CONTACT HANDLE DECOMPOSITIONS

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ABSTRACT. We review Giroux's contact handles and contact handle attachments in dimension three and show that a bypass attachment consists of a pair of *contact* 1 and 2-handles. As an application we describe explicit contact handle decompositions of infinitely many pairwise non-isotopic overtwisted 3-spheres. We also give an alternative proof of the fact that every compact contact 3-manifold (closed or with convex boundary) admits a contact handle decomposition, which is a result originally due to Giroux.

0. INTRODUCTION

Emmanuel Giroux announced the following result in a series of lectures he delivered at Stanford University in the year 2000: "Every contact 3-manifold is convex" — which signified the closure of the program he initiated in his convexity paper published in 1991, where he proved that every oriented 3-manifold has *some* convex contact structure ([7], Theorem III. 1.2). Apparently, an essential motivating factor for studying convexity in contact topology is the following straightforward consequence of the convexity theorem: "*Every contact 3-manifold (closed or with convex boundary) admits a contact handle decomposition*". We should point out that for a closed contact 3-manifold the existence of a contact handle decomposition and the existence of an adapted open book decomposition are equivalent. Despite the fact that several explicit examples of adapted open book decompositions of closed contact 3-manifolds have been published and fruitfully used in various other constructions since Giroux's breakthrough in 2000, explicit examples of contact handle decompositions of closed contact 3-manifolds have not yet appeared in the literature. In this article, we show that a bypass attachment [11] consists of a (topologically cancelling) pair of *contact* 1 and 2-handles. As an application, for each positive integer n, we describe an explicit contact handle decomposition of the overtwisted 3-sphere whose d_3-invariant is $(2n + 1)/2$. Recall that two overtwisted contact structures are isotopic if and only if they are homotopic as oriented 2-plane fields [1]. Moreover the homotopy classes of oriented 2-plane fields on S^3 are classified by their d_3-invariants (see [9] or [14] for a detailed discussion).

For the sake of completeness, we also offer a alternative proof of Giroux's handle decomposition theorem for compact contact 3-manifolds (closed or with convex boundary). Our proof is based on a recent result due to Honda, Kazez and Matić ([12], Theorem 1.1), which asserts that every compact contact 3-manifold with convex boundary has an adapted partial open book decomposition. The technique that Honda, Kazez and Matić apply in constructing adapted partial open book decompositions of contact 3-manifolds with convex boundary is a generalization of Giroux's method of constructing adapted open book decompositions of closed contact 3-manifolds. Giroux's construction, in turn, is based on contact cell decompositions of contact 3-manifolds [8]. Hence the existence of contact *handle* decompositions of compact contact 3-manifolds can be viewed as a consequence of the existence of contact *cell* decompositions. Although we do not delve into the details here, it seems feasible to set up a more direct connection between the two existence results just as in the topological case. The reader is advised to turn to [10] or [13] for necessary background on handle decompositions of manifolds and to [4], [6], [9] and [14] for the related material on contact topology.

1. CONTACT HANDLES IN DIMENSION THREE

We first review Giroux's contact handles in dimension three [7]. The contact structure $\zeta_0 = \ker \alpha_0$, where $\alpha_0 = dz - y dx + x dy$, is the standard tight contact structure in $\mathbb{R}^3$ and the flow of the vector field

$$Z_0 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$$

preserves ζ_0. Let $B^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 1\}$. Then ∂B^3 is a convex surface since Z_0 is transverse to ∂B^3. It is clear that Z_0 lies in the contact planes ζ_0 whenever $\alpha_0(Z_0) = 0$, i.e., when $z = 0$. In other words, the disk $B^3 \cap \{z = 0\}$ is the characteristic surface in B^3 and $\partial B^3 \cap \{z = 0\}$ is the dividing curve on ∂B^3.

A model for a *contact* 0-*handle* is given as (B^3, ζ_0), where Z_0 is used in gluing this handle. Here the orientation of the contact 0-handle coincides with the usual orientation of $\mathbb{R}^3$ (given by $dx \wedge dy \wedge dz$) and its boundary has the induced orientation. The dividing curve divides the convex sphere ∂B^3 into its positive and negative regions: $R_+ = \partial B^3 \cap \{z > 0\}$ and $R_- = \partial B^3 \cap \{z < 0\}$. The characteristic foliation on ∂B^3 appears as in Figure 1,
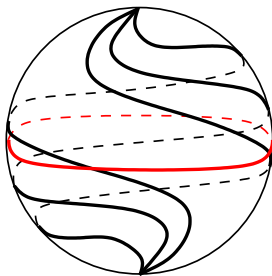
where the "equator" is the dividing curve.

FIGURE 1. Characteristic foliation and the dividing curve on ∂B^3

Giroux's criterion [7] implies that the dividing curve on any tight 3-ball with convex boundary is connected. Moreover there is a unique tight contact structure on the 3-ball with a connected dividing set on its convex boundary up to isotopy fixing the dividing set [1]. Hence we make the following definition.

Definition 1.1. *A standard contact 3-ball is a tight contact 3-ball with convex boundary.*

As a matter of fact, a contact 0-handle is a model for a standard contact 3-ball and when we want to glue such a handle, we use the vector field Z_0 in the model to obtain a "contact" collar neighborhood. A model for a *contact 3-handle*, on the other hand, is also defined as (B^3, ζ_0), where we give opposite orientation to its boundary and use $-Z_0$ to glue this handle.

