obtained by attaching this symplectic 2-handle to (W', ω') by (W'', ω''). Moreover the monodromy of the open book compatible with (Y'', ξ'') is given by the identity map. This implies that (Y'', ξ'') is contactomorphic to the standard tight contact $(\sharp k\ S^1 \times S^2, \xi_{st})$. Note that there is a standard convex filling $(\sharp k\ S^1 \times D^3, \omega_{st})$ of $(\sharp k\ S^1 \times S^2, \xi_{st})$. Hence

$$(W'', \omega'') \bigcup_{(\sharp k\ S^1 \times S^2, \xi_{st})} (\sharp k\ S^1 \times D^3, \omega_{st})$$

is a concave filling of (Y, ξ) by Lemma 11.

Alternatively, a general method on how to find a natural open book on the boundary of any plumbed 4-manifold is given in [Ga-03]. Moreover Gay explains how to construct a symplectic structure on a "positive" plumbing 4-manifold whose concave boundary is compatible with this open book. In the situation above note that U is obtained by a positive plumbing of a $D^2 \times \widehat{F}$ with a disk bundle over S^2 with Euler number -1.

Proposition 13 (Etnyre-Honda [EH-02], Gay [Ga-02b])**.** *If (W, ω) is a strong filling of (Y, ξ) then W can be symplectically embedded into a closed symplectic 4-manifold.*

Proof. Suppose that (W, ω) is a strong filling of (Y, ξ). Consider a concave filling (W_1, ω_1) of (Y, ξ). Then we can glue (cf. Lemma 11) the symplectic manifolds (W, ω) and (W_1, ω_1) along their common boundary (Y, ξ) to get a closed symplectic 4-manifold including (W, ω) as a symplectic subdomain.

□

6. Embedding a weak filling

In this section we will give the most general embedding result that will cover the cases in Sections 4 and 5.

Theorem 14 (Eliashberg [El-04], Etnyre [Et-04])**.** *If (W,ω) is a weak filling of (Y,ξ) then W can be symplectically embedded into a closed symplectic 4-manifold.*

6.1. **Eliashberg's construction:** We first briefly outline Eliashberg's construction: Let (W,ω) be a weak filling of a contact 3-manifold (Y,ξ) and let $\mathfrak{ob}_\xi$ be an open book decomposition of Y (with a connected binding B) compatible with the contact structure ξ. Attach a symplectic 2-handle along $B \subset Y = Y \times \{1\}$ to an appropriate symplectic collar $Y \times I$ to obtain a cobordism with boundary $-Y \cup Y'$ such that Y' fibers over S^1 with symplectic fibers. Then fill in $Y' \to S^1$ by a symplectic Lefschetz fibration over D^2 to complete $W \cup H$ into a closed symplectic 4-manifold.

Eliashberg's idea above is to reduce the question of embedding a weak filling to a question of embedding a symplectic surface fibration over the circle. Notice that the binding B of $\mathfrak{ob}_\xi$ is transverse to ξ, so the crucial point of Eliashberg's construction is the way that he attaches a symplectic 2-handle along the transverse binding B. We would like to mention here that in [Ga-02a], Gay gives a general construction of attaching symplectic 2-handles along transverse knots.

Eliashberg's construction is "topologically" equivalent to the construction that was given in [AO-02] to embed a Stein filling *smoothly* into a closed symplectic 4-manifold.

Now we proceed with the details of Eliashberg's construction. We start with describing the symplectic 2-handle H to be attached along the transverse binding B. We identify $\mathbb{C}^2(z_1,z_2)$ with $\mathbb{R}^4(x_1,y_1,x_2,y_2)$ as usual: $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Let (r_i,φ_i) denote the polar coordinates in the z_i-plane for $i = 1,2$. Then the standard symplectic 2-form ω_0 on $\mathbb{R}^4$ is given by

$$\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = r_1 dr_1 \wedge d\varphi_1 + r_2 dr_2 \wedge d\varphi_2.$$

Let a be a positive real number and let $P = \{r_1 \le a, r_2 \le 1\} \subset \mathbb{C}^2$ be a polydisc. Now we define a domain $\widetilde{P} = \{r_1 \le g(r_2) : r_2 \in [0,1]\} \subset P$ for some non-increasing smooth function $g(t) : [0,1] \to [0,a]$ as shown in Figure 6, where $g([0,0.5]) = a$ and $g'(t) < 0$ for $t \in (0.5,1)$. We will determine the real number a and the particular form of the function $g(t)$ near $t = 1$ later in the proof. Here we can view $\widetilde{P}$ as obtained from the polydics P by smoothing its corners as shown in Figure 7.

