SINGULARITY LINKS WITH EXOTIC STEIN FILLINGS

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ABSTRACT. In [4], it was shown that there exist infinitely many contact Seifert fibered 3-manifolds each of which admits infinitely many exotic (homeomorphic but pairwise non-diffeomorphic) simply-connected Stein fillings. Here we extend this result to a larger set of contact Seifert fibered 3-manifolds with many singular fibers and observe that these 3-manifolds are singularity links. In addition, we prove that the contact structures induced by the Stein fillings are the canonical contact structures on these singularity links. As a consequence, we verify a prediction of András Némethi [26] by providing examples of isolated complex surface singularities whose links with their canonical contact structures admitting infinitely many exotic simply-connected Stein fillings. Moreover, for infinitely many of these contact singularity links and for each positive integer n, we also construct an infinite family of exotic Stein fillings with fixed fundamental group $\mathbb{Z} \oplus \mathbb{Z}_n$.

1. INTRODUCTION

The link of a normal complex surface singularity carries a Milnor fillable (also known as canonical) contact structure ξ_{can} which is uniquely determined up to isomorphism [9]. A Milnor fillable contact structure is Stein fillable since a regular neighborhood of the exceptional divisor in a resolution of the surface singularity provides a holomorphic filling which can be deformed to be a blow-up of a Stein surface without changing the contact structure ξ_{can} on the boundary [6]. Moreover, if a singularity admits a smoothing then the corresponding Milnor fiber is also a Stein filling of ξ_{can}.

In this paper, we generalize the main result in [4] to a larger family of contact Seifert fibered 3-manifolds admitting many singular fibers. We also observe an additional feature of these contact 3-manifolds: They appear as the links of some isolated complex surface singularities, and the contact structures are the canonical ones on these singularity links. The following theorems verify a prediction of Némethi [26] filling a gap in the literature.

Theorem 4.4. *There exist infinitely many Seifert fibered singularity links each of which admits infinitely many exotic simply-connected Stein fillings of its canonical contact structure.*

Theorem 5.3. *There exists an infinite family of Seifert fibered singularity links such that for each positive integer n, each member of this family equipped with its canonical contact structure admits infinitely many exotic Stein fillings whose fundamental group is $\mathbb{Z} \oplus \mathbb{Z}_n$.*

One should contrast our result with what is known for links of some other isolated complex surface singularities. For example, Ohta and Ono showed that the diffeomorphism type of any minimal strong symplectic filling of the link of a simple singularity is unique which implies that the minimal resolution of the singularity is diffeomorphic to the Milnor fiber [30]. They also showed that any minimal strong symplectic filling of the link of a simple elliptic singularity is diffeomorphic either to the minimal resolution or to the Milnor fiber of the smoothing of the singularity [29] .

Moreover, Lisca showed that the canonical contact structure on a lens space (the oriented link of some cyclic quotient singularity) has only *finitely* many distinct Stein fillings, up to diffeomorphism [22] (see also earlier work of McDuff [25]). Recently, it was shown that these Stein fillings correspond bijectively to the Milnor fibres coming from all possible distinct smoothings of the singularity [27].

In summary, in all the previously studied examples in the literature, it was shown that an isolated complex surface singularity with its canonical contact structure admits finitely many diffeomorphism types of Stein fillings such that each Stein filling is diffeomorphic either to the minimal resolution or to the Milnor fiber of one of the smoothings of the singularity.

We should mention that in [31, Theorem 1.2] Ohta and Ono showed the existence of singularity links which admit infinitely many distinct minimal symplectic fillings distinguished by their b_2^+. These fillings, however, are *not necessarily Stein or simply-connected*.

On the other hand, using log transforms, Akbulut and Yasui [3, Theorem 1.1] constructed contact 3-manifolds admitting infinitely many exotic simply-connected Stein fillings with $b_2 = 2$, inspired by an earlier paper by Akbulut [1]. In these articles, however, the contact 3-manifolds in question are *not singularity links*.

Finally, we would like to point out that in [5], *using very different methods*, we were able to prove the statement of Theorem 5.3 by replacing $\mathbb{Z} \oplus \mathbb{Z}_n$ by any finitely presented group G.

2. Milnor fillable contact structures on Seifert fibered 3-manifolds

In this section we identify the isomorphism class of the canonical contact structure on a singularity link which admits a Seifert fibration. A topological characterization of such 3-manifolds was given by Neumann [28]: A closed and oriented Seifert fibered 3-manifold is a singularity link if and only if it has a Seifert fibration over an orientable base such that the Euler number of this fibration is negative.