Let ζ_1 denote the contact structure in $\mathbb{R}^3$ given by the kernel of the 1-form

$$\alpha_1 = dz + ydx + 2xdy,$$

and consider the vector field

$$Z_1 = 2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

whose flow preserves ζ_1. Observe that ζ_1 is isotopic to the standard tight contact structure ζ_0 in $\mathbb{R}^3$. Moreover, for any $\epsilon > 0$, Z_1 is transverse to the surfaces

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 = \epsilon^2\} \text{ and } \{(x, y, z) \in \mathbb{R}^3 \mid y^2 = 1\}.$$

Note that the intersection of these convex surfaces is not Legendrian. Let

$$H_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 \leq \epsilon^2, \ y^2 \leq 1\} \text{ and } F_1 = H_1 \cap \{y = \pm 1\}.$$

A model for a *contact* 1-*handle* is given as (H_1, ζ_1), where Z_1 is used in gluing this handle. Here the contact 1-handle acquires the usual orientation of $\mathbb{R}^3$ and ζ_1 orients F_1 as the outward pointing normal vector field. The characteristic surface in H_1 is given by $H_1 \cap \{z = 0\}$. The dividing curve $\partial H_1 \cap \{z = 0\}$ divides ∂H_1 into its positive and negative regions: $R_+ = \partial H_1 \cap \{z > 0\}$ and $R_- = \partial H_1 \cap \{z < 0\}$. The characteristic foliation on F_1 is linear with slope ∓ 1 on $H_1 \cap \{y = \pm 1\}$ (viewed in a copy of the xz-plane) as depicted in Figure 2.
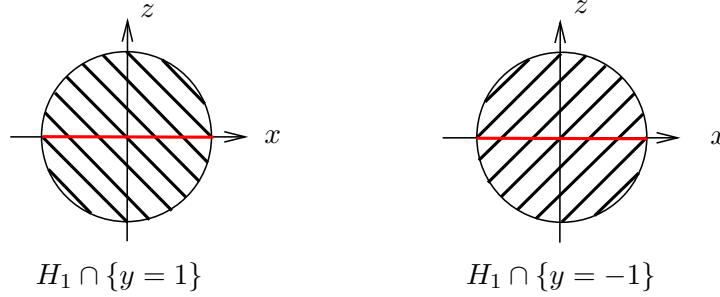

$$H_1 \cap \{y = 1\} \qquad\qquad H_1 \cap \{y = -1\}$$

FIGURE 2. Characteristic foliation and the dividing set on $F_1 = H_1 \cap \{y = \pm 1\}$

A model for a *contact* 2-*handle* is defined as (H_2, ζ_1), where $H_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 \leq 1, \ y^2 \leq \epsilon^2\}$. Note that the intersection of the convex surfaces ($\epsilon > 0$),

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 = 1\} \text{ and } \{(x, y, z) \in \mathbb{R}^3 \mid y^2 = \epsilon^2\}$$

is not Legendrian. The characteristic surface in H_2 is given by $H_2 \cap \{z = 0\}$ and the dividing curve on the boundary of the contact 2-handle is given by $\partial H_2 \cap \{z = 0\}$. Let $F_2 = H_2 \cap \{x^2 + z^2 = 1\}$. Here the contact 2-handle is oriented by the usual orientation of $\mathbb{R}^3$; $-Z_1$ orients F_2 as the outward normal vector field and we use $-Z_1$ when we glue such a handle along F_2.

If we parametrize F_2 by $(\theta, y) \to (x = \sin\theta, y, z = \cos\theta)$ for $(\theta, y) \in [0, 2\pi] \times [-\epsilon, \epsilon]$, then the equation for determining the characteristic foliation on F_2 becomes

$$(y\cos\theta - \sin\theta)d\theta + 2\sin\theta \, dy = 0,$$

where the orientation of F_2 is given by $d\theta \wedge dy$. Therefore the characteristic foliation is the singular foliation which is given as the integral curves of the equation

$$\frac{dy}{d\theta} = \frac{1}{2}(1 - y\cot\theta).$$
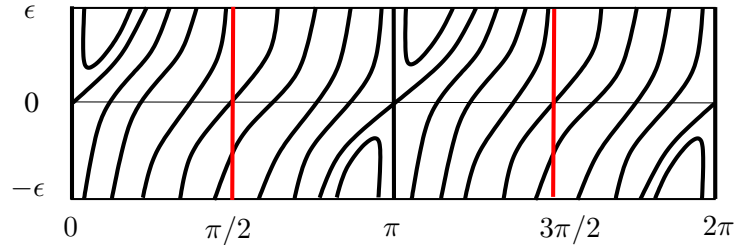
FIGURE 3. Characteristic foliation and the dividing set on F_2

It follows that the characteristic foliation on F_2 appears as in Figure 3. Note that there are two hyperbolic singular points corresponding to $(\theta, y) \in \{(0, 0), (\pi, 0)\}$ and the dividing set on F_2 consists of the lines $\theta = \pi/2$ and $\theta = 3\pi/2$.