We define $\Gamma = \{r_1 = g(r_2) : r_2 \in [0.5,1]\}$ as part of $\partial\widetilde{P}$. (There is a typo here in [El-04], $r_2 \in [0.5,1]$ not r_1.) We observe that Γ is diffeomorphic to $S^1 \times D^2$: As r_2 increases from 0.5 to 1 in the z_2-plane (with polar coordinates (r_2,φ_2)) the boundary of the disks $\{r_1 \le g(r_2)\}$ in the z_1-plane will shrink smoothly to a point according to the function g

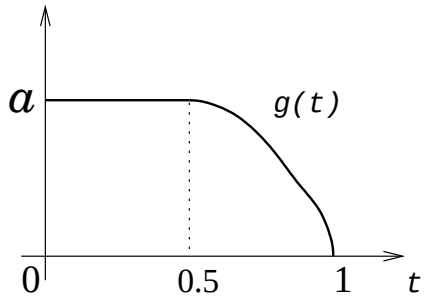

FIGURE 6. The graph of the smooth function g

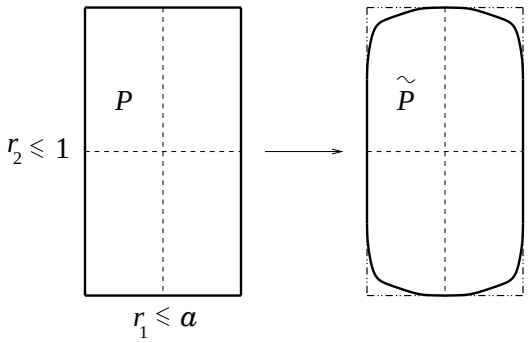

FIGURE 7. Smoothing the corners of the polydics

sweeping out a disk for each fixed φ_2. Here note that the core circle of Γ is parametrized by $\varphi_2 \in [0, 2\pi]$ for $r_2 = 1$ (cf. Figure 8).

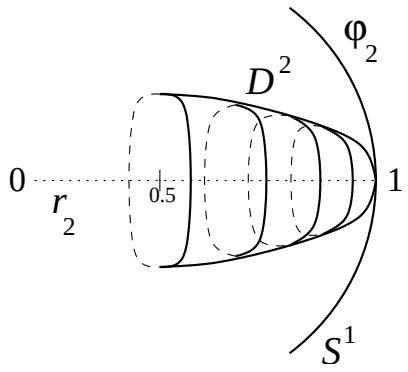

FIGURE 8. $\Gamma \simeq S^1 \times D^2$

Then we observe that $\gamma = \frac{1}{2}(r_1^2 d\varphi_1 + r_2^2 d\varphi_2)$ is a primitive of ω_0 on $\mathbb{R}^4$ and that

$$\gamma|_\Gamma = \frac{1}{2}(g^2(r_2)d\varphi_1 + r_2^2 d\varphi_2) = \frac{r_2^2}{2}(\frac{g^2(r_2)}{r_2^2}d\varphi_1 + d\varphi_2)$$

is a contact 1-form on Γ. This can be verified by a direct calculation where $r_2\ d\varphi_1 \wedge dr_2 \wedge d\varphi_2$ is a volume form on Γ. Also observe that the core circle of Γ is transverse to the contact structure $\ker(\gamma|_\Gamma)$ since $\gamma(\frac{\partial}{\partial\varphi_2})|_{\{r_2=1\}} = 0.5$.

Moreover $(\Gamma, \gamma|_\Gamma)$ is a convex boundary component of $(\widetilde{P}, \omega_0)$ but we would like to convert it to a concave component. So we apply the following trick. We embed $\widetilde{P}$ into a symplectic $S^2 \times D^2$ by a symplectomorphism and take the complement of the image in $S^2 \times D^2$. Let (S^2, σ_1) be a symplectic sphere with area 2π and (D^2, σ_2) be symplectic disk with area πa^2. Denote by $S^2_\pm$ the upper and lower hemispheres of area π, respectively. Then $\sigma_1 \oplus \sigma_2$ induces a symplectic form on $S^2_+ \times D^2$. Let

$$\phi : P \cong D^2 \times D^2 \to S^2_+ \times D^2 \subset S^2 \times D^2$$

be a symplectomorphism. From now on we will identify the symplectic form on P induced from ω_0 on $\mathbb{R}^4$ with the symplectic form $\sigma_1 \oplus \sigma_2$ on $S^2_+ \times D^2$ by the above symplectomorphism ϕ. Define the 2-handle H (see Figure 9) as

$$H = \overline{S^2 \times D^2 - \phi(\widetilde{P})}.$$

Now consider the boundary ∂H of the 2-handle H. We will denote $\phi(\Gamma)$ also by Γ to simplify the notation. Let

$$\Delta = \overline{\partial H \setminus \Gamma}.$$

Observe that Δ is fibered by discs $D_x = \widetilde{S^2_-} \times \{x\}$ for $x \in \partial D^2$, where we have $S^2_- \subset \widetilde{S^2_-} \subset S^2$. This is illustrated in Figure 9: Imagine the complement of $\phi(\widetilde{P})$ in $S^2 \times D^2$ restricted to ∂D^2. Notice here that $\widetilde{S^2_-}$ is symplectic (with respect to ω_0) with fixed area $\frac{7\pi}{4}$ (not $\frac{9\pi}{4}$ as mistakenly typed in [El-04]) for each $x \in \partial D^2$. This is precisely because of our identification of ω_0 with $\sigma_1 \oplus \sigma_2$.