Proposition 2.1. *The isomorphism class of the Milnor fillable contact structure ξ_{can} on a closed and oriented 3-manifold Y which has a Seifert fibration with negative Euler number over an orientable base coincides with the unique isomorphism class of the S^1-invariant transverse contact structures.*

Proof. It is known that any Milnor fillable contact structure ξ_{can} on a singularity link is universally tight [21]. According to [24, Corollary 4], there exist a locally free S^1-action on

Y such that ξ_{can} is either transverse to the orbits or invariant under the S^1-action. Moreover, a contact structure which is both invariant and transverse to the orbits of a locally free S^1-action exists on a Seifert fibered 3-manifold Y exactly when the Euler number of Y is negative (cf. [20, 23]). Furthermore, there is only one isomorphism class of such contact structures as indicated in the last paragraph on page 1356 in [24] and hence the result follows since a Milnor fillable contact structure is unique up to isomorphism [9]. □

3. EXTENDING DIFFEOMORPHISMS

Let $\overline{p} = (p_1, p_2, \ldots, p_r)$ denote an r-tuple of *positive* integers and let Σ_h denote a closed oriented surface of genus $h \geq 0$. Let $Z_{h,\overline{p}}$ denote the oriented smooth 4-manifold-with-boundary obtained by plumbing oriented disk bundles according to the star-shaped graph with $r+1$ vertices described as follows: The central vertex represents $\Sigma_h \times D^2$ and if we label the remaining r vertices by $i = 1, \ldots, r$, the ith vertex—connected by an edge to the central vertex—represents a D^2-bundle over S^2 whose Euler number is $-p_i$.

Proposition 3.1. *Any orientation preserving self-diffeomorphism of $\partial Z_{h,\overline{p}}$ extends over $Z_{h,\overline{p}}$.*

Proof. We sketch the proof of this proposition which is a simple extension of the proof of [4, Lemma 3.1], where the case $r = 1$ was treated in full details. The strategy of the proof there, was to find the required extension in two steps, where the first step was to find an extension to the part $\mathring{\Sigma}_h \times D^2$ of the plumbing and then complete the extension on the remaining part. Here $\mathring{\Sigma}_h$ denotes Σ_h with a disk removed.

In order to prove our result, we apply the same strategy, where we remove several disks from Σ_h and the second paragraph in the proof of [4, Lemma 3.1] works verbatim as the initial step. To complete the extension to the r disk-bundles over S^2, we rely on a result of Bonahon [7], since the boundary of the oriented D^2-bundle over S^2 with Euler number $-p_i$ is the oriented S^1-bundle over S^2 with the same Euler number, which in turn, is orientation preserving diffeomorphic to the lens space $L(p_i, 1)$. □

4. SINGULARITY LINKS WITH SIMPLY-CONNECTED EXOTIC STEIN FILLINGS

The boundary $\partial Z_{h,\overline{p}}$ has an orientation induced from the orientation on the smooth 4-manifold-with-boundary $Z_{h,\overline{p}}$ described in Section 3. Let $Y_{h,\overline{p}}$ denote $\partial Z_{h,\overline{p}}$ with the *opposite* orientation. In other words, $Y_{h,\overline{p}}$ is the closed, oriented 3-manifold which is obtained by plumbing of oriented circle bundles according to the star-shaped graph with $r+1$ vertices as illustrated on the left in Figure 1: The central vertex represents $\Sigma_h \times S^1$ and if we label the remaining r vertices by $i = 1, \ldots, r$, the ith vertex—connected by an edge to the central vertex—represents an S^1-bundle over S^2 whose Euler number is p_i.

Lemma 4.1. *The 3-manifold $Y_{h,\overline{p}}$ is the link of an isolated complex surface singularity.*

Proof. The 3-manifold $Y_{h,\overline{p}}$ is obtained by plumbing of circle bundles according to the star-shaped graph illustrated on the left in Figure 1 with $r+1$ vertices, where the weight on a vertex represents the Euler number of the corresponding oriented circle bundle as usual.

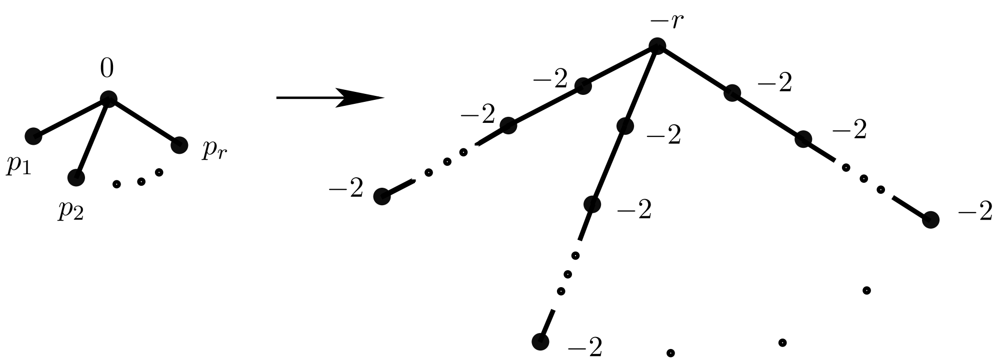

FIGURE 1. All except the central vertex represent circle bundles over S^2.