Roughly speaking, a 3-dimensional contact k-handle is a topological k-handle which carries a tight contact structure whose diving set on the boundary is depicted in Figure 4. Moreover the characteristic foliations on the gluing regions of these handles are shown in Figures 1, 2 and 3.
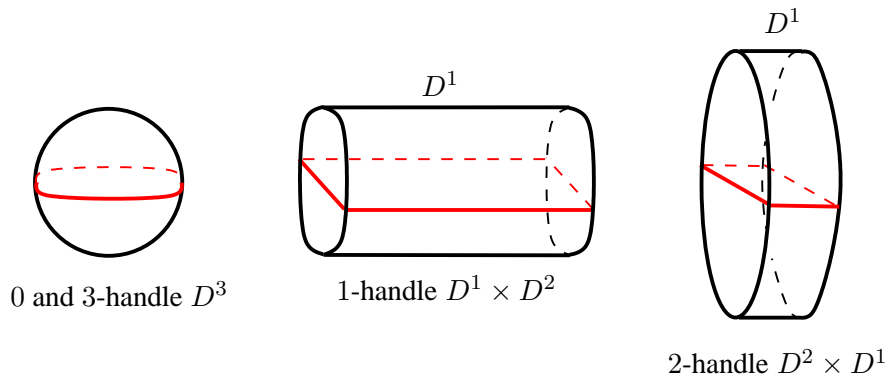

FIGURE 4. 3-dimensional contact handles

Recall [11] that if two convex surfaces inside an ambient contact 3-manifold admit a Legendrian curve as their common boundary, then the diving curves on these convex surfaces will intersect that Legendrian curve in an "alternating" fashion. In the description of the contact k-handle, for $k = 1, 2$, however, the diving curves on the convex surfaces which make up the boundary of the contact k-handle do not meet the intersection of these convex surfaces at an alternating fashion (see Figure 4). This is not a contradiction since the intersection of those convex surfaces is not Legendrian.

Next we would like to discuss contact handle attachments [7]. By attaching contact 0-handles we will just mean taking a disjoint union of some contact 0-handles. In order to attach a contact 3-handle to a contact 3-manifold (M, ξ) with convex boundary, we require that ∂M has at least one component which is a 2-sphere with a connected dividing set. Then a contact 3-handle attachment is just filling in this 2-sphere by a standard contact 3-ball. The key point is that the image of the characteristic foliation on the boundary $F_3 = \partial H_3$ of the 3-handle under the attaching map is adapted to the dividing curve $\Gamma_{\partial M}$ and therefore Giroux's Theorem ([7], Proposition II.3.6) allows us to glue the corresponding contact structures.

Suppose that (M, ξ) is a contact 3-manifold with convex boundary, where $\Gamma_{\partial M}$ denotes the dividing set on ∂M. In order to attach a contact 1-handle to M along two points p and q on $\Gamma_{\partial M}$ we identify the attaching region $F_1 \cong D^0 \times D^2$ of the 1-handle $H_1 \cong D^1 \times D^2$ with regular neighborhoods of these points in ∂M. The difference from just a topological 1-handle attachment is that we require the dividing set on the attaching region of the contact 1-handle to coincide with $\Gamma_{\partial M}$ on ∂M so that we can glue the contact structures on M and the contact 1-handle again by Giroux's Theorem ([7], Proposition II.3.6). The idea here is that once we initially identify the dividing curves then we can match the characteristic foliations on the *convex* pieces that we glue by appropriate isotopies in the collar neighborhoods given by the contact vector fields. Also we need to make sure that the positive and the negative regions on the corresponding convex boundaries match up so that the new convex boundary after the handle attachment has well-defined $\pm$ regions divided by the new dividing set.
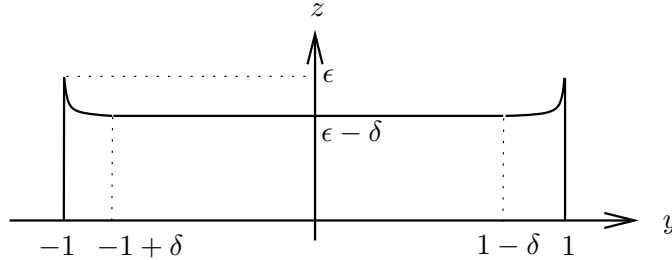

FIGURE 5. Modification of the contact 1-handle

Note that a contact 1-handle is a manifold with *corners*. To get a smooth contact manifold as a result of a contact 1-handle attachment we propose the following modification (similar to Honda's edge rounding technique [11]) to the handle: Let $\delta < \epsilon$ be a sufficiently small positive real number and let $f : [0, 1] \to \mathbb{R}$ be a function defined as follows:

- $f(y) = \epsilon - \delta$ for $y \in [-1 + \delta, 1 - \delta]$,
- f is smooth on $(-1, 1)$,
- f is concave up on both $(-1, -1 + \delta)$ and $(1 - \delta, 1)$,

- $\lim_{y \to \pm 1} f'(y) = \pm \infty$, and
- $f(\pm 1) = \epsilon$.