Next we would like to find an appropriate way to attach this 2-handle H to $Y \times I$ by identifying Γ with a neighborhood U of the binding B of the compatible open book $\mathfrak{ob}_\xi$ in $Y \times \{1\} \subset Y \times I$. By Giroux [Gi-02], we can find coordinates (r, φ, u) near the binding B of $\mathfrak{ob}_\xi$ such that

$$U \cong [0, R] \times (\mathbb{R}^2/2\pi\mathbb{Z}) \times (\mathbb{R}^2/2\pi\mathbb{Z})$$

satisfying the following conditions:

(1) $\alpha|_U = h(r)(du + r^2 d\varphi)$ for some positive function h defined on $[0, R]$ such that $h(r) - h(0) = -r^2$ near $r = 0$, and $h'(r) < 0$ for all $r > 0$,

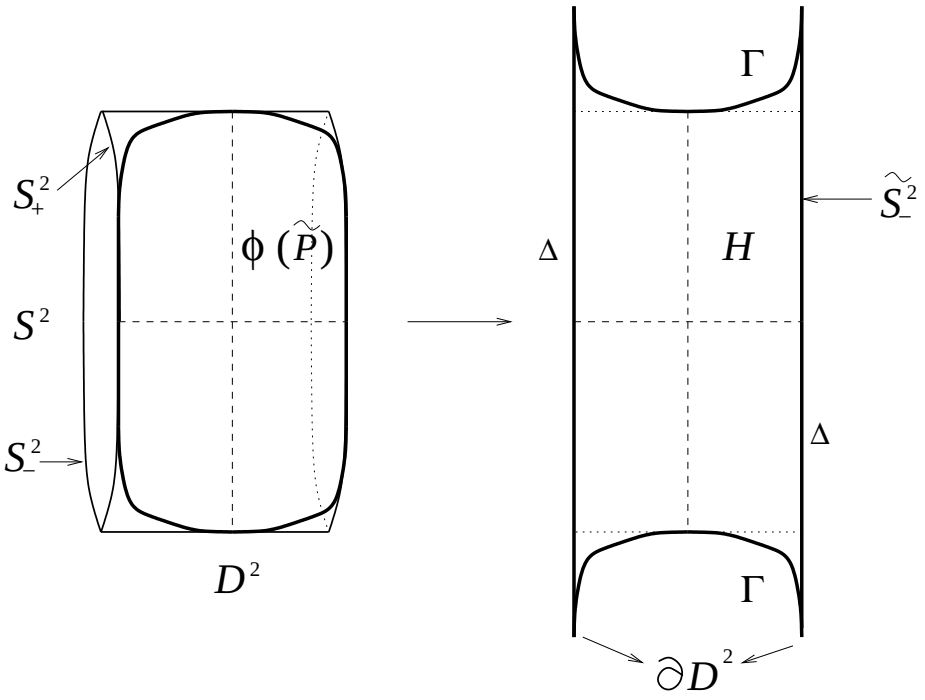

FIGURE 9. The 2-handle H

(2) $d\alpha$ is symplectic on the pages of $\mathfrak{ob}_\xi$ and

(3) pages of $\mathfrak{ob}_\xi$ in U are given by φ=constant.

Now we fix R and set a (in the definition of the function g) equal to $\frac{R}{2}$. Consider the following map $F : \Gamma \to U$ (cf. Figure 10) given by the following identifications of coordinates:

$$r = \frac{g(r_2)}{r_2}, \quad \varphi = \varphi_1, \quad u = \varphi_2.$$

Notice that under this map the core circle of Γ parametrized by φ_2 is mapped onto the binding B parametrized by u. It is clear that F is a diffeomorphism but we would like F to be a contactomorphism which takes the contact structure $\ker(\gamma|_\Gamma)$ onto the contact structure $\xi|_U$. Well, we will simply choose our function g (depicted in Figure 6) accordingly near $t = 1$ so that F becomes a contactomorphism. The function

$$t \to \frac{g(t)}{t}$$

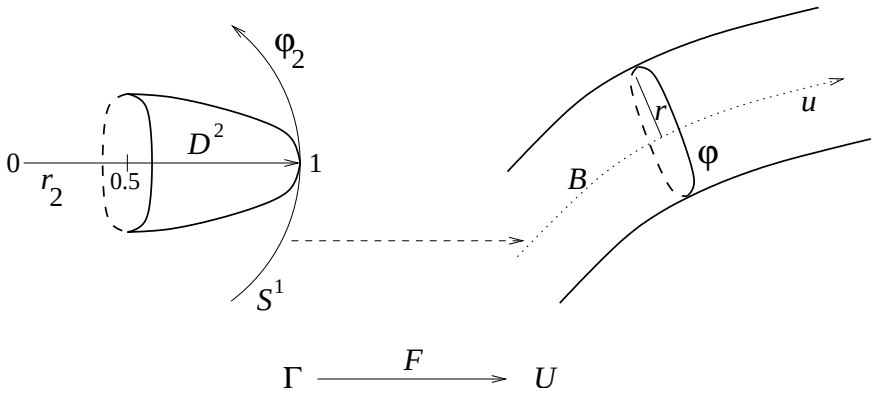

FIGURE 10. The diffeomorphism $F : \Gamma \to U$

is a decreasing function from $[0.5, 1]$ to $[0, 2a] = [0, R]$. Let $\psi : [0, R] \to [0.5, 1]$ be the inverse of this function. Recall that

$$\gamma|_\Gamma = \frac{r_2^2}{2}\big(\frac{g^2(r_2)}{r_2^2}d\varphi_1 + d\varphi_2\big).$$

