EXOTIC STEIN FILLINGS WITH ARBITRARY FUNDAMENTAL GROUP

ANAR AKHMEDOV AND BURAK OZBAGCI

ABSTRACT. Let G be a finitely presentable group. We provide an infinite family of homeomorphic but pairwise non-diffeomorphic, symplectic but non-complex closed four-manifolds with fundamental group G such that each member of the family admits a Lefschetz fibration of the same genus over the two-sphere. As a corollary, we also show the existence of a contact three-manifold which admits infinitely many homeomorphic but pairwise non-diffeomorphic Stein fillings such that the fundamental group of each filling is isomorphic to G. Moreover, we observe that the contact three-manifold above is contactomorphic to the link of some isolated complex surface singularity equipped with its canonical contact structure.

1. INTRODUCTION

In his ground-breaking work, Donaldson [15] proved that every closed symplectic 4-manifold admits a Lefschetz pencil over S^2 and Gompf [21] showed that every finitely presentable group G can be realized as the fundamental group of some closed symplectic 4-manifold. Since a Lefschetz pencil can be turned into a Lefschetz fibration by blowing up its base locus—that has no effect on the fundamental group of the underlying 4-manifold—one immediately obtains the existence of a closed symplectic 4-manifold with fundamental group G, which admits a Lefschetz fibration over S^2. An alternative method of proof of this result was given in [10, 27], where a Lefschetz fibration was constructed over S^2 via an explicit description of its vanishing cycles on a surface of genus greater than one, so that the fundamental group of the total space is isomorphic to G. Note that the total space of any Lefschetz fibration over S^2 admits a symplectic structure, provided that its fiber genus is greater than one [22]. Our first goal here is to prove the following result.

Theorem 1.1. *For any finitely presentable group G, there exists an infinite family of homeomorphic but pairwise non-diffeomorphic, symplectic but non-complex closed 4-manifolds with fundamental group G such that each member of this family admits a Lefschetz fibration of the same genus over S^2.*

2000 *Mathematics Subject Classification.* 57R17.

The first author was partially supported by the NSF grant DMS-1005741. The second author was partially supported by a research grant of the Scientific and Technological Research Council of Turkey.

Note that the Lefschetz fibrations we construct are necessarily *non-holomorphic*, since their total spaces are non-complex. In order to prove Theorem 1.1, we first design an initial closed symplectic 4-manifold with fundamental group G using some standard techniques of 4-manifold theory such as Luttinger surgery and symplectic sum in an intricate way so that the well-known Lefschetz fibrations on the symplectic building blocks involved in the construction fit together properly to yield a Lefschetz fibration (with some extra properties) on this 4-manifold. Then, by performing Fintushel-Stern knot surgery—using an infinite family of pairwise inequivalent fibered knots of some fixed genus—along a homologically essential torus of square zero in our initial manifold, we obtain an infinite family of closed symplectic 4-manifolds. The crux of the matter is that the torus above is strategically embedded relative to the Lefschetz fibration structure on the initial 4-manifold so that this structure is retained after the knot surgery. Moreover, we show that the 4-manifolds in this family are all homeomorphic, and we rely on their Seiberg-Witten invariants, which are determined by the Alexander polynomials of the fibered knots, to distinguish their diffeomorphism types pairwise (cf. [18]).

Finally, we use the classification of complex surfaces, symplectic/complex Kodaira dimension, and the Seiberg-Witten invariants to show that these symplectic 4-manifolds can not carry any complex structures, which clearly implies that no Lefschetz fibrations on them (including the ones we constructed above) can be holomorphic. Note that Parshin and Arakelov's proofs of the Geometric Shafarevich Conjecture show that there are only finitely many holomorphic Lefschetz fibrations with fixed fiber genus and singular set over S^2 (cf. [11, 41]).

Next, we prove the existence of a contact 3-manifold which admits an infinite family of homeomorphic but pairwise non-diffeomorphic Stein fillings with fundamental group G—which is essentially the content of Corollary 1.2 below—by removing a regular fiber and some number of sections from each Lefschetz fibration over S^2 given in Theorem 1.1. This simple-minded approach has some delicate issues that has to be dealt with such as to show that the fundamental group remains the same after the removal of the aforementioned pieces from each Lefschetz fibration and that the Seiberg-Witten invariants are still effective to distinguish the diffeomorphism types of the remaining Stein fillings pairwise. The non-holomorphicity of the Lefschetz fibrations given in Theorem 1.1 does not play any role in the proof of Corollary 1.2, although some other extra properties they possess turn out to be essential.

Corollary 1.2. *For any finitely presentable group G, there exists a contact 3-manifold which admits infinitely many homeomorphic but pairwise non-diffeomorphic Stein fillings such that the fundamental group of each filling is isomorphic to G. Moreover, we observe that the contact 3-manifold above is contactomorphic to the link of some isolated complex surface singularity equipped with its canonical contact structure.*

We would like to point out that the word *exotic* in the title of the present paper refers to *homeomorphic but non-diffeomorphic*, which is commonly used among 4-manifold topologists. The second statement in Corollary 1.2 follows immediately from [6, Lemmata 4.1 & 4.2], since the proof presented there for a certain type of Seifert fibered 3-manifolds holds true verbatim for the case at hand in the present paper. By including the second statement in Corollary 1.2, we intend to emphasize the fact that although many examples of isolated complex surface singularity links which admit only finitely many Stein/symplectic fillings, up to diffeomorphism or symplectic deformation equivalence, appeared in the literature (see, for example, [14, 29, 33, 35, 36, 37, 39]), very few examples of singularity links with infinitely many Stein fillings are known.

Note that Corollary 1.2 may be considered as a vast generalization of [5, Theorem 1.1] and [6, Theorem 5.3], where the first statement was proved for $G = 1$ in the former and both statements were proved for $G \in \{\mathbb{Z} \oplus \mathbb{Z}_m | \ m \in \mathbb{N} \}$ in the latter. The method of proof in the present paper, however, is very similar to the ones cited above.

We would like to point out that none of the previous constructions of Lefschetz fibrations in the literature could be effectively utilized instead of our Theorem 1.1 to prove Corollary 1.2. For example, the Lefschetz fibrations over S^2 described in [27] do not carry the homologically essential tori we need for producing exotic copies using the knot surgery operation. This is due to the fact that the examples in [27] are obtained by performing many symplectic sums along higher genus surfaces, in contrast to the examples in the present article, where we perform *only two* symplectic sums. Moreover, the existence of *homologically trivial vanishing cycles* in those Lefschetz fibrations rules out the Steinness of the remaining piece after the removal of some sections and a regular fiber.

The direct approach using Donaldson's Lefschetz pencils would not work for us either, since any of the sections of a Lefschetz fibration obtained by blowing up the base locus of a Lefschetz pencil has self-intersection -1 and therefore is not suitable for our construction of exotic Stein fillings of an *isolated complex surface singularity.*

Note that the total space of any of the Lefschetz fibrations that we construct in this paper is symplectically minimal, which follows from Usher's theorem in [46] (see also [13]), and has $b_2^+ \geq 2$. The case $b_2^+ = 1$ has been studied separately in [9]. The examples there, however, do not necessarily yield Stein fillings.

Outline of the paper: In Section 2, we briefly review Luttinger surgery, knot surgery and symplectic sum. In Section 3, we discuss a positive Dehn twist factorization of a certain involution on a closed orientable surface, which leads to the description of a set of Lefschetz fibrations that we use in our constructions. In Section 4, using Luttinger surgery and symplectic sum we design the aforementioned initial closed symplectic 4-manifold with $\pi_1 = G$ and $b_2^+ \geq 2$ which admits a Lefschetz fibration over S^2 that has some additional features. In Section 5, we prove Theorem 1.1, while in Section 6, we prove Corollary 1.2.

2. LUTTINGER SURGERY, SYMPLECTIC SUM AND KNOT SURGERY

Luttinger surgery (cf. [31], [12]), symplectic sum (cf. [21]) and knot surgery (cf. [18]) are the fundamental tools for constructing exotic smooth structures on 4-manifolds. In this section, we briefly recall these operations.