Such a function is depicted in Figure 5. Now consider the region in the upper half yz-plane under the graph of the function f defined over $-1 \leq y \leq 1$. By revolving this region around the y-axis, topologically we get a 1-handle (which looks like a vase). One can verify that the contact vector field Z_1 is still transverse to the side surface as well as the top and the bottom disks. When we glue this (modified) contact 1-handle to a contact manifold with convex boundary we get a smooth manifold carrying a contact structure which makes the resulting boundary convex. In Figure 6, we illustrated two possible contact 1-handle attachments (taking into account the compatibility of the $\pm$ regions), where corners should be smoothed as explained above.
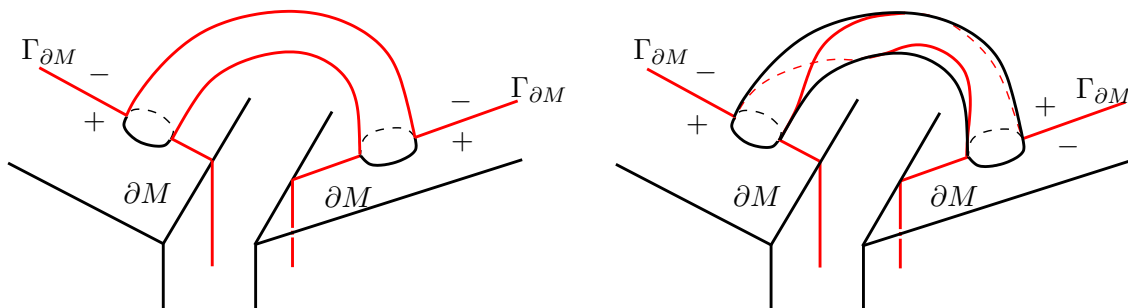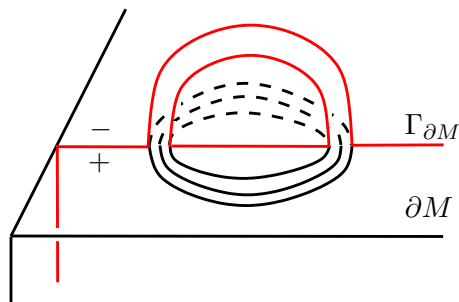

FIGURE 6. Attaching contact 1-handles

FIGURE 7. Attaching a contact 2-handle

Next we explain how to attach a contact 2-handle on top of a contact 3-manifold (M, ξ) with convex boundary. As expected, attachment of a contact 2-handle requires more work compared to the other contact handles. The attaching curve is the image of the core circle of the annulus $F_2 \cong \partial D^2 \times D^1$ under the attaching map $F_2 \to \partial M$ of the 2-handle. It is well-known that in order to attach a topological 2-handle one only has to specify the attaching

curve on ∂M. To attach a contact 2-handle, however, we require the attaching curve to intersect $\Gamma_{\partial M}$ transversely at two distinct points. This will allow one to glue the contact structures on M and the 2-handle as explained in great details by Giroux in ([7], Lemma III.3.2). The idea here is that one can construct a singular foliation adapted to $\Gamma_{\partial M}$ which conjugates to the characteristic foliation on F_2 (see Figure 3) in an annulus neighborhood of the attaching curve on the convex surface ∂M. In addition, just as in attaching a contact 1-handle, we need to pay attention so that the $\pm$ regions in the corresponding boundaries match up appropriately. Moreover, one can smooth the corners of the contact 2-handle by a modification which preserves the convexity of the boundary—similar to the modification we explained for contact 1-handles.

2. CONTACT HANDLE DECOMPOSITIONS

Theorem 2.1 (Giroux). *Every compact contact 3-manifold (closed or with convex boundary) admits a contact handle decomposition.*

Proof. Suppose that (M, ξ) is a *connected* contact 3-manifold with convex boundary. It follows that $(M, \Gamma_{\partial M}, \xi)$ admits a compatible partial open book decomposition [12] and, in particular, (M, ξ) can be decomposed into two tight contact handlebodies $(H, \xi|_H)$ and $(N, \xi|_N)$ where H is connected by our assumption that M is connected (see [2] for notation). Now we claim that $(H, \xi|_H)$ has a contact handle decomposition with a unique contact 0-handle and some contact 1-handles. This is because $(H, \xi|_H)$ is *product disk decomposable* [12], i.e., there exist some pairwise disjoint compressing disks in H each of whose boundary intersects $\Gamma_{\partial H}$ transversely in two points, so that when we cut H along these disks we get a standard contact 3-ball. Clearly the resulting standard contact 3-ball can be considered as a contact 0-handle. On the other hand, the thickening of a compressing disk satisfies our definition of a contact 1-handle which is attached to the contact 0-handle. This proves our claim about the tight contact handlebody $(H, \xi|_H)$. Moreover each component of the handlebody N is also product disk decomposable. By turning the handles upside down we conclude that $(M, \xi|_M)$ is obtained from $(H, \xi|_H)$ by attaching some contact 2 and 3-handles. Thus we proved that (M, ξ) admits a contact handle decomposition.

Suppose that (Y, ξ) is a *connected* and *closed* contact 3-manifold. Let p be an arbitrary point in Y. Then, by Darboux's theorem, there is a neighborhood of p in Y which is just a standard contact 3-ball. Now the closure of the complement of this ball in Y is a contact 3-manifold $(M, \xi|_M)$ whose boundary is a convex 2-sphere with a connected dividing set $\Gamma_{\partial M}$. We proved above that (M, ξ) admits a contact handle decomposition. Furthermore we can obtain (Y, ξ) from $(M, \xi|_M)$ by gluing back the standard contact 3-ball that we deleted at the beginning, which is indeed equivalent to a contact 3-handle attachment. Hence we proved that (Y, ξ) has a contact handle decomposition with a unique contact 0-handle and

some contact $1, 2$ and 3-handles. If (Y, ξ) is not connected then we can apply the above argument to each of its components to obtain a contact handle decomposition.