Hence by the change of coordinates which describes the diffeomorphism F we get

$$(F^{-1})^*(\gamma|_\Gamma) = \frac{\psi^2(r)}{2}(r^2 d\varphi + du)$$

which is a 1-form defined on U. Then we have $h(r) = \frac{1}{2}\psi^2(r)$ where $h(r) - h(0) = -r^2$ near $r = 0$ so that $\frac{1}{2}(\psi^2(r) - \psi^2(0)) = -r^2$ and thus $\psi(r) = \sqrt{1 - 2r^2}$. Recall that $r_2 = \psi(r)$ and $r = \frac{g(r_2)}{r_2}$ under the diffeomorphism F. Considering that $\psi : [0, R] \to [0.5, 1]$ is the inverse of the function $t \to \frac{g(t)}{t}$ we finally obtain

$$r_2 = \big(1 - 2\frac{g^2(r_2)}{r_2^2}\big)^{1/2}$$

which implies that

$$g(t) = \frac{1}{2}t\sqrt{1 - t^2}$$

near $t = 1$. Notice that $g(t)$ has a vertical tangent at $t = 1$. This calculation determines the particular form of the function $g(t)$ near $t = 1$. (Our calculation of the function g is slightly different from the one given in [El-04].)

While preparing our 2-handle for gluing we also have to equip the cobordism $Y \times I$ by an appropriate symplectic form. Let $C > 0$ be an arbitrary constant. It is easy to see (cf.

[El-04]) that there is a symplectic form Ω on $Y \times I$ (see Figure 11) which "extends" ω and agrees with $\omega + Cd(t\alpha)$ for $t \in [\varepsilon, 1]$.

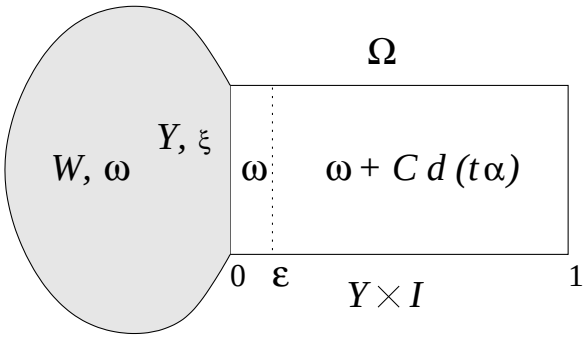

FIGURE 11. Extending ω to $\Omega \in H^2(Y \times I)$

Now observe that $Y \times \{1\} \subset Y \times I$ is convex with respect to $d(t\alpha)$. Recall that (H, ω_0) is a concave filling of $(\Gamma, \gamma|_\Gamma)$. Thus by Lemma 11, we can glue (H, ω_0) to $(Y \times I, d(t\alpha))$ identifying Γ and U by the contactomorphism F which extends to a symplectomorphism in some neighborhoods of $\Gamma \subset H$ and $U \subset Y \times I$. Consequently we have $F^*(d(t\alpha)) = \omega_0$. Notice that $F^*\omega$ is exact in a regular neighborhood $\nu(\Gamma)$ of Γ in H because the second cohomology group of $\nu(\Gamma) \cong I \times S^1 \times D^2$ is trivial. Since $F^*\omega$ is exact, there is a 1-form θ on $\nu(\Gamma)$ such that $F^*\omega = d\theta$. Take a smooth cut-off function σ on H which vanishes outside of $\nu(\Gamma)$. Then $d(\sigma\theta)$ defines an extension $\widetilde{\omega}$ of $F^*\omega$ from $\nu(\Gamma)$ to H.

Finally we are ready to define a symplectic form on H that will allow us to make this 2-handle attachment in the symplectic category. Let $\Omega_0 = \widetilde{\omega} + C\omega_0$ on H for some constant C. It is not hard to see that Ω_0 will be symplectic for sufficiently large values of C since $C^2\omega_0 \wedge \omega_0 > 0$ will dominate the other terms in $\Omega_0 \wedge \Omega_0$ on a compact manifold. Here notice that we have a well-defined symplectic form on $(Y \times I) \cup_{U=F(\Gamma)} H$ since $\Omega = \omega + Cd(t\alpha)$ on $(Y \times I)$ is identified with Ω_0 on H in the gluing region. This is because $\widetilde{\omega}$ is an extension of $F^*\omega$ and $F^*(Cd(t\alpha)) = C\omega_0$ so that $F^*\Omega = \Omega_0$.

On the other hand, by attaching the 2-handle H we perform a Dehn surgery on the 3-manifold Y to yield a 3-manifold Y' which fibers over the circle. This should be clear since we take out a neighborhood U from Y and glue in $\widetilde{S^2_-} \times \partial D^2$ to cap off each page F of $\mathfrak{ob}_\xi$ by a disk $D_x = \widetilde{S^2_-} \times \{x\}$. Let $\widehat{F}$ denote the closed surface obtained by capping off a page F by gluing a 2-disk D_x along its boundary. Consider the 2-form $\Omega|_Y = \Omega|_{Y \times \{1\}} = \omega + Cd\alpha$. We know that $d\alpha$ is symplectic on every page F of $\mathfrak{ob}_\xi$. Thus $\omega + Cd\alpha$ will be a symplectic form on F for sufficiently large values of C.

Recall that we identified the symplectic forms Ω and Ω_0 when we attached the symplectic 2-handle H. Also note that D_x is symplectic with respect to ω_0. Consequently since every page F of $\mathfrak{ob}_\xi$ is symplectic (with respect to Ω) and the disk D_x is symplectic as well (with respect to Ω_0) we get a fibration over S^1 for which $\omega' = \Omega_0|_{Y'}$ restricts to a symplectic form on each fiber $\widehat{F}$ for sufficiently large values of C. We will call such a surface fibration over S^1 a *symplectic* fibration over S^1. Note that we have the freedom to choose C as large as we wish. Also note that in order to prove that we have a symplectic fibration over S^1 after surgery we had to use the compatibility of ξ and $\mathfrak{ob}_\xi$.