2.1. **Luttinger surgery.** Let L be a Lagrangian torus embedded in a closed symplectic 4-manifold (X, ω). It follows that L has a trivial normal bundle. In addition, by the Lagrangian neighborhood theorem of Weinstein, a tubular neighborhood νL of L in X can be identified *symplectically* with a neighborhood of the zero-section in the cotangent bundle $T^*L \simeq T \times \mathbb{R}^2$ with its standard symplectic structure. This identification gives a framing to L, which is called the Lagrangian framing. Let γ be any simple closed curve on L. The Lagrangian framing determines uniquely, up to homotopy, a push-off of γ in $\partial(X - \nu L)$, which we denote again by γ.

Definition 2.1. For any integer m, the (L, γ, m) *Luttinger surgery* on X is defined as

$$X(L, \gamma, m) = (X - \nu L) \cup_\phi (S^1 \times S^1 \times D^2),$$

where, for a meridian μ_L of L, the gluing map $\phi : S^1 \times S^1 \times \partial D^2 \to \partial(X - \nu L)$ satisfies $\phi([\partial D^2]) = [\mu_L] + m[\gamma]$ in $H_1(\partial(X - \nu L)$.

Note that for $m = 0$, the Luttinger surgery is trivial, which means that $X(L, \gamma, 0) = X$.

Remark 2.2. A salient feature of Luttinger surgery is that it can be done symplectically, i.e., the symplectic form ω on $X - \nu L$ can be extended to a symplectic form on $X(L, \gamma, m)$ as shown in [12].

Lemma 2.3. *The fundamental group $\pi_1(X(L, \gamma, m))$ is obtained as the quotient of the group $\pi_1(X - \nu L)$ by the normal subgroup generated by the product $\mu_L \gamma^m$. Moreover, we have $\sigma(X) = \sigma(X(L, \gamma, m))$, and $\chi(X) = \chi(X(L, \gamma, m))$, where σ and χ denote the signature and the Euler characteristic, respectively.*

Proof. The result about the fundamental group follows from the Seifert-van Kampen's theorem. The fact that the signature is preserved under Luttinger surgery is a consequence of the Novikov additivity. The result about the Euler characteristic is evident. □

2.2. **Symplectic sum.** Suppose that X_1 and X_2 are closed symplectic four manifolds. For each $i = 1, 2$, let F_i be a 2-dimensional, smooth, closed, connected symplectic submanifold of X_i. Assume that F_1 and F_2 have the same genus and $[F_1]^2 + [F_2]^2 = 0$. Let νF_i denote the disk normal bundle of F_i in X_i.

Definition 2.4. For any orientation-reversing diffeomorphism $\psi : \partial\nu F_1 \to \partial\nu F_2$ that is lifted from an orientation-preserving diffeomorphism from F_1 to F_2, the *symplectic sum* of X_1 and X_2 is defined as the closed 4-manifold

$$X_1\#_\psi X_2 = (X_1 - \nu F_1) \cup_\psi (X_2 - \nu F_2).$$

This gluing is called a symplectic sum since there is a natural isotopy class of symplectic structures on $X_1\#_\psi X_2$ extending the symplectic structures on $X_1 - \nu F_1$ and $X_2 - \nu F_2$ as shown in [21].

2.3. **Knot surgery.** Let K be an arbitrary knot in S^3 and let $N(K)$ denote a tubular neighborhood of $K \subset S^3$. Suppose that T is an embedded torus with a tubular neighborhood $T \times D^2$ in some smooth 4-manifold X. Let X_K denote the 4-manifold obtained by gluing $X \setminus (T \times D^2)$ with $(S^1 \times (S^3 \setminus N(K))$ along their boundaries, where we identify the boundary of a disk normal to T with a longitude of $N(K)$. The 4-manifold X_K is said to be obtained from X by a Fintushel-Stern knot surgery [18].

3. Positive factorizations of some involutions on surfaces

In this section, we briefly introduce a set of Lefschetz fibrations over S^2, which will be one of the main ingredients in our proofs. It is a standard fact that an expression of the identity in the mapping class group $Map(\Sigma)$ of some closed orientable surface Σ as a product of positive (a.k.a. right-handed) Dehn twists along some simple closed curves on Σ induces a Lefschetz fibration over S^2. Here the regular fiber of the Lefschetz fibration is the surface Σ at hand, while the simple closed curves are the vanishing cycles.

An obvious factorization of the identity in $Map(\Sigma)$ can be obtained by taking the square of a factorization of some involution on Σ. Consider, for example, the involution θ on the surface Σ_{2g+n-1} of genus $2g + n - 1$ depicted in Figure 1. The involution θ can be viewed as a combination of the hyperelliptic involution on the horizontal surface of genus $n - 1$ with a "vertical" involution on a surface of genus $2g$. Both the hyperelliptic and the vertical involution of surfaces have well-known explicit positive factorizations in the respective mapping class groups. When $n = 1$, a positive factorization of the vertical involution θ is given in [32] for $g = 1$ and in [27] for $g \geq 2$.

Since the involution θ in Figure 1 is obtained as a combination of two involutions in the mapping class group with well-known positive factorizations, a factorization of θ, in principle, should be a combination of those factorizations. As a matter of fact, an explicit factorization of θ was worked out in [23, Theorem 2.0.1]. The method of proof in this unpublished manuscript is a straightforward application of the Alexander's trick. Namely, one first fixes a finite set of curves whose complement is a disk on the surface and shows that the image of each curve under the involution and (the potential) positive factorization is isotopic.

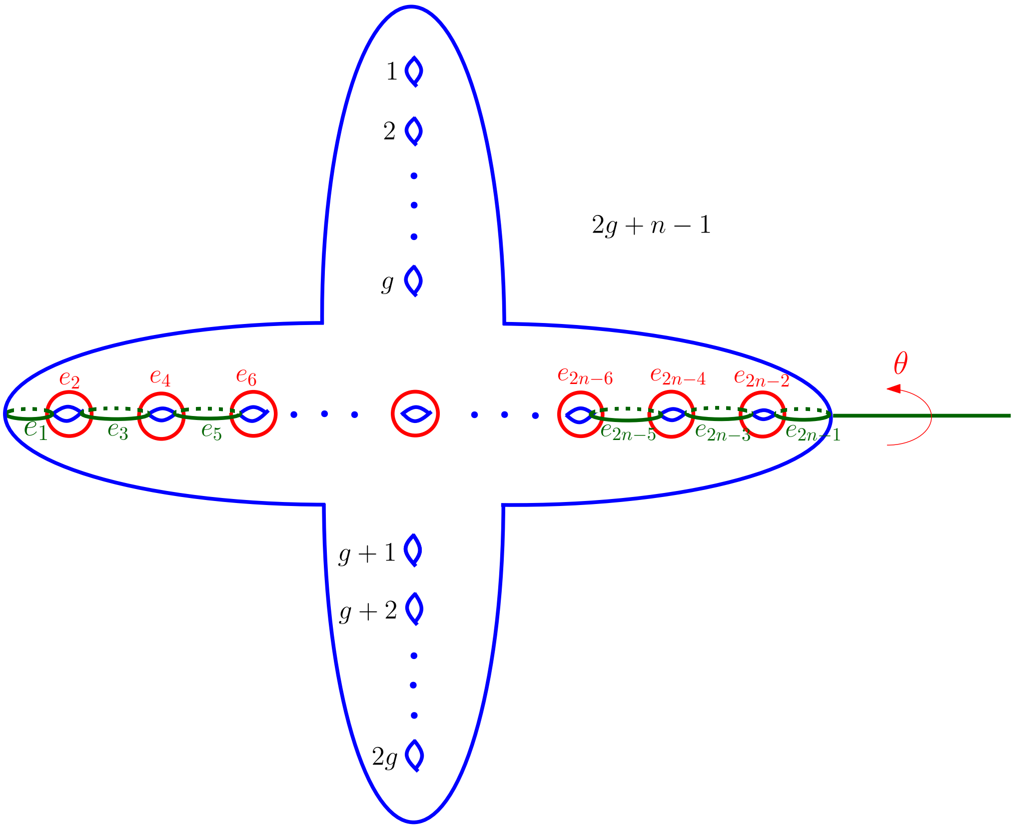

FIGURE 1. The involution θ on the surface Σ_{2g+n-1}

The positive factorization of θ in [23] was also verified employing rather conceptual methods in [47, Theorem 5], up to Hurwitz equivalence. In both papers, it was shown that θ can be expressed as a product of $4n + 2g - 2$ positive Dehn twists. Let $Y(n, g)$ denote the total space of the Lefschetz fibration defined by $\theta^2 = 1 \in Map(\Sigma_{2g+n-1})$. It follows that the 4-manifold $Y(n, g)$ has a genus $2g + n - 1$ Lefschetz fibration over S^2 with $s = 8n + 4g - 4$ singular fibers, all of which are induced by nonseparating vanishing cycles.