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3. BYPASS ATTACHMENT

Recall that a bypass [11] for a convex surface Σ in a contact 3-manifold is an oriented embedded half-disk D with Legendrian boundary, satisfying the following:

- ∂D is the union of two arcs γ_1 and γ_2 which intersect at their endpoints,
- D intersects Σ transversely along γ_2,
- D (or D with the opposite orientation) has the following tangencies along ∂D:
    (1) positive elliptic tangencies at the endpoints of γ_2,
    (2) one negative elliptic tangency on the interior of γ_2, and
    (3) only positive tangencies along γ_1, alternating between elliptic and hyperbolic,
- γ_2 intersects the dividing set Γ exactly at three points, and these three points are the elliptic points of γ_2.

In this section we show that a bypass attachment consists of a pair of *contact* 1 and 2-handles—which cancel each other out only topologically. Here by a bypass attachment we mean attaching a thickened neighborhood of the bypass disk D. The attaching arc γ_2 of an *exterior* bypass is a Legendrian arc on the convex boundary of a contact 3-manifold, where γ_2 intersects Γ transversely at p_2, and $p_1, p_3 \in \Gamma$ are the endpoints of γ_2, as we depict in Figure 8.
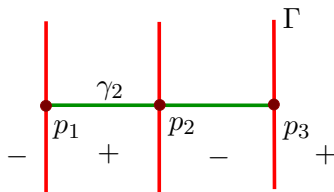

FIGURE 8. The attaching arc of a bypass intersecting the dividing set Γ at $\{p_1, p_2, p_3\}$

In order to attach a bypass along the arc γ_2 indicated in Figure 8, we first attach a contact 1-handle whose feet are identified with the neighborhoods of p_1 and p_3, respectively. Here we pay attention to the compatibility of the $\pm$ regions in the surfaces that we glue together. To be more precise, we describe the gluing map ϕ which identifies the gluing region F_1 of the contact 1-handle with two disjoint disks around p_1 and p_3 as follows: ϕ takes $(0, -1, 0)$ to p_1, the dividing arc $\{-1 \leq x \leq 1, y = -1, z = 0\}$ to an arc around p_1 in Γ and the arc $\{x = 0, y = -1, -1 \leq z \leq 0\}$ to an arc on γ_2. Similarly, ϕ takes $(0, 1, 0)$ to p_3, the dividing arc $\{-1 \leq x \leq 1, y = 1, z = 0\}$ to an arc around p_3 in Γ and the arc $\{x = 0, y = 1, 0 \leq z \leq 1\}$ to an arc on γ_2 (see Figure 9). Now we claim that we can attach

a topologically cancelling contact 2-handle so that the union of the contact 1 and 2-handles that we attach has the same effect as attaching a bypass along γ_2. Hence this procedure gives the contact anatomy of a bypass attachment, which is depicted (locally) in Figure 9. In the following, we explain how to glue the contact 2-handle so that the union of the contact 1 and 2-handles can be viewed as a neighborhood of a bypass disk $D = D_1 \cup D_2$, where D_i is a disk in the contact i-handle, for $i = 1, 2$.
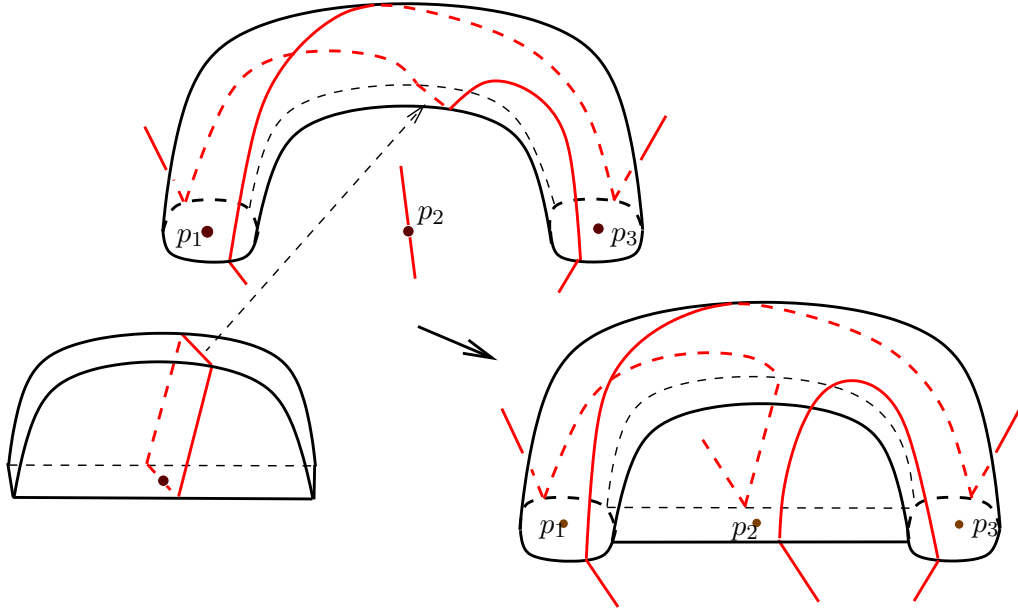

FIGURE 9. Anatomy of a bypass attachment

Construction of D_1: The idea here is to perturb the (rectangular) disk $\{z = 0\} \cap \{x \leq 0\}$ in the contact 1-handle (H_1, ζ_1) so that the boundary of that disk is a "Legendrian" curve on which there are one positive hyperbolic and two positive elliptic singular points. To be more precise, let a_1 denote the Legendrian arc $\{x = z = 0\}$ in H_1; a_2 denote the Legendrian arc $\{y = 1\} \cap \{x = -z\} \cap \{z \geq 0\}$; a_3 denote a Legendrian arc connecting the points $(-\frac{\epsilon}{\sqrt{2}}, 1, \frac{\epsilon}{\sqrt{2}})$ and $(-\frac{\epsilon}{\sqrt{2}}, -1, -\frac{\epsilon}{\sqrt{2}})$ on ∂H_1 (see Figure 10) and a_4 denote the Legendrian arc $\{y = -1\} \cap \{x = z\} \cap \{z \leq 0\}$. Then $a_1 \cup a_2 \cup a_3 \cup a_4$ bounds a surface D_1 in (H_1, ζ_1), where $(0, 0, 0)$ is a hyperbolic singular point and $(0, \pm 1, 0)$ are elliptic singular points on ∂D_1. Moreover we orient D_1 such that all the singularities on ∂D_1 are positive.