Denote by (W', ω') the resulting symplectic 4-manifold obtained by attaching the symplectic 2-handle H to the given weak filling (W, ω) of (Y, ξ). To finish Eliashberg's construction we need to cap off the symplectic fibration $\partial W' = Y' \to S^1$ by a symplectic 4-manifold. Let ϕ be the topological monodromy of this surface fibration. Then we can *smoothly* fill in $-Y'$ (see [AO-02]) by a symplectic Lefschetz fibration over D^2 with regular fiber $\widehat{F}$ since the monodromy ϕ^{-1} of $-W'$ can be written as a product of *right-handed* Dehn twists by Lemma 15.

Lemma 15. *Any element in Map$(\widehat{F})$ can be expressed as a product of non-separating right-handed Dehn twists.*

Proof. We repeat the proof described in [AO-02] for this elementary result. Recall that the relation (cf. Lemma 7)

$$(t_{a_1} t_{a_2} \cdots t_{a_{2g}})^{4g+2} = t_c$$

in Γ_F induces a relation

$$(t_{a_1} t_{a_2} \cdots t_{a_{2g}})^{4g+2} = 1$$

in $\Gamma_{\widehat{F}}$. We conclude that $t_{a_1}^{-1}$ is a product of non-separating right-handed Dehn twists. Therefore any left-handed non-separating Dehn twist — being conjugate to $t_{a_1}^{-1}$ — is a product of non-separating right-handed Dehn twists. This finishes the proof of the lemma combined with the fact that $\Gamma_{\widehat{F}}$ is generated by (right and left-handed) non-separating Dehn twists. □

In fact, Eliashberg proves a "symplectic" version of Lemma 15 in [El-04] so that we can actually fill in $-\partial W' = -Y'$ *symplectically* by a symplectic 4-manifold. The point here is that when we measure the topological monodromy of a symplectic fibration $Y' \to S^1$ we do not take into account the symplectic structure on the fiber. But to fill in such a symplectic fibration *symplectically* we need to measure the holonomy (i.e., "symplectic" monodromy) of this fibration, which we describe below. Suppose that the symplectic fibration $Y' \to S^1$ is normalized so that $\int_{\widehat{F}} \omega' = 1$. Since the 2-form ω' is positive on the fibers its kernel $\ker \omega'$ is a 1-dimensional line field on Y' transverse to the fibers. The flow generated by a vector field which directs this line field determines a holonomy automorphism $\mathrm{Hol}(\omega')\colon \widehat{F}_0 \to \widehat{F}_0$

of a fixed fiber $\widehat{F}_0$. This is an area and orientation preserving diffeomorphism (i.e., a symplectomorphism) which defines (Y', ω') uniquely up to fiber preserving diffeomorphism fixed on $\widehat{F}_0$.

Now let (V, η) denote the symplectic Lefschetz fibration over D^2 mentioned above with regular fiber $\widehat{F}$ which will be used to fill in the symplectic fibration $-Y' \to S^1$. Since $(V, \eta) \to D^2$ is a symplectic Lefschetz fibration, the symplectic 2-form η restricts to a symplectic form on each regular fiber and moreover we can assume that $\eta|_{\partial V}$ integrates to 1 on the fibers of the symplectic fibration $\partial V \to S^1$. If we can choose (V, η) such that $\text{Hol}(\omega'|_{Y'})^{-1} = \text{Hol}(\eta|_{\partial V})$ then we are done since we can glue (W', ω') to (V, η) symplectically. Eliashberg constructs such a symplectic Lefschetz fibration over D^2 in [El-04]. In fact it is shown in [KrMr-04] that it suffices to prove Lemma 16 below. (See also Section 7 for another argument for the sufficiency of Lemma 16.)

Lemma 16 (Kronheimer-Mrowka [KrMr-04])**.** *Let Σ be a closed symplectic surface of area 1 and genus $g > 1$. Let $\phi : \Sigma \to \Sigma$ be an area preserving diffeomorphism that is smoothly isotopic to the identity. Then there is a symplectic Lefschetz fibration $p : (V, \eta) \to D^2$ such that $p^{-1}(1) = \Sigma$ and $Hol(\eta|_{\partial V}) = \phi$.*

As it is pointed out in [El-04], we could alternatively use Lemma 17 to cap off a symplectic fibration over S^1 by a symplectic surface bundle over a surface with boundary. Recall that a group G is said to be perfect if it is equal to its commutator subgroup $[G, G]$. In other words, G is perfect if and only if every element in G can be expressed as a product of commutators. Yet another way of characterizing the perfectness of a group is given by the triviality of its first homology group $H_1(G) = G/[G, G]$.

It is well-known that the mapping class group of a surface of genus greater than two is perfect. This is a consequence of the lantern relation (cf. [Jo-79]) in the mapping class groups which essentially says "three equals four". The fact that one can smoothly fill in a smooth surface bundle over S^1 by a smooth surface (of genus > 2) bundle over a surface with boundary easily follows from the perfectness of the corresponding mapping class group (of genus > 2). Here we need a symplectic version of this fact which is provided by Kotschick and Morita [KoMo-05]. Let $Symp_\sigma\Sigma$ denote the group of all symplectomorphisms of the closed symplectic surface (Σ, σ) with respect to a prescribed symplectic form σ on Σ which is normalized such that $\int_\Sigma \sigma = 1$.