The Euler characteristic of the symplectic 4-manifold $Y(n, g)$ can be easily computed using the following formula:

$$\chi(Y(n,g)) = \chi(S^2)\chi(\Sigma_{2g+n-1}) + s = 2(2-2(n+2g-1)) + 8n + 4g - 4 = 4n - 4g + 4.$$

The signature $\sigma(Y(n, g))$ was calculated to be $-4n$ in [47, Theorem 1].

The Lefschetz fibration defined by $\theta^2 = 1$ can also be described with a different point of view as follows. Take a double branched cover of $S^2 \times \Sigma_g$ along the union of $2n$

disjoint copies of $S^2 \times \{pt\}$ and two disjoint copies of $\{pt\} \times \Sigma_g$ as illustrated in Figure 2. The resulting branched cover has $4n$ singular points corresponding to the number of the intersection points of the $2n$ horizontal spheres and the two vertical genus g surfaces in the branch set. By desingularizing these $4n$ singular points, one obtains the symplectic 4-manifold $S^2 \times \Sigma_g \# 4n\overline{\mathbb{C}P^2}$. Note that by projecting onto the S^2 factor we obtain a (vertical) fibration over S^2 whose generic fiber is the double cover of Σ_g, branched over $2n$ points. Thus, a generic fiber of the vertical fibration has genus $n+2g-1$. Furthermore, each of the two singular fibers of the vertical fibration, arising from two disjoint copies of $\Sigma_g \times \{pt\}$, can be perturbed into $4n+2g-2$ Lefschetz type singular fibers, which is equivalent to the positive factorization of the involution θ, as shown in the proof of [47, Theorem 27]. As an immediate corollary one obtains that the 4-manifold $Y(n,g)$ is in fact diffeomorphic to $S^2 \times \Sigma_g \# 4n\overline{\mathbb{C}P^2}$.

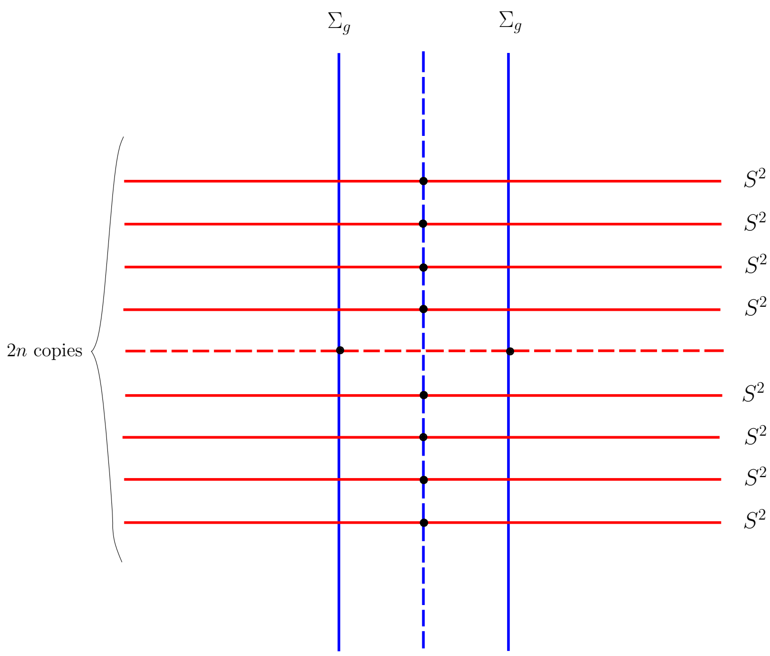

FIGURE 2. The branched cover description of the Lefschetz fibrations

4. LEFSCHETZ FIBRATIONS WITH ARBITRARY FUNDAMENTAL GROUP

In this section, for each finitely presentable group G, we construct a closed symplectic 4-manifold with $\pi_1 = G$ and $b_2^+ \geq 2$ which admits a Lefschetz fibration over S^2 having some additional properties (see Proposition 4.7). Parts of this section overlaps with certain parts of [9], where the case $b_2^+ = 1$ has been studied.

4.1. **Construction for a finitely generated free group.** In the following, we first explain our construction for the case of a finitely generated free group of arbitrary rank $g \geq 1$, before we deal with the general case.

The product $\Sigma_g \times T^2$ admits a symplectic structure, where Σ_g is a closed symplectic genus g surface and T^2 is a symplectic torus. Suppose that $\{a_i, b_i : 1 \leq i \leq g\}$ is the set of standard generators of $\pi_1(\Sigma_g)$ and $\{c, d\}$ is the set of standard generators of $\pi_1(T^2)$. Let $\{p_i, q_i \geq 0 : 1 \leq i \leq g\}$ be a set of nonnegative integers and let $\overline{p} = (p_1, \ldots, p_g)$ and $\overline{q} = (q_1, \ldots, q_g)$.

We denote by $M_g(\overline{p}, \overline{q})$ the symplectic 4-manifold obtained by performing a Luttinger surgery on the symplectic 4-manifold $\Sigma_g \times T^2$ along each of the $2g$ Lagrangian tori with the associated framings belonging to the set

$$\mathcal{L} = \{(a_i' \times c', a_i', -p_i), (b_i' \times c'', b_i', -q_i) \mid 1 \leq i \leq g\}.$$

The reader can consult [8, Figure 1] for more on the prime and double prime notation, where these loops are explicitly depicted. Here, a_i' for example, is a free simple loop on the surface Σ_g parallel to the generator a_i of $\pi_1(\Sigma_g)$. Therefore $a_i' \times c'$ is a Lagrangian torus in $\Sigma_g \times T^2$, along which a Luttinger surgery is possible. Note that every Luttinger surgery in a symplectic 4-manifold is determined by a triple: a Lagrangian torus, a simple closed curve on that torus and an integer (see Section 2.1).

This family of symplectic 4-manifolds $M_g(\overline{p}, \overline{q})$ has been studied in [8] (see the discussion on pages 579–580, and 592–593). For further details, we refer the reader to [8] and references therein. The proof of the following result follows from the Example in [12, page 189].

Lemma 4.1. *The 4-manifold $M_g(\overline{p}, \overline{q})$ admits a locally trivial genus g fibration over T^2.*

Proof. The $(a_i' \times c', a_i', -p_i)$ or $(b_i' \times c'', b_i', -q_i)$ Luttinger surgery in the trivial bundle $\Sigma_g \times T^2$ preserves the fibration structure over T^2 introducing a monodromy of the fiber Σ_g along the curve c' and c'', respectively, in the base. Depending on the type of the surgery the monodromy is either $(t_{a_i})^{p_i}$ or $(t_{b_i})^{q_i}$, where t denotes a Dehn twist. □

The proof of the next result—which essentially follows from Lemma 2.3—can be found in [8].

Lemma 4.2. *The fundamental group of $M_g(\overline{p}, \overline{q})$ is generated by a_i, b_i $(i = 1, \ldots, g)$ and c, d, with the following relations:*

(1) $[b_i^{-1}, d^{-1}] = a_i^{p_i}$, $[a_i^{-1}, d] = b_i^{q_i}$, *for all* $1 \leq i \leq g$,
(2) $[a_i, c] = 1$, $[b_i, c] = 1$, *for all* $1 \leq i \leq g$,
(3) $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1$, *and*
(4) $[c, d] = 1$.

The torus $\{\mathrm{pt}\} \times T^2 \subset \Sigma_g \times T^2$ induces a torus $\mathbb{T}$, which is a section of the fibration in Lemma 4.1, with trivial normal bundle in $M_g(\overline{p}, \overline{q})$. On the other hand, a regular fiber of the elliptic fibration on the complex surface $E(n)$ is also a torus of square zero.

Definition 4.3. Let $X_{g,n}(\overline{p}, \overline{q})$ denote the symplectic sum of $M_g(\overline{p}, \overline{q})$ along the torus $\mathbb{T}$ with the elliptic surface $E(n)$ along a regular elliptic fiber.