Construction of D_2: The idea here is to perturb the disk $\{y = 0\} \cap H_2$ in the contact 2-handle (H_2, ζ_1) into a disk whose boundary is a Legendrian circle on which there is a unique
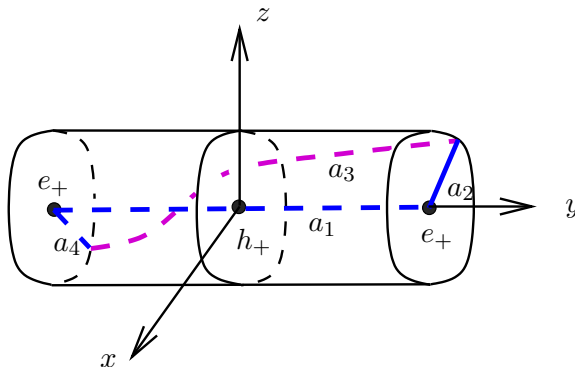
FIGURE 10. The disk D_1 has boundary $a_1 \cup a_2 \cup a_3 \cup a_4$.

elliptic singularity. To achieve this we first perturb the curve $\{y = 0\}$ on F_2 as follows:
Fix the points $(\theta, y) \in \{(\pi/2, 0), (3\pi/2, 0)\}$ and push the arc $\{\pi/2 \leq \theta \leq 3\pi/2, y = 0\}$
slightly in the upward direction and the arc $\{0 \leq \theta \leq \pi/2, y = 0\} \cup \{3\pi/2 \leq \theta \leq 2\pi, y = 0\}$ slightly in the downward direction as shown in Figure 11; Legendrian realize the per-
turbed curve and then consider the spanning disk D_2. With a little bit of care, we can make
sure that ∂D_2 has a unique elliptic singular point at $\theta = \pi/2$. More precisely, to have an
elliptic singularity at $\theta = \pi/2$, the slope of the perturbed curve should agree with the slope
of the characteristic foliation at that point on F_2, which certainly can be arranged.
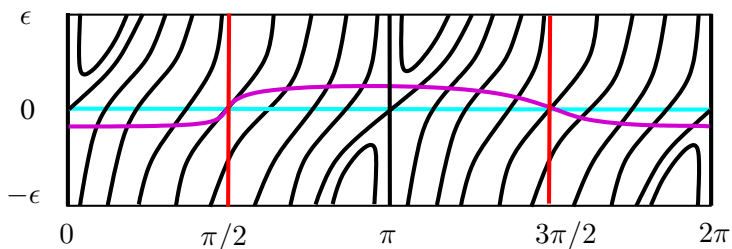
FIGURE 11. Perturbation of the curve $\{y = 0\} \cap F_2$

In order to exhibit the bypass disk D, we glue the disks D_1 and D_2 along some parts
of their boundaries as follows. Let us express ∂D_2 as a union of two arcs b_1 and b_2 where
$b_1 = \partial D_2 \cap \{0 \leq \theta \leq \pi\}$ and $b_2 = \partial D_2 \cap \{\pi \leq \theta \leq 2\pi\}$ on F_2. Then D is obtained by
gluing D_1 and D_2 where we simply identify a_3 and b_2. This can be achieved if the attaching
diffeomorphism takes the core $\{y = 0\}$ of the attaching region F_2 of the 2-handle H_2 to the
attaching curve that is indicated in Figure 12. Note that the boundary of the disk D consists

of the Legendrian arcs $\gamma_1 = a_1$ and $\gamma_2 = a_2 \cup b_1 \cup a_4$ and hence we can isotope D to be convex. If we orient D keeping the orientation of D_1, then the sign of the unique elliptic point on ∂D_2 becomes negative. The characteristic foliation on the convex disk D appears as in Figure 12, since the negative elliptic singular point is a source whereas the positive elliptic singular points are sinks and there is a unique hyperbolic singular point on ∂D.
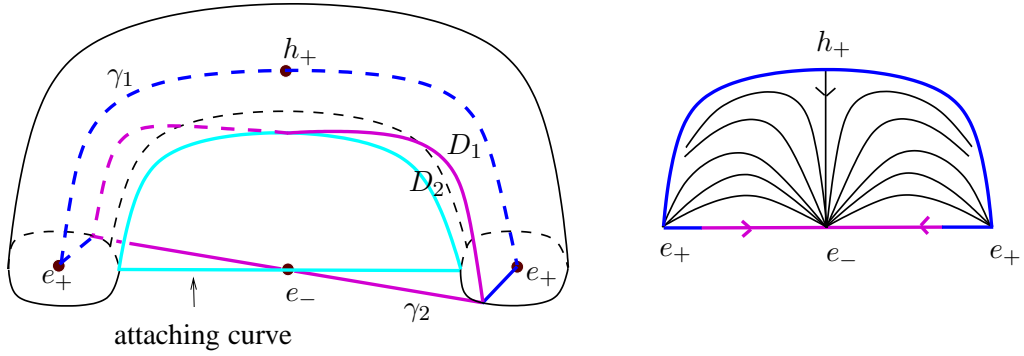