By blowing up and down this plumbing graph several times we see that $Y_{h,\overline{p}}$ is orientation-preserving diffeomorphic to the 3-manifold given by the star-shaped plumbing graph depicted on the right in Figure 1, where there are r legs emanating from the central vertex of weight $-r$ (and genus h) and the i-th leg is a chain of $p_i - 1$ vertices (excluding the central vertex) each with weight -2. Since the intersection matrix of this last graph is negative definite, we conclude that $Y_{h,\overline{p}}$ is orientation-preserving diffeomorphic to the link of a normal and hence isolated surface singularity by Grauert's theorem. $\square$

Let $\mathcal{OB}_{h,\overline{p}}$ denote the open book on $Y_{h,\overline{p}}$ whose page is a genus $h \geq 0$ surface with $r \geq 1$ boundary components and monodromy is given as

$$t_1^{p_1} t_2^{p_2} \ldots t_r^{p_r}$$

where t_i is a right-handed Dehn twist along a curve parallel to the i-th boundary component and let $\xi_{h,\overline{p}}$ denote the contact structure which is supported by $\mathcal{OB}_{h,\overline{p}}$.

Lemma 4.2. *The contact structure $\xi_{h,\overline{p}}$ is the canonical contact structure on $Y_{h,\overline{p}}$.*

Proof. First we observe that $Y_{h,\overline{p}}$ admits a Seifert fibration over a closed oriented surface of genus h with r singular fibers with multiplicities $p_1, p_2, \ldots, p_r$, respectively. Note that an explicit open book transverse to the fibers of such a Seifert fibration was constructed in [32], which is indeed isomorphic to the open book $\mathcal{OB}_{h,\overline{p}}$ on $Y_{h,\overline{p}}$. Moreover, it was also shown [32] that the contact structure supported by this open book is transverse to the Seifert fibration. Furthermore, it is easy to see that this contact structure is invariant under the natural S^1 action induced by the fibration. This is because the pages of the open book are S^1-invariant by construction and contact planes can be perturbed to be arbitrarily close to tangents of the pages by allowing an isotopy of the contact structure [10]. Therefore $\xi_{h,\overline{p}}$ has to be the unique Milnor fillable contact structure on $Y_{h,\overline{p}}$ by Proposition 2.1. $\square$

The following was proved in [2]:

Proposition 4.3. *Suppose that the closed 4-manifold X admits a genus h Lefschetz fibration over S^2 with homologically nontrivial vanishing cycles. Let $S_1, S_2, \dots, S_r$ be r disjoint sections of this fibration with squares $-p_1, -p_2, \dots, -p_r$, respectively. Let V denote the 4-manifold with boundary obtained from X by removing a regular neighborhood of these r sections union a nonsingular fiber. Then V admits a PALF (positive allowable Lefschetz fibration over D^2) and hence a Stein structure such that the induced contact structure $\xi_{h,\overline{p}}$ on $\partial V = Y_{h,\overline{p}}$ is compatible with the open book $\mathcal{OB}_{h,\overline{p}}$ induced by this PALF, where $\overline{p} = (p_1, p_2, \dots, p_r)$. In other words, V is a Stein filling of the contact 3-manifold $(Y_{h,\overline{p}}\ ,\ \xi_{h,\overline{p}})$.*

Now we are ready to state and prove the main result of this section:

Theorem 4.4. *There exist infinitely many Seifert fibered singularity links each of which admits infinitely many exotic (homeomorphic but pairwise non-diffeomorphic) simply connected Stein fillings of its canonical contact structure.*

Proof. We will give a proof of this result in the following four parts:

Part 1. *A genus g Lefschetz fibration on $\mathbb{C}P^2\#(4g+5)\overline{\mathbb{C}P^2}$*: Let Σ_g be a closed orientable surface of genus $g \geq 1$. Let $\gamma_1, \gamma_2, \dots, \gamma_{2g+1}$ denote the collection of simple closed curves on Σ_g depicted in Figure 2, and c_i denote the right handed Dehn twists along the curve γ_i. Let $X(g,1)$ denote $\mathbb{C}P^2\#(4g+5)\overline{\mathbb{C}P^2}$. The next result is well-known (cf. [15, Exercises 7.3.8(b) and 8.4.2(a)]).