Lemma 4.4. *The symplectic* 4*-manifold* $X_{g,n}(\overline{p}, \overline{q})$ *admits a genus* $2g + n - 1$ *Lefschetz fibration over* S^2 *with at least* $4n$ *pairwise disjoint sphere sections of self intersection* -2*. Moreover,* $X_{g,n}(\overline{p}, \overline{q})$ *contains a homologically essential embedded torus of square zero disjoint from these sections which intersects each fiber of the Lefschetz fibration twice.*

Proof. By definition, $X_{g,n}(\overline{p}, \overline{q})$ is obtained as the symplectic sum of the complex surface $E(n)$ along a regular elliptic fiber with the symplectic 4-manifold $M_g(\overline{p}, \overline{q})$ along the section $\mathbb{T}$ defined above.

Since the complex surface $E(n)$ can be obtained (see [22, Section 7.3]) as a desingularization of the branched double cover of $S^2 \times S^2$ with the branching set being 4 copies of $\{pt\} \times S^2$ and $2n$ copies of $S^2 \times \{pt\}$, it admits a genus $n - 1$ fibration over S^2 as well as an elliptic fibration over S^2, both of which are obtained by the projection of $S^2 \times S^2$ onto one of the S^2 factors. In fact, both fibrations can be realized as Lefschetz fibrations and a regular fiber of the elliptic fibration on $E(n)$ intersects every genus $n - 1$ fiber of the other Lefschetz fibration twice.

Consequently, when performing the symplectic sum of $E(n)$ along a regular elliptic fiber with the surface bundle $M_g(\overline{p}, \overline{q})$ along the section $\mathbb{T}$, the fibration structures in both pieces can be glued together to yield a genus $2g + n - 1$ Lefschetz fibration on $X_{g,n}(\overline{p}, \overline{q})$ over S^2. The reason that this works is that if one composes the projection $M_g(\overline{p}, \overline{q}) \to T^2$ with a hyperelliptic quotient $T^2 \to S^2$, one obtains a fibration of $M_g(\overline{p}, \overline{q})$ over S^2, with disconnected generic fiber and four singular fibers, such that the fibration near the torus $\mathbb{T}$ is equivalent to the fibration on $E(n)$ near a regular fiber.

Moreover, we observe that a sphere section of the genus $n - 1$ Lefschetz fibration $E(n) \to S^2$ induce a section of the genus $2g + n - 1$ Lefschetz fibration $X_{g,n}(\overline{p}, \overline{q}) \to S^2$. Since $E(n)$ can be realized as a fiber sum of two copies of $\mathbb{C}P^2 \# (4n + 1)\overline{\mathbb{C}P^2}$ along a genus $n - 1$ surface (see [22, Remark 7.3.9(b)]), and since there is a Lefschetz fibration $\mathbb{C}P^2 \# (4n+1)\overline{\mathbb{C}P^2} \to S^2$ with at least $4n$ pairwise disjoint sphere sections of self intersection -1 (cf. [26, 43, 44]), we conclude that the genus $n - 1$ Lefschetz fibration $E(n) \to S^2$ has at least $4n$ pairwise disjoint sphere sections of self intersection -2.

Furthermore, the homologically essential embedded torus $\mathbb{T}$ of square zero in $X_{g,n}(\overline{p}, \overline{q})$ which is disjoint from these sections intersects each fiber of the Lefschetz fibration twice.

□

Lemma 4.5. *The fundamental group of the symplectic 4-manifold $X_{g,n}(\overline{p}, \overline{q})$ is generated by the set $\{a_i, b_i : 1 \leq i \leq g\}$ subject to the relations:*

(1) $a_i^{p_i} = 1, b_i^{q_i} = 1$, *for all* $1 \leq i \leq g$, *and*
(2) $\Pi_{j=1}^{g} [a_j, b_j] = 1$.

Proof. Choose a base point x on $\partial(\nu\mathbb{T})$ such that $\pi_1(M_g(\overline{p}, \overline{q}) \backslash \nu\mathbb{T}, x)$ is normally generated by a_i, b_i $(i = 1, \cdots, g)$ and c, d. Notice that the symplectic torus $\mathbb{T}$ is disjoint from the neighborhoods of $2g$ Lagrangian tori in $\mathcal{L}$ above. Consequently, all but relation (3) in Lemma 4.2 holds in $\pi_1(M_g(\overline{p}, \overline{q}) \setminus \nu\mathbb{T})$. The product $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g]$ is no longer trivial, and it represents a meridian of $\mathbb{T}$ in $\pi_1(M_g(\overline{p}, \overline{q}) \setminus \nu\mathbb{T})$. Now since the complement of a neighborhood of a regular fiber in the elliptic fibration on $E(n)$ is simply connected, after the fiber sum which by definition identifies the regular fiber with the section $\mathbb{T}$, we have $c = d = 1$ in the fundamental group of $X_{g,n}(\overline{p}, \overline{q})$. Hence we obtain the desired presentation for $\pi_1(X_{g,n}(\overline{p}, \overline{q}))$. □

Corollary 4.6. *The fundamental group of $X_{g,n}((1, 1, \ldots, 1), (0, 0, \ldots, 0))$ is a free group rank g.*

Proof. The result follows immediately from Lemma 4.5, by setting $p_i = 1$ and $q_i = 0$, for all $1 \leq i \leq g$. □

To summarize, combining Lemma 4.4 with Corollary 4.6, we obtain a closed symplectic 4-manifold whose fundamental group is a free group of rank g, which has the desired properties we listed at the beginning of this section.

4.2. **Construction for an arbitrary finitely presentable group.** In this section, we give a construction for the general case. Let G be a finitely presentable group with a given presentation $\langle x_1, \ldots, x_k \mid r_1, \ldots, r_s \rangle$. Suppose that Σ_k is a closed orientable surface of genus k and let $\{a_j, b_j : 1 \leq j \leq k\}$ denote the set of standard generators of $\pi_1(\Sigma_k)$. Since r_i is a word in the generators $x_1, \ldots, x_k$, there is a smooth immersed oriented circle γ_i on F representing the corresponding word in $\pi_1(\Sigma_k)$, obtained by replacing each x_j with b_j. We may choose the loop γ_i (up to homotopy) such that at each self-intersection point, only two segments of γ_i intersect transversely. In order to carry out some Luttinger surgeries we have in mind, we first need to resolve the self-intersection points of γ_i by a trick that was initially introduced in [10], and refined further in [27]. In the following, we will use the version discussed in [27].

For each self-intersection point of the immersed curve γ_i where two segments intersect locally, we glue a 1-handle to F and modify γ_i so that one of the intersecting segments goes through the handle while the other remains under it. The modified curve on the new surface will be denoted by γ_i as well. Notice that we only need a finite number of such handles to resolve all the self-intersections of γ_i. After these handle additions, the surface Σ_k is changed to a surface Σ_g of genus $g \geq k$ so that each (modified) γ_i is now an embedded curve on Σ_g.

Let $\{a_i, b_i : 1 \leq j \leq g\}$ represent the set of standard generators of $\pi_1(\Sigma_g)$, extending the standard generators of $\pi_1(\Sigma_k)$. We perform Luttinger surgeries on the standard symplectic 4-manifold $\Sigma_g \times T^2$, along the following Lagrangian tori

$$\{(a'_i \times c', a'_i, -1),\ (b'_i \times c'', b'_i, -1),\ \ k+1 \leq i \leq g\}.$$

Moreover, just as in the free group case, we perform Luttinger surgeries on $\Sigma_g \times T^2$ along the $2k$ Lagrangian tori belonging to the set

$$\{(a'_i \times c', a'_i, -1),\ (b'_i \times c'', b'_i, 0) \mid 1 \leq i \leq k\}.$$

Let $M(G)$ denote the symplectic 4-manifold obtained by the total of $2g$ Luttinger surgeries on $\Sigma_g \times T^2$. Notice that k of these Luttinger surgeries have a surgery coefficient 0. As in Section 4.1, we take a symplectic sum of $M(G)$ along the torus $\mathbb{T}$ descending from $\{pt\} \times T^2$ with $E(n)$ along a regular elliptic fiber (here we assume $n \geq 2$ for reasons which will be clear in Section 6), and denote the resulting symplectic 4-manifold by $Y_n(G)$. Note that $\pi_1(Y_n(G))$ is a free group of rank k, by Lemma 4.5.

Since a regular fiber of a genus $n-1$ hyperelliptic Lefschetz fibration on $E(n)$ intersects a regular fiber of an elliptic fibration on $E(n)$ at two points, $Y_n(G)$ admits a genus $2g+n-1$ Lefschetz fibration over S^2, just as we explained in the proof of Lemma 4.4.