FIGURE 12. Left: Bypass disk $D = D_1 \cup D_2$ inside a bypass attachment; Right: The characteristic foliation on D

4. AN INFINITE FAMILY OF OVERTWISTED CONTACT 3-SPHERES

An overtwisted contact 3-sphere: In the following we describe a contact handle decomposition of an overtwisted contact structure ξ_0 in S^3. We start with attaching a bypass to a contact 0-handle along the Legendrian arc depicted in Figure 13 on the convex sphere ∂B^3, where the *southern* hemisphere is the $+$ region.
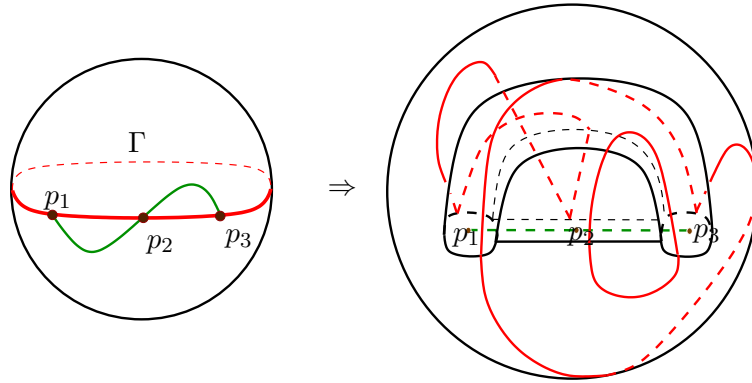

FIGURE 13. The result of a bypass attachment to a contact 0-handle

The Legendrian arc has its endpoints at $p_1, p_3 \in \Gamma$ and intersects Γ transversely at p_2. The diving set on the convex boundary of the resulting 3-ball B_{ot}^3 after the bypass attachment consists of three connected components (see Figure 13) and it follows, by Giroux's criterion [7], that B_{ot}^3 is overtwisted. Moreover we claim that B_{ot}^3 is the standard neighborhood of an overtwisted disk. To prove our claim we first describe a partial open book of B_{ot}^3. The partial open book of a contact 0-handle is described in [3]. The page S is an annulus, P is a neighborhood of a trivial arc connecting the distinct components of the boundary of this annulus, and the monodromy is a right-handed Dehn twist along the core of the annulus. In [12], Honda, Kazez and Matić describe how to obtain a partial open book of the resulting contact 3-manifold after a bypass attachment. According to their recipe, a 1-handle is attached to the page S to obtain the new page S' as depicted in Figure 14. Moreover $P' = P \cup P_1$, and the embedding of the new piece P_1 into S' is described explicitly in Figure 14: The solid arc in P_1 is mapped to the dashed arc going once over the new 1-handle. It follows that when we attach a bypass to a contact 0-handle along the arc given in Figure 13, the resulting partial open book (see Figure 14) is nothing but a positive stabilization of the partial open book of a standard neighborhood of an overtwisted disk ([12], see also [3]).
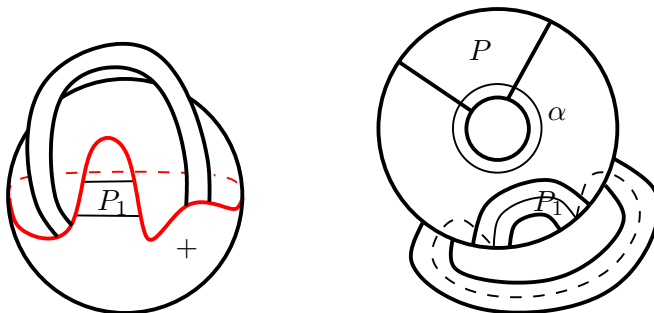

FIGURE 14. Left: The new page S' is $S \cup$ the attached 1-handle and $P' = P \cup P_1$; Right: The 1-handle P and a right-handed Dehn twist around α can be viewed as a stabilization of the rest.

Next we attach another bypass to B_{ot}^3 along the given arc on ∂B_{ot}^3 as depicted in Figure 15. The diving set on the convex boundary of the resulting 3-ball is connected as shown in Figure 15 and therefore we can cap off the convex boundary by a contact 3-handle. The resulting contact 3-sphere (S^3, ξ_0), which consists of a contact 0-handle, two contact 1-handles, two contact 2-handles and a contact 3-handle, is indeed overtwisted. In fact, we will show that $d_3(\xi_0) = 1/2$, which determines the homotopy (and hence the isotopy) class of the overtwisted contact structure ξ_0 in S^3.