Lemma 4.5. *There is a hyperelliptic genus g Lefschetz fibration $f_1 : X(g,1) \to S^2$ with global monodromy $(c_1c_2\cdots c_{2g-1}c_{2g}c_{2g+1}^2c_{2g}c_{2g-1}\cdots c_2c_1)^2 = 1$.*

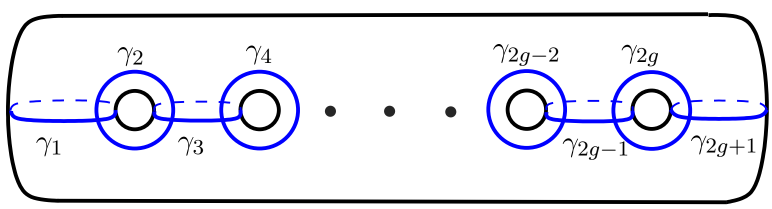

FIGURE 2. Vanishing cycles of the hyperelliptic genus g Lefschetz fibration $f_1 : X(g,1) = \mathbb{C}P^2\#(4g+5)\overline{\mathbb{C}P^2} \to S^2$

The 4-manifold $X(g,1)$ is diffeomorphic to the desingularization of the double branched cover of $S^2 \times S^2$ with branch locus given as two copies of $S^2 \times pt$ and $2g+2$ copies of $pt \times S^2$. Based on this description, it is easy to see that $X(g,1)$ admits a "vertical" genus g fibration with two singular fibers and a "horizontal" fibration with S^2 as a regular fiber. Moreover the vertical fibration can be locally perturbed so that it becomes isomorphic to the Lefschetz fibration $f_1 : X(g,1) \to S^2$ as explained in [15, Exercise 8.4.2(c)]).

Lemma 4.6. [35, Corollary 4.6] *For any $g \geq 1$, $f_1 : X(g,1) \to S^2$ admits at least $4g+4$ disjoint (-1)-sphere sections.*

We claim that the exceptional sphere s_i of the i-th blow up is a section of the Lefschetz fibration $f_1 : X(g,1) \to S^2$ for $2 \leq i \leq 4g+5$. Let h denote the canonical generator of $H_2(\mathbb{C}P^2, \mathbb{Z})$ and let $[F] \in H_2(X(g,1), \mathbb{Z})$ denote the homology class of the fiber F of the Lefschetz fibration $f_1 : X(g,1) \to S^2$. Then, by [11, Lemma 3.3], we have $[F] = (g+2)h - ge_1 - e_2 - \cdots - e_{4g+5}$, where $e_i = [s_i]$ denotes the homology class of the sphere s_i. Since, $[F] \cdot e_i = 1$ (for $2 \leq i \leq 4g+5$) and the fiber F and sphere e_i can be chosen to be pseudo-holomorphic (so that they only intersect positively), we conclude that the exceptional sphere s_i intersects each genus g fiber of the Lefschetz fibration $f_1 : X(g,1) \to S^2$ geometrically ones—which proves our claim.

Note that the fiber of the horizontal fibration above is a square zero sphere in $X(g,1)$ given by the homology class $h - e_1$, which intersects every fiber of $f_1 : X(g,1) \to S^2$ twice.

Definition 4.7. We denote the n-fold fiber sum of the genus g Lefschetz fibration $f_1 : X(g,1) \to S^2$ by $f_n : X(g,n) \to S^2$.

By sewing together the disjoint (-1)-sphere sections of $f_1 : X(g,1) \to S^2$ we obtain $4g+4$ disjoint $(-n)$-sphere sections of $f_n : X(g,n) \to S^2$. In order to prove Theorem 4.4, we just focus on $f_2 : X(g,2) \to S^2$ for $g \geq 2$. When we fiber sum two copies of $f_1 : X(g,1) \to S^2$ to obtain $f_2 : X(g,2) \to S^2$, we can also glue together square-zero sphere fibers of the horizontal fibrations on each summand to construct an embedded essential torus T of square zero in $X(g,2)$. The outcome of Part 1 of our proof is that

Lemma 4.8. *There is an embedded torus T in $X(g,2)$ with two key properties: (i) T intersects every fiber of the genus g Lefschetz fibration $f_2 : X(g,2) \to S^2$ at two points and (ii) T has no intersection with the $4g+4$ disjoint (-2)-sphere sections of this fibration.*

Part 2. *Fintushel-Stern knot surgery*: Let $X(g,2)_K$ denote the 4-manifold obtained from $X(g,2)$ by performing a Fintushel-Stern knot surgery on the torus T (see Lemma 4.8) in $X(g,2)$ using a knot $K \subset S^3$ (cf. [12]). More precisely,

$$X(g,2)_K = (X(g,2) \setminus (T \times D^2)) \cup (S^1 \times (S^3 \setminus N(K)),$$

where we identify the boundary of a disk normal to T with a longitude of a tubular neighborhood $N(K)$ of K in S^3. Next we observe that,