Note that $Y_n(G)$ can also be constructed as the twisted fiber sum of two copies of the genus $2g+n-1$ Lefschetz fibration on $Y(n,g) \cong S^2 \times \Sigma_g \# 4n\overline{\mathbb{C}P^2}$ over S^2, that we discussed in Section 3. This essentially follows from the fact that the symplectic sum of $E(n)$ along a regular elliptic fiber with $\Sigma_g \times T^2$ along the square zero torus $\mathbb{T}$ is diffeomorphic to the untwisted fiber sum of two copies of the genus $2g+n-1$ fibration on $S^2 \times \Sigma_g \# 4n\overline{\mathbb{C}P^2}$, which in turn follows from the branched cover description of these 4-manifolds (cf. [19, page 1466]). When performing the Luttinger surgeries, the gluing diffeomorphism of the genus $2g+n-1$ fibration, which is an identity map initially, turns into the product of a certain Dehn twists. This gluing diffeomorphism ϕ can be described explicitly using the curves along which we perform our Luttinger surgeries: $\phi = t_{a_1} \cdots t_{a_k} t_{a_{k+1}} t_{b_{k+1}} \cdots t_{a_g} t_{b_g}$. Here we view the Dehn twists t_{a_i}'s and t_{b_i}'s as self-diffeomorphisms of the genus $2g+n-1$ surface by extending them from the genus g surface by the identity. Note that an analogues construction is given in [48].

The global monodromy of the genus $2g+n-1$ Lefschetz fibration on $Y_n(G)$ is given by the following word: $\theta^2\phi^{-1}\theta^2\phi = 1$, where a positive factorization $\theta \in Map(\Sigma_{2g+n-1})$ is described in Section 3. Here, we would like to point out that the global monodromy consists of Dehn twists in the factorization of θ^2 and their images by the diffeomorphism ϕ, whose particular factorization above has nothing to do with the vanishing cycles.

In addition to applying the above Luttinger surgeries on the "standard" tori, we apply s more surgeries along tori:

$$\{(\gamma_i' \times c''', \gamma_i', -1),\ 1 \leq i \leq s\}.$$

Recall that γ_i is the simple closed curve obtained by resolving the self-intersection points of the immersed curve γ_i which comes from the presentation of the group G given at the first paragraph of Section 4.2, where s is the number of relations. Note that these Lagrangian tori descend from $M(G)$ and survive in $Y_n(G)$ after the fiber sum with $E(n)$. Let $X_n(G)$ denote the symplectic 4-manifold obtained by performing these Luttinger surgeries in $Y_n(G)$.

In the fundamental group of $X_n(G)$ we have the following relations—which we explain below—that come from the last set of Luttinger surgeries,

$$[e_{k_1}^{-1}, d] = \gamma_1, \ldots, [e_{k_s}^{-1}, d] = \gamma_s$$

where $e_{k_i} \times d$ is a dual torus of $\gamma_i' \times c'''$. Here each e_{k_i} (see Figure 1) is a carefully chosen disjoint vanishing cycle of the genus $2g + n - 1$ fibration (see Section 3) coming from the hyperelliptic part of the involution θ, and each γ_i' is modified so that it intersects the vanishing cycles e_{k_i} in a single point. Since after fiber summing with $E(n)$, we have $c = d = 1$, it follows that $\pi_1(X_n(G))$ admits a presentation with generators $\{a_i, b_i : 1 \leq i \leq g\}$ and relations:

$$a_1 = 1, \ldots, a_g = 1,$$
$$b_{k+1} = 1, \ldots, b_g = 1,$$
$$\gamma_1 = 1, \ldots, \gamma_s = 1.$$

In other words, $\pi_1(X_n(G)) = \langle b_1, \ldots, b_k \mid \gamma_1, \ldots, \gamma_s \rangle$, which is indeed isomorphic to the given group G. The above presentation follows from the following facts: (i) $c = d = 1$ in $\pi_1(X(G))$, (ii) for each torus $T_i = \gamma_i' \times c'''$ there is at least one vanishing cycle e_{k_i} of the genus $2g + n - 1$ Lefschetz fibration on $Y_n(G)$ (see Section 3) such that γ_i' intersects e_{k_i} precisely at one point, and γ_i' does not intersect with e_{k_j} for any $j \neq i$.

After the s Luttinger surgeries on tori $\gamma_i' \times c'''$, we obtain the following set of relations:

$$[e_{k_1}^{-1}, d] = \gamma_1, \ldots, [e_{k_s}^{-1}, d] = \gamma_s.$$

Since $e_{k_i} = 1$, and $d = 1$ in the fundamental groups of $Y_n(G)$ and $X_n(G)$, because e_{k_i} are the vanishing cycles of the genus $2g + n - 1$ fibrations on them, we obtain $\gamma_i = 1$ for any i. We can easily write down the global monodromy of the genus $2g + n - 1$ Lefschetz fibration on $X_n(G)$: $\theta^2 {\phi'}^{-1} \theta^2 \phi' = 1$, where $\phi' = t_{a_1} \cdots t_{a_k} t_{a_{k+1}} t_{b_{k+1}} \cdots t_{a_g} t_{b_g} t_{\gamma_1'} \cdots t_{\gamma_s'}$. In conclusion, we proved the main goal of this section which we state below.

Proposition 4.7. *Let G be any finitely presentable group. Then for any integer $n > 1$, there exists a closed symplectic 4-manifold $X_n(G)$ satisfying the following properties:*

(1) *The fundamental group of $X_n(G)$ is isomorphic to G.*
(2) *There exists a Lefschetz fibration $X_n(G) \to S^2$ that admits at least $4n$ pairwise disjoint sphere sections of self intersection -2.*
(3) *There is a homologically essential embedded torus $\mathbb{T} \subset X_n(G)$ of square zero such that $\mathbb{T}$ is disjoint from the sections of the Lefschetz fibration in (2), and it intersects each fiber twice.*
(4) $b_2^+(X_n(G)) > 1$.

Note that (4) in Proposition 4.7 simply follows from our assumption $n > 1$, using the fact that $b_2^+(E(n)) = 2n - 1$. The genus of the Lefschetz fibration $X_n(G) \to S^2$ is $2g + n - 1$, where g depends on the presentation of G as explained at the second paragraph of Section 4.2.

Remark 4.8. Our construction for $n = 1$ yields a genus $2g$ Lefschetz fibrations over S^2 with $b_2^+ = 1$ and $c_1^2 = 0$. For the reader's convenience, we discuss a few cases below

(1) By setting $p_i = \pm 1$ and $q_i = \pm 1$ in Subsection 4.1, we see that the fundamental group of $X_{g,1}(\overline{p}, \overline{q})$ is a trivial group. Simple computation shows that $\chi(X_{g,1}(\overline{p}, \overline{q})) = \chi(E(1)) = 12$, $\sigma((X_{g,1}(\overline{p}, \overline{q})) = \sigma(E(1)) = -8$, and consequently we have $b_2^+(X_{g,1}(\overline{p}, \overline{q})) = b_2^+(E(1)) = 1$. In fact, the 4-manifold $X_{g,1}(\overline{p}, \overline{q})$ in this special case is an exotic copy of $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$.
(2) By setting $g = 1$ and letting $|p| \geq 1$ and $|q| \geq 1$ vary, we see that the fundamental group of $X_{1,1}(\overline{p}, \overline{q})$ is generated by two elements $\{a, b\}$ subject to the relations: $a^p = 1, b^q = 1$, and $[a, b] = 1$, thus $\pi_1(X_{1,1}(\overline{p}, \overline{q})) \cong \mathbb{Z}_p \times \mathbb{Z}_q$. Since $b_1(X_{1,1}(\overline{p}, \overline{q})) = 0$, we similarly have $b_2^+(X_{1,1}(\overline{p}, \overline{q})) = b_2^+(E(1)) = 1$.
(3) By setting $g = 1$, $p = 0$, and $q = \pm m$ for any integer $m \geq 1$, we obtain the Lefschetz fibration with the fundamental group $\mathbb{Z} \times \mathbb{Z}_m$. Note that these examples have $\chi = 12$, $\sigma = -8$, $c_1^2 = 0$, and $b_2^+ = b_2^+(E(1)) + 1 = 2$.