Lemma 17 (Kotschick-Morita [KoMo-05])**.** *If the genus of Σ is greater than two then $Symp_\sigma\Sigma$ is perfect.*

The restriction in Lemma 17 on the genus of the fiber is not a serious one since in the construction above one can arbitrarily increase the genus of the page of $\mathfrak{ob}_\xi$ (which is compatible with (Y, ξ)) by positively stabilizing $\mathfrak{ob}_\xi$ (cf. [Gi-02]) to begin with.

6.2. **Etnyre's construction:** We first briefly outline Etnyre's construction: In Section 5 we showed that to find an embedding of a strong filling one can use an embedding of a Stein filling. Etnyre's idea in [Et-04] was to find an embedding of a weak filling using an embedding of a strong filling. Suppose that (W,ω) is a weak filling of a contact 3-manifold (Y,ξ). Etnyre showed that (W,ω) can be embedded into a symplectic 4-manifold (W',ω') which weakly fills its boundary $(\partial W'=Y',\xi')$, where Y' happens to be a integral homology sphere. Now by a homological argument the symplectic structure ω' can be perturbed near the boundary so that (W',ω') strongly fills (Y',ξ'). Therefore (W',ω') can be embedded into a closed symplectic 4-manifold (X,ω_X) by Proposition 13 and hence $(W,\omega)\subset(W',\omega')$ can be embedded symplectically into (X,ω_X). Below we proceed with the details.

Let (W,ω) be a weak filling of (Y,ξ) and let $\mathfrak{ob}_\xi$ be an open book compatible with (Y,ξ). We can assume that the binding B of $\mathfrak{ob}_\xi$ is connected. Let ϕ be the monodromy of this open book and use Lemma 8 to express ϕ as

$$\phi = t_c^m \circ t_{\gamma_1}^{-1} \circ \cdots \circ t_{\gamma_n}^{-1}.$$

Now Legendrian Realize γ_n (cf. Lemma 9) on a convex page of $\mathfrak{ob}_\xi$ and perform contact (-1)-surgery on γ_n. The new open book will have monodromy

$$\phi \circ t_{\gamma_n} = t_c^m \circ t_{\gamma_1}^{-1} \circ \cdots \circ t_{\gamma_{n-1}}^{-1}.$$

Repeat this for all the curves γ_i (for $i=n-1,\cdots,1$) to get down to t_c^m as in the proof of Proposition 5. Denote by (Y',ξ') the contact 3-manifold obtained as a result of the contact (-1)-surgeries above. Then (Y',ξ') is compatible with the open book whose monodromy is given by t_c^m, by Lemma 10.

Recall that by Theorem 6 we can attach a symplectic 2-handle to a strong filling along a Legendrian knot in its convex boundary in such a way that the symplectic structure extends to the 2-handle and the new symplectic 4-manifold strongly fills its boundary. In this gluing process, however, the Liouville (i.e., symplectically dilating) vector field is used only in a neighborhood of the attaching circle. It turns out that if $L\subset(Y,\xi)$ is Legendrian and (W,ω) is a weak filling of (Y,ξ) then there is always a symplectically dilating vector field near L, implying

Proposition 18. *Suppose that (Y',ξ') is given by contact (-1)-surgery along $L\subset(Y,\xi)$. If (Y,ξ) is weakly fillable then so is (Y',ξ').*

The result above was first explicitly stated in [EH-02] although it was probably known to the experts, and certainly to Eliashberg.

Hence there exists a weak filling (W',ω') of (Y',ξ') obtained by attaching symplectic 2-handles to (W,ω). The page of the compatible open book $\mathfrak{ob}_{\xi'}$ is a genus g surface with

one boundary component. Consider the curves a_i depicted in Figure 2. Legendrian realize a_i's and perform contact (-1)-surgery on each a_i to get (Y'', ξ'') compatible with the open book $\mathfrak{ob}_{\xi''}$ whose monodromy is given by

$$t_c^m \circ t_{a_1}^{-1} \circ \cdots \circ t_{a_{2g}}^{-1}.$$

It is not hard to see that Y'' is an integral homology sphere. Moreover, by Proposition 18, there exists a weak filling (W'', ω'') of (Y'', ξ''). Then we use Proposition 19 to modify the symplectic form ω'' near the boundary so that it is a strong filling of (Y'', ξ''). Note that (Y'', ξ'') has a concave filling by Proposition 5. Thus we cap off (W'', ω'') by this concave filling using Lemma 11 to get a closed symplectic 4-manifold (X, ω_X) in which (W, ω) sits as a symplectic subdomain.

Proposition 19 (Eliashberg [El-91], [El-04]; Ohta-Ono [OO-99])**.** *Any weak filling of a rational homology sphere can be deformed into a strong filling by modifying the symplectic form near the boundary.*

The main step in Etnyre's construction is embedding a weak filling of an arbitrary contact 3-manifold into a weak filling of an integral homology sphere. We would like to point out here that this follows also from a result that was obtained by Stipsicz in [St-03]. Namely, Stipsicz showed the existence of a Stein cobordism from an arbitrary contact 3-manifold to an integral homology sphere. Stipsicz's construction (which we describe below) can be slightly modified to imply the main step above.