Lemma 4.9. *For any genus k fibered knot K, the surgered 4-manifold $X(g,2)_K$ admits a genus $(g+2k)$-Lefschetz fibration with $4g+4$ disjoint (-2)-sphere sections.*

Proof. The torus $T \subset X(g,2)$ on which we perform knot surgery intersects every fiber of $f_2 : X(g,2) \to S^2$ twice and a fiber of the Lefschetz fibration $X(g,2)_K \to S^2$ is obtained by gluing one copy of the Seifert surface of the fibered knot K to each puncture of the twice punctured fiber of $f_2 : X(g,2) \to S^2$ (cf. [13]).

Recall that $e_2, e_3, \ldots, e_{4g+5}$ denote the homology classes of the disjoint (-1)-sphere sections of $f_1 : X(g,1) \to S^2$. When we fiber sum two copies of $f_1 : X(g,1) \to S^2$, we can glue corresponding (-1)-sphere sections in the two summands to obtain $4g+4$ disjoint (-2)-sphere sections $S_2, S_3, \ldots, S_{4g+5}$ of $f_2 : X(g,2) \to S^2$. Note that these (-2)-sphere sections will remain as sections of the Lefschetz fibration $X(g,2)_K \to S^2$, since they are disjoint from the surgery torus T. □

Part 3. *Construction of the simply-connected Stein fillings*:

Definition 4.10. For any $1 \leq r \leq 4g+3$ and for any genus k fibered knot K in S^3, the 4-manifold-with-boundary $V(g,r)_K \subset X(g,2)_K$ is obtained by removing a regular neighborhood of r disjoint sections $S_2, S_3, \ldots, S_{r+1}$ union a nonsingular genus $g+2k$ fiber of the Lefschetz fibration $X(g,2)_K \to S^2$ given in Lemma 4.9.

We would like to emphasize that we do not remove the section S_{4g+5}.

Lemma 4.11. *The 4-manifold $V(g,r)_K$ is simply-connected.*

Proof. Observe that, by the Seifert-Van Kampen's theorem, the fundamental group of $V(g,r)_K$ is generated by the homotopy classes of loops based at some point $q \in S_{4g+5}$ that are conjugate to loops $\mu_2, \mu_3, \ldots, \mu_{r+1}$ and η normal to $S_2, S_3, \ldots, S_{r+1}$, and to the regular fiber we remove, respectively. Since all the loops $\mu_2, \mu_3, \ldots, \mu_{r+1}$, and η can be deformed to a point using the spheres represented by the homology classes $e_{4g+5} - e_2, e_{4g+5} - e_3, \ldots, e_{4g+5} - e_{r+1}$ and the section S_{4g+5}, respectively, we conclude that $V(g,r)_K$ is simply-connected. □

For any positive integer r, let $\overline{r}$ denote the r-tuple $(2,2,\ldots,2)$ for the rest of this section. Then Proposition 4.3 coupled with Lemma 4.2 imply that

Lemma 4.12. *The 4-manifold $V(g,r)_K$ is a Stein filling of $(Y_{g+2k,\overline{r}}\ ,\ \xi_{g+2k,\overline{r}})$, where $\xi_{g+2k,\overline{r}}$ is the canonical contact structure on the Seifert-fibered singularity link $Y_{g+2k,\overline{r}}$.*

Part 4. *An infinite family of exotic Stein fillings*: For $k \geq 2$, let $\mathcal{F}_k = \{K_{k,i} : i \in \mathbb{N}\}$ denote an infinite family of genus k fibered knots in S^3 with pairwise distinct Alexander polynomials, which exists by [17]. Then the infinite family $\{X(g,2)_{K_{k,i}} : K_{k,i} \in \mathcal{F}_k\}$ consists of smooth 4-manifolds homeomorphic to $X(g,2)$ which are pairwise non-diffeomorphic by a theorem of Fintushel and Stern [12]. Now we claim that for fixed $g \geq 2$, $k \geq 2$, and $1 \leq r \leq 4g+3$, the infinite set

$$\mathcal{S}_{g,k,r} = \{V(g,r)_{K_{k,i}} : K_{k,i} \in \mathcal{F}_k\}$$

indexed by $i \in \mathbb{N}$, includes infinitely many homeomorphic but pairwise non-diffeomorphic simply-connected Stein fillings of the Seifert fibered singularity link $(Y_{g+2k,\overline{r}}\ ,\ \xi_{g+2k,\overline{r}})$.