5. Non-holomorphic Lefschetz fibrations with arbitrary fundamental group

In this section we give a proof of Theorem 1.1.

Theorem 1.1. *For any finitely presentable group G, there exists an infinite family of homeomorphic but pairwise non-diffeomorphic, symplectic but non-complex closed 4-manifolds with fundamental group G such that each member of this family admits a Lefschetz fibration of the same genus over S^2.*

Proof. The statement in Theorem 1.1 follows by combining Proposition 5.6 and Lemma 5.4. □

In the following, we use $X_n(G)$ to denote the 4-manifold given in Proposition 4.7, and for any knot $K \subset S^3$, we use $X_n(G)_K$ to denote the 4-manifold obtained from $X_n(G)$ by performing knot surgery along the torus $\mathbb{T}$ specified in Proposition 4.7, using the knot K. We begin with some preliminary results.

Lemma 5.1. *For any knot K, we have $\pi_1(X_n(G)_K) = G$.*

Proof. We know that $\pi_1(X_n(G)) = G$ by Proposition 4.7. Then the result follows from the Seifert-Van Kampen's theorem, since all the loops on the torus $\mathbb{T}$ given above are null-homotopic in $X_n(G)$, and the homology class of a longitude of the knot K in $S^3 \setminus \nu(K)$ is trivial. □

Lemma 5.2. *For any knot K, the 4-manifold $X_n(G)_K$ is homeomorphic to $X_n(G)$.*

Proof. Recall that there are three disjoint copies of the Gompf nuclei $N(n)$ (cf. [20]) in $E(n)$ (for $n \geq 2$) and only one of them is used during the construction of $X_n(G)$, and the remaining two descend to $X_n(G)$. Using any one of the nuclei that was not used to obtain $X_n(G)$, we get the following decomposition $X_n(G) = N(n) \cup W(G, n, g)$, where $W(G, n, g)$ is diffeomorphic to the Milnor fiber of the singularity whose link is the Brieskorn homology 3-sphere $\Sigma(2, 3, 6n - 1)$. Consequently, we see that $X_n(G)_K$ contains a copy of an exotic nucleus $N(n)_K$ and we have $X_n(G)_K = N(n)_K \cup W(G, n, g)$. Since the boundary of $N(n)_K$ is the Brieskorn homology 3-sphere, the argument which was elaborated in details in the last paragraph of the proof of [6, Proposition 5.4] shows that for any choice of K, the 4-manifold $X_n(G)_K$ is homeomorphic to $X_n(G)$ such that this homeomorphism is the identity on $W(G, n, g)$. Note that there is still a cusp neighborhood in the third nucleus. Similar arguments were used in [40] and [7].

□

Lemma 5.3. *For any fibered knot K, the smooth 4-manifold $X_n(G)_K$ is symplectic.*

Proof. This is indeed a well-known result [18, page 368]. It simply follows by the fact that the knot surgery 4-manifold $X_n(G)_K$ can be obtained as a symplectic fiber sum of the symplectic manifolds $X_n(G)$ and $M_K \times S^1$ along the symplectic torus $\mathbb{T}$, where M_K is obtained by 0-framed surgery on $K \subset S^3$. □

Lemma 5.4. *For any fibered knot K, there exists a Lefschetz fibration $X_n(G)_K \to S^2$ induced from the Lefschetz fibration $X_n(G) \to S^2$ described in Proposition 4.7.*

Proof. This follows from the fact that the torus $\mathbb{T}$ along which we perform knot surgery intersects each fiber of the Lefschetz fibration $X_n(G) \to S^2$ twice. The genus $2g + n - 1$ fiber of the Lefschetz fibration $X_n(G) \to S^2$ extends to become a genus $2g + 2h + n - 1$ fiber of the induced Lefschetz fibration $X_n(G)_K \to S^2$ by gluing in two copies of the genus h fiber surface of the knot K along the two punctures obtained by the removal of a tubular neighborhood of $\mathbb{T}$. For the details of this classical trick we refer to [19]. □

Lemma 5.5. *The symplectic Kodaira dimension $\kappa^s(X_n(G))$ is equal to* 1.

Proof. First of all, since the Luttinger surgery preserves the symplectic Kodaira dimension κ^s [24], we conclude that $\kappa^s(X_n(G)) = \kappa^s(Y_n(G))$. Note that $Y_n(G)$ is obtained as a fiber sum of $M(G)$ with $E(n)$ and $M(G)$ is obtained from $\Sigma_g \times T^2$ by Luttinger surgeries. Thus, $\kappa^s(X_n(G)) = \kappa^s(E(n,g))$, where the Kahler surface $E(n,g)$ is by definition the fiber sum $E(n)\#_{id}\Sigma_g \times T^2$. But according to [16, page 350], we have $\kappa^s(E(n,g)) = 1$. Here we assume that $n \geq 2$ and $g \geq 1$. □

Proposition 5.6. *There exists an infinite family $\mathcal{F} = \{K_i : i \in \mathbb{N}\}$ of fibered knots of some fixed genus such that $\{X_n(G)_{K_i} : K_i \in \mathcal{F}\}$ consists of homeomorphic, pairwise non-diffeomorphic, symplectic but non-complex smooth closed* 4*-manifolds.*

Proof. The following facts are some of the main ingredients of our proof.

(a) The complex Kodaira dimension κ^h is equal to the symplectic Kodaira dimension κ^s for a smooth 4-manifold which admits a symplectic structure as well as a complex structure, where these structures are not necessarily required to be compatible [16, Theorem 1.1].

(b) A complex surface with $\kappa^h = 1$ is properly elliptic.

(c) The diffeomorphism type of an elliptic complex surface with non-cyclic fundamental group is determined by its topological type [45] (see also [22, Remark 8.3.13]).

(d) The diffeomorphism type of a complex elliptic surface S with $\chi(S) > 0$ and $|\pi_1(S)| = \infty$ is determined by its fundamental group (cf. [22, Theorem 8.3.12], and [45]).

Fix any integer $h \geq 2$. Suppose that $\mathcal{F}_h = \{K_i : i \in \mathbb{N}\}$ is any infinite family of genus h fibered knots in S^3 with pairwise distinct Alexander polynomials. Such families of knots exist by the work of T. Kanenobu [25].

Recall that $\pi_1(X_n(G)) \cong G$ by Proposition 4.7. The infinite family $\{X_n(G)_{K_i} : K_i \in \mathcal{F}_h\}$ consists of closed symplectic 4-manifolds with $\pi_1 \cong G$, which are all homeomorphic to $X_n(G)$. These assertions indeed follow from Lemmata 5.1–5.3. Moreover, we observe that the members of this family are pairwise non-diffeomorphic. This is because of the fact that the Seiberg-Witten invariant of $X_n(G)_{K_i}$ is determined by the Alexander polynomial of K_i (cf. [18]). Although $X_n(G)$ is not simply-connected, this formula still works for knot surgery on $X_n(G)$, since $b_2^+(X_n(G)) > 1$ and $X_n(G)$ has non-trivial SW-invariants which is guaranteed by its symplectic structure. To finish the proof of Proposition 5.6, we show that at most finitely many of the symplectic 4-manifolds in the infinite family $\{X_n(G)_{K_i} : K_i \in \mathcal{F}_h\}$ can carry a complex structure.

Case 1: Suppose that G is finite, but not cyclic. We claim that at most one of the symplectic 4-manifolds in the family $\{X_n(G)_{K_i} : K_i \in \mathcal{F}_h\}$ can be complex. This follows by combining the fact that $\kappa^s(X_n(G)) = \kappa^s(X_n(G)_{K_i}) = 1$, items (a),(b) and (c) above.

Case 2: Suppose that G is infinite. Then by comparing the Seiberg-Witten invariants of the members of the family $\{X_n(G)_{K_i} : K_i \in \mathcal{F}_h\}$, which are pairwise distinct, and using (a), (b) and (d) above, we conclude that at most one of the symplectic 4-manifolds in the family $\{X_n(G)_{K_i} : K_i \in \mathcal{F}_h\}$ can be complex.

Case 3: Finally, assume that G is finite cyclic. Note that complex elliptic surfaces with finite cyclic fundamental group consist of the following family: $E(n)_{p,q}$ ($1 \leq p \leq q$), for which $\pi_1 \cong \mathbb{Z}_{gcd(p,q)}$. We refer the reader to [22, Theorem 3.3.6] for the computation of the Seiberg-Witten invariants of $E(n)_{p,q}$ (for complete details, see the original paper [17]). In order to complete the discussion, we need to impose a slight restriction on the family $\mathcal{F}_h$.