Let (W, ω) be a weak filling of (Y, ξ). Consider the right-handed Legendrian trefoil knot K as depicted in Figure 12 in the standard contact S^3, having $\mathrm{tb}(K) = 1$. To construct such a cobordism start with a contact surgery diagram $\mathbb{L}$ of (Y, ξ) and for every knot L_i in $\mathbb{L}$ add a copy K_i of K into the diagram linking L_i once, not linking the other knots in $\mathbb{L}$. Adding symplectic 2-handles along K_i we get (W', ω') and the resulting 3-manifold Y' is an integral homology sphere. To see this just convert the contact surgery diagram into a smooth handlebody diagram and calculate the first homology. Observe that the topological framing of K is 0. Denote by μ_i a small circle meridional to K_i and μ_i' a small circle meridional to L_i for $i = 1, \cdots, n$. Recall that $H_1(Y', \mathbb{Z})$ is generated by $[\mu_i]$ and $[\mu_i']$ and the relations are $[\mu_i'] = 0$ and $[\mu_i] + \sum_{j \neq i} lk(L_i, L_j)[\mu_j'] = 0$. It follows that $H_1(Y', \mathbb{Z}) = 0$.

Although it was not considered in [St-03], Stipsicz's construction immediately implies the main step above because one can add a symplectic 2-handle along a Legendrian knot in the boundary of a weak filling to extend it to another weak filling by Proposition 18.

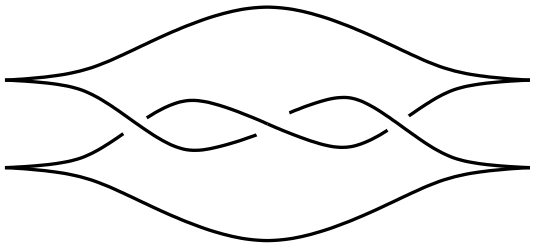

FIGURE 12. A right-handed Legendrian trefoil knot K

7. A HYBRID SOLUTION

In this section we suggest another symplectic embedding of a weak filling into a closed symplectic 4-manifold which is obtained by a mixture of the ideas we discussed so far. First we note that it is possible to attach symplectic 1-handles (as well as symplectic 2-handles) to a weak filling to extend it to another weak filling. Suppose that (W, ω) is a weak filling of a contact 3-manifold (Y, ξ). Now we proceed as in the second proof of Proposition 5 to embed (W, ω) into a weak filling (W', ω') by attaching symplectic 1- and 2-handles so that the resulting contact structure on the boundary $\partial W'$ has a compatible open book whose page has only one boundary component and whose monodromy is just one right-handed boundary-parallel Dehn twist. Then we attach a symplectic 2-handle to (W', ω') along the binding of this open book and we get a symplectic fibration over a circle with topologically trivial monodromy on the other end of the cobordism given by this 2-handle attachment. Finally we cap off this surface bundle by a symplectic Lefschetz fibration over D^2 using Lemma 16.

8. FINAL COMMENTS

The presentation in this article may suggest that Eliashberg's method is unnecessarily long but he constructs from scratch a symplectic 2-handle to be attached to the binding of a compatible open book—which is crucial. Note that a construction of attaching symplectic 2-handles along transverse knots was also given in [Ga-02a]. It would be very interesting to interpret this symplectic surgery in terms of contact surgery. Unfortunately, there does not seem to exist a natural contact structure on the symplectic fibration over S^1 obtained by this surgery. This is exactly the point where Eliashberg's method differs from the method of Etnyre. In Etnyre's construction one always makes use of the contact structures on the boundaries of symplectic 4-manifolds to glue them symplectically. In fact, based on Giroux's correspondence, Etnyre mostly deals with open books compatible with these contact structures rather than the contact structures directly. In Eliashberg's construction,

however, at one point or another we have to glue a symplectic (Lefschetz) fibration to a symplectic 4-manifold whose boundary symplectically fibers over S^1. This is achieved by matching up the holonomy diffeomorphisms on the boundaries and contact structures are not visible in this picture. It might be worth pointing out that the proof of the non-triviality of the contact Heegaard Floer invariant of a fillable contact structure follows from Eliashberg's embedding but it is not clear whether or not it follows from Etnyre's construction.

Also it is intriguing to note that most of the constructions in this article rely on the relation

$$t_c = (t_{a_1} \circ t_{a_2} \circ \cdots \circ t_{a_{2g-1}} \circ t_{a_{2g}})^{4g+2} \in \Gamma_F$$

given in Lemma 7 which implies

$$1 = (t_{a_1} \circ t_{a_2} \circ \cdots \circ t_{a_{2g-1}} \circ t_{a_{2g}})^{4g+2} \in \Gamma_{\widehat{F}},$$

where $\widehat{F}$ denotes the closed surface obtained by capping off the surface F by gluing a 2-disk along ∂F. This latter relation says that identity can be expressed as a product of right-handed Dehn twists in the mapping class group of a closed surface. It is not possible, however, to express the identity as a product of right-handed Dehn twists in the mapping class group of a surface with non-empty boundary (cf. [OzSt-04]).

Acknowledgement: We would like to thank Selman Akbulut for encouragement, Tolga Etgü for helpful conversations, András Stipsicz and especially the referee for reading a preliminary version, correcting a few mistakes and suggesting numerous improvements. The author was partially supported by the Turkish Academy of Sciences.