In order to prove that these Stein fillings are pairwise non-diffeomorphic we just appeal to Proposition 3.1, by observing that what we delete from $X(g,2)_{K_{k,i}}$ to obtain $V(g,r)_{K_{k,i}}$ is indeed diffeomorphic to $Z_{g+2k,\overline{r}}$.

Next we prove that infinitely many of the Stein fillings in $\mathcal{S}_{g,k,r}$ are homeomorphic. We first observe that all of these Stein fillings have the same Euler characteristic by elementary facts and the same signature by Novikov additivity. It follows that the rank of the second homology group of the fillings is fixed as well because the fillings are simply-connected. Moreover, since the boundary of any Stein filling in $\mathcal{S}_{g,k,r}$ is diffeomorphic to $Y_{g+2k,\overline{r}}$ and $H_1(Y_{g+2k,\overline{r}}\ ;\mathbb{Z})$ is infinite, we conclude that the determinant of the intersection form of any filling in $\mathcal{S}_{g,k,r}$ is zero. It follows that intersection forms of all the Stein fillings in $\mathcal{S}_{g,k,r}$ are isomorphic (see [15, Corollary 5.3.12 and Exercise 5.3.13(f)]). Furthermore, a fixed symmetric bilinear form is realized as an intersection form by only finitely many homeomorphism types of simply-connected compact oriented 4-manifolds with a given boundary [8, Corollary 0.4]. Therefore the infinitely many Stein fillings in $\mathcal{S}_{g,k,r}$ belong to finitely many homeomorphism types—which finishes the proof of Theorem 4.4. □

5. Exotic Stein fillings with non-trivial fundamental groups

Our aim in this section is to explore the existence of non-simply connected exotic Stein fillings of some singularity links. Let n be a positive integer. In this paper, we only study the case when the fundamental group of the Stein fillings is $\mathbb{Z} \oplus \mathbb{Z}_n$.

As an essential ingredient we use the family of non-holomorphic genus g Lefschetz fibrations with fundamental group $\mathbb{Z} \oplus \mathbb{Z}_n$ constructed in [33] for $g = 2$ and generalized to the case $g \geq 3$ in [18]. For the purposes of this article we focus on the case where $g \geq 3$ is odd and provide the necessary background for the convenience of the reader.

Definition 5.1. Let $W(m) := \Sigma_m \times \mathbb{S}^2 \# 8\overline{\mathbb{C}P^2}$, where Σ_m denotes a closed oriented genus m surface.

Note that $W(m)$ is the total space of a genus $g = 2m + 1$ Lefschetz fibration over $\mathbb{S}^2$, which was proved in [18, Remark 5.2] generalizing a classical result for $g = 2$ due to Y. Matsumoto. The branched-cover description of this Lefschetz fibration can be given as follows [13]: Take a double branched cover of $\Sigma_m \times \mathbb{S}^2$ along the union of four disjoint copies of $pt \times \mathbb{S}^2$ and two disjoint copies of $\Sigma_m \times pt$ as shown in Figure 3.

The resulting branched cover has eight singular points, corresponding to the intersection points of the horizontal spheres and the vertical genus m surfaces in the branch set. By desingularizing this singular manifold one obtains $W(m)$. Observe that a generic fiber of the vertical fibration is the double cover of Σ_m branched over four points. Thus, a generic fiber is a genus $g = 2m + 1$ surface and each of the two singular fibers of the vertical fibration can be perturbed into $2m + 6$ Lefschetz type singular fibers.

Proposition 5.2. [18] *The 4-manifold $W(m)$ admits a genus g Lefschetz fibration over S^2 with $2g + 10$ singular fibers such that the monodromy of this fibration is given by the relation*

$$(b_0 b_1 b_2 \dots b_g a^2 b^2)^2 = 1$$

where b_i denotes a right-handed Dehn twists along β_i, for $i = 0, 1, \dots, g$ and a and b denote right-handed Dehn twists along α and β respectively (see Figure 4).