Lemma 5.7. *For any integer $h > 1$, there exists an infinite family $\mathcal{F}'_h \subset \mathcal{F}_h$ such that the Alexander polynomial of any member of $\mathcal{F}'_h$ is different than that of any torus knot.*

Proof. Since the equation $(p-1)(q-2)/2 = h$ has only finitely many solutions, the number of (p,q)-torus knots of genus h is finite. Hence we can get the desired family $\mathcal{F}'_h$ by removing at most finitely many members of $\mathcal{F}_h$. □

As a result, for any family $\mathcal{F}'_h$ as in Lemma 5.7, we conclude that none of symplectic 4-manifolds in the family $\{X_n(G)_{K_i} : K_i \in \mathcal{F}'_h\}$ can be complex, since we can guarantee that the Seiberg-Witten invariants of each $X_n(G)_{K_i}$ is different from the Seiberg-Witten invariants of any $E(n)_{p,q}$. □

6. Exotic Stein fillings with arbitrary fundamental group

In this final section we prove Corollary 1.2.

Corollary 1.2. *For any finitely presentable group G, there exists a contact 3-manifold which admits infinitely many homeomorphic but pairwise non-diffeomorphic Stein fillings such that the fundamental group of each filling is isomorphic to G. Moreover, we observe that the contact 3-manifold above is contactomorphic to the link of some isolated complex surface singularity equipped with its canonical contact structure.*

Proof. By Proposition 4.7, there is a closed symplectic 4-manifold $X_n(G)$ whose fundamental group is G, which admits a Lefschetz fibration over S^2 that has at least $4n$ pairwise disjoint sphere sections of square -2. By removing tubular neighborhoods of some of these sections and a neighborhood of a regular fiber we obtain a PALF (positive allowable Lefschetz fibration) over D^2 which is a Stein filling of the contact structure induced on its boundary (cf. [2, 30]). As shown in [6, Lemmata 4.1 & 4.2], the boundary 3-manifold is a Seifert fibered *singularity link* and the induced contact structure is the *canonical* contact structure on this singularity link. For more on the canonical contact structures and a discussion about their Stein/symplectic fillings we advise the reader to turn to [34, Section 9] and [42, Section 6].

To produce an infinite family of exotic Stein fillings of the same Seifert fibered singularity link with its canonical contact structure, we use the family of Lefschetz fibrations obtained via knot surgery along the aforementioned torus $\mathbb{T}$ as in the proof of Theorem 1.1. To be more precise, we take an infinite family $\mathcal{F}_h = \{K_i : i \in \mathbb{N}\}$ of fibered knots of some fixed genus $h \geq 2$ with pairwise distinct Alexander polynomials and apply knot surgery to $X_n(G)$ along $\mathbb{T}$, to produce an infinite family of mutually non-diffeomorphic closed symplectic 4-manifolds $\{X_n(G)_{K_i} : i \in \mathbb{N}\}$ all homeomorphic to $X_n(G)$.

Moreover, by Lemma 5.4, $X_n(G)_{K_i}$ admits a genus $2g + 2h + n - 1$ Lefschetz fibration over S^2 induced from the genus $2g + n - 1$ Lefschetz fibration on $X_n(G)$ described in Proposition 4.7. Furthermore, the disjoint (-2)-sphere sections of the Lefschetz fibration $X_n(G) \to S^2$ remains as disjoint (-2)-sphere sections of the Lefschetz fibration $X_n(G)_{K_i} \to S^2$, since the former sections are disjoint from the surgery torus $\mathbb{T}$.

As a consequence, for each $i \in \mathbb{N}$, by removing a tubular neighborhood of only one of these (-2)-sphere sections and a neighborhood of a regular fiber of the Lefschetz fibration $X_n(G)_{K_i} \to S^2$, we obtain an infinite family of Stein fillings of the *same* Seifert fibered singularity link equipped with its canonical contact structure on the boundary . In fact, we fix one (-2)-sphere section of the Lefschetz fibration $X_n(G) \to S^2$ from the beginning and remove the "same" section in each $X_n(G)_{K_i} \to S^2$. Using the facts that (i) any diffeomorphism of the boundary of the union of a tubular neighborhood of a (-2)-sphere section and a neighborhood of a regular fiber extends to this union (cf. [5, Lemma 3.1]) and (ii) the members of the family $\{X_n(G)_{K_i} \mid i \in \mathbb{N}\}$ are mutually non-diffeomorphic (see the proof of Proposition 5.6), we conclude that the members of our infinite family of Stein fillings are mutually non-diffeomorphic as well.

Now we show that the fundamental group of each filling is isomorphic to G, which is a consequence of the Seifert-Van Kampen's Theorem. Notice that the normal circles resulting from the removal of the (-2)-sphere section and the genus $2g + 2h + n - 1$ fiber of the Lefschetz fibration $X_n(G)_{K_i} \to S^2$ are both nullhomotopic. For the former we use any of the remaining (-2)-sphere sections and for the latter we use some (-2)-sphere transversal to the removed (-2)-sphere section. For the full details on the existence of such disjoint (-2)-spheres, we refer the reader to [6, Lemma 4.9].

Finally, we claim that the Stein fillings that we constructed above are all homeomorphic, which finishes the proof of Corollary 1.2. First we note that, for some fixed $h \geq 2$, the 4-manifold $X_n(G)_{K_i}$ is homeomorphic to $X_n(G)$ for any choice of $K_i \in \mathcal{F}_h$, by Lemma 5.2. Therefore, for any $i \neq j$, we deduce that $X_n(G)_{K_i}$ is homeomorphic to $X_n(G)_{K_j}$. We would like to show that after we remove from both $X_n(G)_{K_i}$ and $X_n(G)_{K_j}$, a tubular neighborhood of a (-2)-sphere section and a neighborhood of a regular fiber in their respective Lefschetz fibrations over S^2, the remanning Stein fillings are homeomorphic.

Let us recall from Lemma 5.2 that $X_n(G)$ contains a copy of the nucleus $N(n)$, and each $X_n(G)_{K_i}$ contains a copy of an exotic nucleus $N(n)_{K_i}$. Moreover, we have the following decompositions: $X_n(G) = N(n) \cup W(G,n,g)$ and $X_n(G)_{K_i} = N(n)_{K_i} \cup W(G,n,g)$. Furthermore, for any choice of K_i, the 4-manifold $X_n(G)_{K_i}$ is homeomorphic to $X_n(G)$, which can be assumed to restrict to the identity map on $W(G,n,g)$.

Next, we would like to see that this homeomorphism descends to a homeomorphism of the Stein fillings. First, we observe that in $X_n(G)$ a tubular neighborhood of any (-2)-sphere section is disjoint from the cusp neighborhood of the torus $\mathbb{T}$ in $N(n)$ and the tubular neighborhood of a regular fiber intersects this cusp neighborhood and consequently $N(n)$ along two disjoint copies of $D^2 \times D^2$. Using the fact that the homeomorphism described in the previous paragraph is identity on $W(G,n,g)$, we can remove the configuration consisting of the union of a section and a regular fiber entirely, except the two disjoint copies of $D^2 \times D^2$, by not affecting the homeomorphism. Performing knot surgery operation on $\mathbb{T}$ in $N(n)$ using any genus h fibered knot, changes these two disk bundles to $D^2 \times \Sigma(h,1)$, where $\Sigma(h,1)$ denotes genus h surface with one puncture. Since any homeomorphism of $\partial(D^2 \times \Sigma(h,1))$ extends, the identity map on the boundary extends over these two copies of $D^2 \times \Sigma(h,1)$. Thus, our homeomorphism on the complement of the removed neighborhoods of the regular genus $2g+2h+n-1$ fiber and the (-2)-sphere section is well defined. □

Acknowledgement: The authors would like to thank the anonymous referee for his careful reading of the manuscript and his/her suggestions that improved the presentation greatly. The authors would also like to thank R. İ. Baykur for helpful comments.

REFERENCES

[1] M. Akaho, *A connected sums of knots and Fintushel-Stern knot surgery*, Turkish. J. Math. 30 (2006), 87–93.

[2] S. Akbulut and B. Ozbagci, *Lefschetz fibrations on compact Stein surfaces*, Geom. Topol. 5 (2001), 319–334.