REFERENCES

[AO-01] S. Akbulut and B. Ozbagci, *Lefschetz fibrations on compact Stein surfaces*, Geom. Topol. **5** (2001), 319–334.

[AO-02] S. Akbulut and B. Ozbagci, *On the topology of compact Stein surfaces*, Int. Math. Res. Not. **15** (2002), 769–782.

[Al-23] J. Alexander, *A lemma on systems of knotted curves*, Proc. Nat. Acad. Sci. USA **9** (1923), 93–95.

[DG-01] F. Ding and H. Geiges, *Symplectic fillability of tight contact structures on torus bundles*, Algebr. Geom. Topol. **1** (2001), 153–172.

[DG-04] F. Ding and H. Geiges, *A Legendrian surgery presentation of contact 3-manifolds*, Proc. Cambridge. Philos. Soc. **136** (2004), 583–598.

[DLS-94] J.P. Demailly, L. Lempert, B. Shiffman, *Algebraic approximations of holomorphic maps from Stein domains to projective manifolds*, Duke Math. J. **76** (1994), no. 2, 333–363.

[El-90] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension* > 2, International J. of Math. **1** (1990), 29–46.

[El-91] Y. Eliashberg, *On symplectic manifolds with some contact properties*, J. Differential Geom. **33** (1991), 233–238.

[El-96] Y. Eliashberg, *Unique holomorphically fillable contact structure on the* 3*-torus*, Int. Math. Res. Not. **2** 1996, 77–82.

[El-04] Y. Eliashberg, *Few remarks about symplectic filling*, Geom. Topol. **8** (2004), 277–293.

[Et-98] J. Etnyre, *Symplectic convexity in low-dimensional topology,* Topology and its Appl. **88** (1998), 3–25.

[Et-04] J. Etnyre, *On symplectic fillings,* Algebr. Geom. Topol. **4** (2004), 73–80.

[Et-05] J. Etnyre, *Lectures on open book decompositions and contact structures,* Lecture notes from the Clay Mathematics Institute summer school "Floer Homology, Gauge Theory, and Low Dimensional Topology" Alfred Renyi Institute; arXiv:math.SG/0409402.

[EH-02] J. Etnyre and K. Honda, *On symplectic cobordisms,* Math. Ann. **323** (2002), 31–39.

[Ga-02a] D. Gay, *Symplectic 2-handles and transverse links,* Trans. Amer. Math. Soc. **354** (2002), no. 3, 1027–1047.

[Ga-02b] D. Gay, *Explicit concave fillings of* 3*-manifolds,* Math. Proc. Cambridge Philos. Soc. **133** (2002), 431–441.

[Ga-03] D. Gay, *Open books and configurations of symplectic surfaces,* Algebr. Geom. Topol. **3** (2003), 569–586.

[Ge-05] E. Geiges, *Contact Dehn surgery, symplectic fillings, and Property P for knots,* preprint, arXiv:math.SG/0506472.

[Gi-02] E. Giroux, *Contact geometry: from dimension three to higher dimensions,* Proceedings of the International Congress of Mathematicians (Beijing 2002), 405–414.

[Go-98] R. Gompf, *Handlebody construction of Stein surfaces,* Ann. of Math. **148** (1998), 619–693.

[GS-99] R. Gompf and A. Stipsicz, 4*-manifolds and Kirby calculus,* Graduate Studies in Mathematics, vol. **20,** American Math. Society, Providence 1999.

[Jo-79] D. Johnson, *Homeomorphisms of a surface which act trivially on homology,* Proc. Amer. Math. Soc. **75** (1979), 119–125.

[Ka-97] Y. Kanda, *The classification of tight contact structures on the* 3*-torus,* Comm. Anal. Geom. **5** (1997), 413–438.

[KoMo-05] D. Kotschick and S. Morita, *Signatures of foliated bundles and the symplectomorphism groups of surfaces*, Topology **44** (2005), no. 1, 131–149.

[KrMr-04] P. Kronheimer and T. Mrowka, *Witten's conjecture and Property* P*,* Geom. Topol. **8** (2004), 295–310.

[LM-97] P. Lisca and G. Matić, *Tight contact structures and Seiberg–Witten invariants,* Invent. Math. **129** (1997), 509–525.

[LP-01] A. Loi and R. Piergallini, *Compact Stein surfaces with boundary as branched covers of* B^4*,* Invent. Math. **143** (2001), 325–348.

[McD-91] D. McDuff, *Symplectic manifolds with contact type boundaries,* Invent. Math. **103** (1991), 651–671.

[OO-99] H. Ohta and K. Ono, *Simple singularities and topology of symplectically filling* 4*-manifolds,* Comment. Math. Helv. **74** (1999), 575–590.

[OzSt-04] B. Ozbagci and A. Stipsicz, *Surgery on contact 3-manifolds and Stein surfaces*, Springer, 2004.

[OzSz-04] P. Ozsváth and Z. Szabó, *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334.

[St-03] A. Stipsicz, *On the geography of Stein fillings of certain* 3*-manifolds*, Michigan Math. J. **51** (2003), 327–337.

[Wa-99] B. Wajnryb, *An elementary approach to the mapping class group of a surface,* Geom. Topol. **3** (1999), 405–466.

[We-91] A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. **20** (1991), 241–51.

DEPARTMENT OF MATHEMATICS, KOÇ UNIVERSITY, ISTANBUL, TURKEY
E-mail address: bozbagci@ku.edu.tr