To prove our claim we observe that

Remark 4.1. *We can turn contact handles upside down and a contact k-handle becomes a contact $(3-k)$-handle when turned upside down. Moreover, a bypass turned upside down is another bypass attached from the other side.*

Thus the second bypass and the last contact 3-handle attached to B^3_{ot} can be viewed as a copy of B^3_{ot} when the contact handles are turned upside down. This is because the upside down bypass is attached to the contact 0-handle along an arc isotopic to the one in Figure 13. Hence we conclude that (S^3, ξ_0) can be obtained by taking the double of the standard neighborhood B^3_{ot} of the overtwisted disk instead of attaching the second bypass and the last contact 3-handle. Since we know a partial open book for B^3_{ot}, we can actually construct an open book for the double by "gluing" the partial open books along their boundaries as explained in [12]. It turns out [3] that the open book for (S^3, ξ_0) has a twice punctured disk as its page and the monodromy is given by a positive and a negative Dehn twists along the two punctures, respectively. It is known (see, for example [5]) that the d_3-invariant of the contact structure corresponding to such an open book is equal to $1/2$.
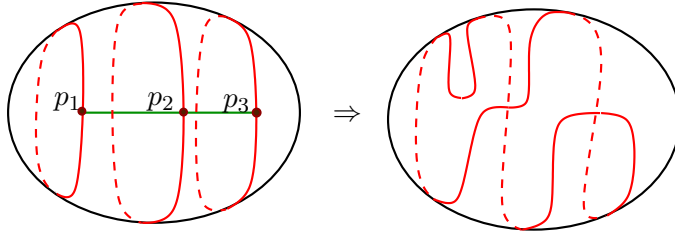

FIGURE 15. Left: The attaching arc of a second bypass; Right: The dividing set after the second bypass attachment

An infinite family of overtwisted contact 3-spheres: We can generalize our discussion above to obtain contact handle decompositions of infinitely many pairwise non-isotopic overtwisted contact 3-spheres. We first fix a positive integer n, and choose a sequence of nearby points $p_1, p_2, \ldots, p_{3n}$ on the dividing set on the boundary ∂B^3 of the contact 0-handle, where the *southern* hemisphere is the $+$ region. For $k = 1, 4, 7, \ldots, 3n - 2$, let γ_k be an arc isotopic to the one depicted in Figure 13 starting at p_k, passing through p_{k+1}, and ending at p_{k+2}. Next we attach a bypass along each γ_k to this contact 0-handle. The result of attaching these bypasses is indeed an overtwisted 3-ball where the dividing set on the convex boundary has $2n + 1$ connected components as shown in Figure 16.

The resulting partial open book can be constructed similar to the $n = 1$ case (that we already discussed), since a bypass attachment is just a local modification. Then by taking the double of the resulting overtwisted 3-ball we get an overtwisted 3-sphere (S^3, ξ_n). The page of the open book compatible with (S^3, ξ_n) is a disk with $2n$-punctures. Let t_m
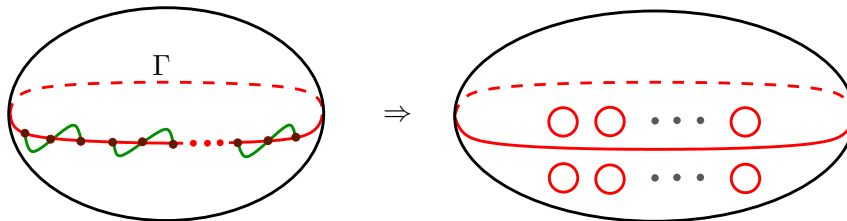
FIGURE 16. Left: The attaching arcs for n bypasses; Right: The dividing set after attaching bypasses along the given arcs

denote a right-handed Dehn twist around α_m, where α_m is a curve around the mth puncture. Then the monodromy of this open book is given by $\prod_{i=1}^{n} t_i t_{n+i}^{-1}$. It follows that, $d_3(\xi_n) = (2n + 1)/2$, since ξ_n can be obtained from ξ_{n-1} by a positive stabilization followed by a negative stabilization, where a negative stabilization increases the d_3-invariant by one while a positive stabilization does not affect the contact structure (see, for example, [14]). Similar to the $n = 1$ case, instead of doubling the overtwisted 3-ball to obtain (S^3, ξ_n), we can attach n more bypasses to this 3-ball along the n arcs shown in Figure 17 and a contact 3-handle to cap off the resulting boundary (see Remark 4.1). Hence, for each positive integer n, we get an explicit contact handle decomposition of the overtwisted 3-sphere (S^3, ξ_n) consisting of a contact 0-handle, $2n$ contact 1-handles, $2n$ contact 2-handles and a contact 3-handle.
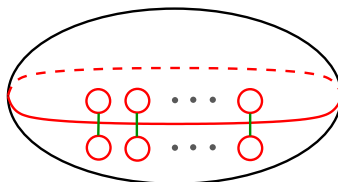

FIGURE 17. The attaching arcs for the second set of bypasses

5. FINAL REMARKS

It is well-known that one can slide handles in a given handle decomposition of a smooth manifold. The natural question which arises from the discussion in this paper is that whether there is an analogue of handle sliding in contact topology. Similarly one can ask whether there is a contact handle cancellation? It seems to us that both questions have affirmative answers and we are planning to investigate such issues in a future work.

In addition, it may be possible to compute the EH-class of a contact 3-manifold via its contact handle decomposition. In order to achieve this goal one can first obtain a partial open book decomposition of the contact 3-manifold based on its handle decomposition.

The idea here is that the page S of a partial open book will acquire a 1-handle once we attach a contact 1-handle to the contact 3-manifold at hand. The attachment of a contact 2-handle (in fact just its attaching curve) will simply determine P and its embedding in S. The attachments of contact 0 and 3-handles will manifest themselves merely as suitable stabilizations. Finally, to compute the EH-class of the resulting contact 3-manifold, we apply the techniques recently developed by Honda, Kazez and Matić [12].

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