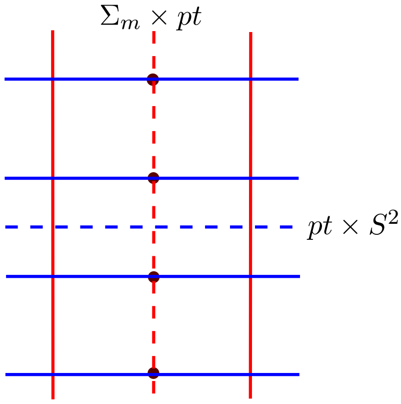

FIGURE 3. The branch set in $\Sigma_m \times \mathbb{S}^2$

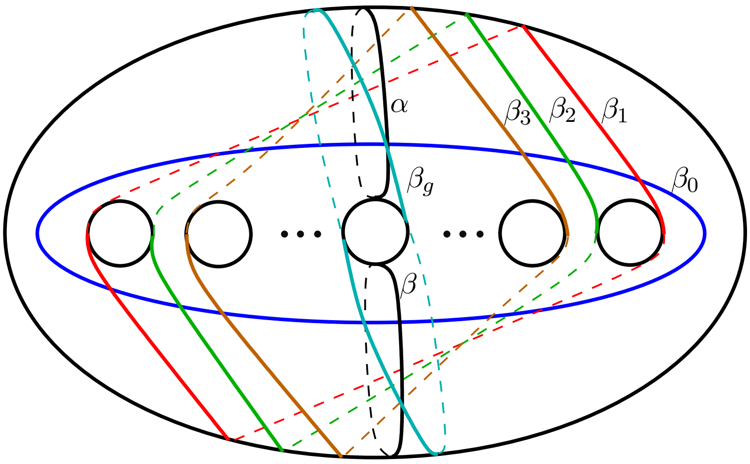

FIGURE 4. Vanishing cycles of the genus $g = 2m + 1$ Lefschetz fibration $W(m) = \Sigma_m \times \mathbb{S}^2 \# 8\overline{\mathbb{C}P^2} \to S^2$ corresponding to $(b_0 b_1 b_2 \dots b_g a^2 b^2)^2 = 1$.

Also, a generic fiber of the horizontal fibration is the double cover of $\mathbb{S}^2$ branched over two points. This gives a sphere fibration on $W(m)$. We are now ready to state the main result of this section.

Theorem 5.3. *There exists an infinite family of Seifert fibered singularity links such that for each positive integer n, each member of this family equipped with its canonical contact structure admits infinitely many exotic (homeomorphic but pairwise non-diffeomorphic) Stein fillings whose fundamental group is $\mathbb{Z} \oplus \mathbb{Z}_n$.*

Proof. For $g = 2m+1 \geq 3$, let $W_n(m)$ denote the total space of the Lefschetz fibration over S^2 obtained by a *twisted* fiber sum of two copies of the Lefschetz fibration $W(m) \to S^2$ along the regular genus g fiber (cf. [33, 18]). Notice that twisted fiber sum refers to the fiber sum where a regular neighborhood of a fixed regular fiber of $W(m) \to S^2$ is identified with a regular neighborhood of a fixed regular fiber of another copy of $W(m) \to S^2$ by a nontrivial diffeomorphism of the fiber. As shown in [5], there is a diffeomorphism involving an n-fold power of a right-handed Dehn twist along a homologically nontrivial curve on the fiber such that $\pi_1(W_n(m)) = \mathbb{Z} \oplus \mathbb{Z}_n$. Since in $W(m)$ the generic fiber of the vertical fibration intersects the generic fiber of the sphere fibration in two points, after the fiber sum we have an embedded homologically essential torus T of self-intersection zero in $W_n(m)$. Notice that a regular fiber of the genus g fibration on $W_n(m)$ intersects T at two points. It was shown in [19] that the Lefschetz fibration on $W(m)$ admits at least two disjoint (-1)-sphere sections, which implies that the Lefschetz fibration on $W_n(m)$ admits at least two disjoint (-2)-sphere sections. The torus T above can be chosen to be disjoint from these (-2)-sphere sections.

Let $W_n(m)_K$ denote the result of the Fintushel-Stern knot surgery along the torus T by a knot K in S^3. We observe that by Seifert-Van Kampen's theorem, $\pi_1(W_n(m)_K) = \mathbb{Z} \oplus \mathbb{Z}_n$, since all the loops on T are nullhomotopic in $W_n(m)$ and $W_n(m)_K$.

Proposition 5.4. *For any pair of positive integers (m, n) and for any knot K in S^3, the 4-manifold $W_n(m)_K$ is homeomorphic to $W_n(m)$.*

Proof. The branched-cover description of the 4-manifold $W(m)$, whose branch locus in $\Sigma_m \times \mathbb{S}^2$ is depicted in Figure 3, shows that $W(m)$ admits a sphere fibration, and the generic fiber of the genus g Lefschetz fibration on $W(m)$ intersects the generic fiber of the sphere fibration in two points. Hence the untwisted fiber sum of two copies of the Lefschetz fibration $W(m) \to S^2$ along the regular genus g fiber, which we denote by $W_0(m)$, admits an elliptic fibration. Alternatively, $W_0(m)$ can be viewed as the fiber sum of $\Sigma_m \times T^2$ and the elliptic surface $E(2)$ where we identify $pt \times T^2 \subset \Sigma_m \times T^2$ with an elliptic fiber of $E(2)$. The elliptic fibration structure on $W_0(m)$ over the genus g surface is induced from the elliptic fibrations of $E(2)$ and $\Sigma_m \times T^2$ via this fiber sum. Moreover, the manifold $W_n(m)$ can be obtained from $W_0(m)$ by a single Luttinger surgery along a Lagrangian torus (for details, see [5]), disjoint from an elliptic fiber. Therefore, $W_n(m)$ contains a Gompf nucleus C_2 of $E(2)$: Use a cusp fiber of the above mentioned elliptic fibration, and a (-2)-sphere section obtained by sewing together (-1)-sphere sections of the sphere fibration on $W(m)$. Furthermore, the torus along which we perform Fintushel-Stern knot surgery can be assumed to lie in this cusp neighborhood.