[3] A. Akhmedov, *Construction of symplectic cohomology* $\mathbb{S}^2 \times \mathbb{S}^2$, Gökova Geometry and Topology Proceedings, 14 (2007), 36–48.

[4] A. Akhmedov, R.İ. Baykur and B.D. Park, *Constructing infinitely many smooth structures on small 4-manifolds*, J. Topol. 1 (2008), no. 2, 409–428.

[5] A. Akhmedov, J.B. Etnyre, T.E. Mark and I. Smith, *A note on Stein fillings of contact manifolds*, Math. Res. Lett. 15 (2008), no. 6, 1127–1132.

[6] A. Akhmedov and B. Ozbagci, *Singularity links with exotic Stein fillings*, J. Singul. 8 (2014), 39–49.

[7] A. Akhmedov and B. D. Park, *Geography of simply connected spin symplectic 4-manifolds*, Math. Res. Lett. 17 (2010), no. 3, 483-492.

[8] A. Akhmedov and B. D. Park, *Exotic Smooth Structures on Small 4-Manifolds with Odd Signatures*, Invent. Math. 181 (2010), no. 3, 577–603.

[9] A. Akhmedov and W. Zhang, *The fundamental group of symplectic 4-manifolds with* $b_2^+ = 1$, preprint.

[10] J. Amorós, F. Bogomolov, L. Katzarkov and T. Pantev, *Symplectic Lefschetz fibrations with arbitrary fundamental groups*, J. Differential Geom. 54 (2000), no. 3, 489–545.

[11] S. J. Arakelov, *Families of algebraic curves with fixed degeneracies*, Izv. Akad. Nauk. SSSR Ser. Mat. 35 (1971) 1269–1293.

[12] D. Auroux, S. K. Donaldson and L. Katzarkov, *Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves*, Math. Ann. 326 (2003), 185–203.

[13] R. İ. Baykur, *Minimality and fiber sum decompositions of Lefschetz fibrations*, Proc. Amer. Math. Soc. 144 (2016), no. 5, 2275–2284.

[14] M. Bhupal and K. Ono, *Symplectic fillings of links of quotient surface singularities*, Nogoya Math. J. 207 (2012), 1–45.

[15] S. K. Donaldson, *Lefschetz pencils on symplectic manifolds*, J. Differential Geom. 53 (1999), no. 2, 205–236.

[16] J. G. Dorfmeister, and W. Zhang, *The Kodaira dimension of Lefschetz fibrations*, Asian J. Math. 13 (2009), 341–358.

[17] R. Fintushel and R. Stern, *Rational blowdowns of smooth* 4*-manifolds*, J. Differential Geom. 46 (1997), 181–235.

[18] R. Fintushel and R. Stern, *Knots, links, and* 4 *-manifolds*, Invent. Math. 134 (1998), no. 2, 363–400.

[19] R. Fintushel and R. Stern, *Families of simply connected 4-manifolds with the same Seiberg-Witten invariants*, Topology 43, (2004), 1449–1467.

[20] R. E. Gompf, *Nuclei of elliptic surfaces*, Topology 30 (1991), no. 3, 479–511.

[21] R. E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2) 142 (1995), no. 3, 527–595.

[22] R. E. Gompf and A. I. Stipsicz, 4*–manifolds and Kirby calculus*, Grad. Stud. Math., Vol. 20, AMS, 1999.

[23] Y. Gurtas, *Positive Dehn twist expressions for some elements of finite order in the mapping class group*, preprint, arXiv:math.GT/0404310.

[24] C.I. Ho and T.J. Li, *Luttinger surgery and Kodaira dimension*, Asian J. Math. 16 (2012), no. 2, 299–318.

[25] T. Kanenobu, *Module d'Alexander des noeuds fibrés et polynme de Hosokawa des lacements fibrés*, (French) [Alexander module of fibered knots and Hosokawa polynomial of fibered links] Math. Sem. Notes Kobe Univ. 9 (1981), no. 1, 75–84.

[26] S. Kitagawa and K. Konno, *Fibred rational surfaces with extremal Mordell-Weil lattices.* Math. Z. 251 (2005) 179–204.

[27] M. Korkmaz, *Lefschetz fibrations and an invariant of finitely presented groups*, Int. Math. Res. Not. IMRN (2009), no. 9, 1547–1572.

[28] M. Korkmaz, *Noncomplex smooth 4-manifolds with Lefschetz fibrations*, Internat. Math. Res. Notices, 3 (2001), 115–128.

[29] P. Lisca, *On symplectic fillings of lens spaces*, Trans. Amer. Math. Soc. 360 (2008), no. 2, 765–799.

[30] A. Loi and R. Piergallini, *Compact Stein surfaces with boundary as branched covers of B^4*, Invent. Math. 143 (2001), 325–348.

[31] K. M. Luttinger, *Lagrangian tori in $\mathbb{R}^4$*, J. Differential Geom. 42 (1995), no. 2, 220–228.

[32] Y. Matsumoto, *Lefschetz fibrations of genus two: a topological approach*, in Topology and Teichmüller spaces (Katinkulta, Finland, 1995), World Sci. Publ., River Edge, NJ, 1996, 123–148.

[33] D. McDuff, *The structure of rational and ruled symplectic 4-manifolds*, J. Amer. Math. Soc. 3 (1990), no. 3, 679–712.

[34] A. Némethi, *Some meeting points of singularity theory and low dimensional topology*, Proceedings of the Deformation Theory Conference, Budapest, 2008; Bolyai Soc. Math. Studies 23, Springer 2013.

[35] H. Ohta and K. Ono, *Simple singularities and topology of symplectically filling 4-manifolds*, Comment. Math. Helv. 74 (1999), 575–590.

[36] H. Ohta and K. Ono, *Symplectic fillings of the link of simple elliptic singularities*, J. Reine und Angew. Math. 565 (2003), 183–205.

[37] H. Ohta and K. Ono, *Simple singularities and symplectic fillings*, J. Differential Geom. 69 (2005), no. 1, 1–42.

[38] B. Ozbagci and A.I. Stipsicz, *Noncomplex smooth 4-manifolds with genus-2 Lefschetz fibrations*, Proc. Amer. Math. Soc. 128 (2000), no. 10, 3125–3128.

[39] H. Park, J. Park, D. Shin and G. Urzúa, *Milnor fibers and symplectic fillings of quotient surface singularities*, arXiv:1507.06756.

[40] J. Park, *The geography of symplectic 4-manifolds with an arbitrary fundamental group*, Proc. Amer. Math. Soc. 135 (2007), no. 7, 2301-2307.

[41] A. N. Parshin, *Algebraic curves over function fields, I*, Izv. Akad. Nauk. SSSR Ser. Mat. 32 (1968), 1191–1219.

[42] P. Popescu-Pampu,*Complex singularities and contact topology.* Winter Braids Lecture Notes, 3 : Winter Braids VI (Lille, 2016) (2016), Exp. No. 3, 74 p., doi: 10.5802/wbln.14

[43] M-H. Sait$\overline{\text{o}}$ and K-I. Sakakibara, *On Mordell-Weil lattices of higher genus fibrations on rational surfaces.* J. Math. Kyoto Univ. 34 (1994) 859–871.

[44] S. Tanaka, *On sections of hyperelliptic Lefschetz fibration*, Algebr. Geom. Topol. 12 (2012), no. 4, 2259–2286.

[45] M. Ue, *On the diffeomorphism types of elliptic surfaces with multiple fibers*, Invent. Math. 84 (1986), no. 3, 633–643.

[46] M. Usher, *Minimality and symplectic sums*, Int. Math. Res. Not. (2006), Art. ID 49857, 17 pp.

[47] K. Yun, *On the signature of a Lefschetz fibration coming from an involution*, Topology Appl. 153 (2006), no. 12, 1994–2012.

[48] K. Yun, *Twisted fiber sums of Fintushel-Stern's knot surgery 4-manifolds.* Trans. Amer. Math. Soc. 360 (2008), no. 11, 5853-5868.

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455, USA
E-mail address: akhmedov@math.umn.edu

DEPARTMENT OF MATHEMATICS, UCLA, LOS ANGELES, CA 90095 AND DEPARTMENT OF MATHEMATICS, KOÇ UNIVERSITY, ISTANBUL, TURKEY
E-mail address: bozbagci@ku.edu.tr