Next we decompose $W_n(m)$ into $C_2 \cup_{\Sigma(2,3,11)} V(n, m)$ along the homology 3-sphere $\Sigma(2, 3, 11)$, where $V(n, m)$ denotes the complement of C_2. Then, for any knot K in S^3, we have a corresponding decomposition of $W_n(m)_K$ into $(C_2)_K \cup_{\Sigma(2,3,11)} V(n, m)$, where $(C_2)_K$ is an exotic copy of C_2 (cf. [14]). Since ∂C_2 is a homology 3-sphere, by [8, Corollary 0.9], there exits a homeomorphism from $(C_2)_K$ to C_2 which restricts to the identity map on the boundary. As a consequence, we have constructed a homeomorphism between the 4-manifolds $W_n(m)_K$ and $W_n(m)$ which extends the identity map on $V(n, m)$. $\square$

Suppose that K is a fibered knot in S^3 of genus k. Then, a simple argument similar to the one used in the proof of Lemma 4.9 shows that $W_n(m)_K$ admits a genus $g + 2k = 2(m + k) + 1$ Lefschetz fibration over S^2 with two disjoint (-2)-sphere sections. Recall that, in Part 4 of the proof of Theorem 4.4, for any $k \geq 2$, we denoted an infinite family of genus k fibered knots in S^3 with pairwise distinct Alexander polynomials by $\mathcal{F}_k = \{K_{k,i} : i \in \mathbb{N}\}$.

Now let us fix a triple of positive integers (m, n, k), where $k \geq 2$. By removing a tubular neighborhood of a regular fiber and only one of the two (-2)-sphere sections of the genus $2(m + k) + 1$ Lefschetz fibration on $W_n(m)_{K_{k,i}}$, we obtain an infinite family (indexed by $i \in \mathbb{N}$) of Stein fillings of the Seifert fibered singularity link $Y_{2(m+k)+1,(2)}$ with its canonical contact structure, such that each filling has $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_n$. We claim that these Stein fillings are exotic copies of each other, i.e., they are all homeomorphic but pairwise non-diffeomorphic. The fact that these fillings are pairwise non-diffeomorphic follows from Proposition 3.1 as in Part 4 in the proof of Theorem 4.4.

Finally, for fixed (m, n, k), we show that the Stein fillings described above with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_n$ belong to the same homeomorphism type. We proved in Proposition 5.4 that for fixed positive integers m and n, all the smooth 4-manifolds in the infinite family $\{W_n(m)_{K_{k,i}} : K_{k,i} \in \mathcal{F}_k\}$ belong to the same homeomorphism type, independent of the knot $K_{k,i}$. Now we simply claim that the knot surgery performed on $W_n(m)$ to obtain $W_n(m)_{K_{k,i}}$ essentially affects the complement of the removed neighborhood of the regular fiber union the (-2)-sphere section, and hence it does not have any effect on the homeomorphism type of the "remaining" Stein fillings. So the strategy is to start with a homeomorphism of the closed 4-manifolds including the Stein fillings, and verify that it will "descend" to a homoeomorphism of the Stein fillings when we remove a piece from each after performing a Fintushel-Stern knot surgery.

More precisely, first note that in $W_n(m)$, a tubular neighborhood of the (-2)-sphere section is disjoint from the cusp neighborhood (see the proof of Proposition 5.4) including the torus T given above. Moreover, the cusp neighborhood intersects with a tubular neighborhood of a regular fiber along two disjoint copies of $D^2 \times D^2$. Since the initial homeomorphism in Proposition 5.4 is identity on the complement of the cusp neighborhood, we can delete these configurations, except the two copies of $D^2 \times D^2$, without affecting our homeomorphism. Performing knot surgery on T turns these two disk bundles into $\mathring{\Sigma}_k \times D^2$, where $\mathring{\Sigma}_k$ denotes a genus k surface with one disk removed. Since any homeomorphism of $\partial(\mathring{\Sigma}_k \times D^2)$ extends, we can delete these two $D^2 \times \mathring{\Sigma}_k$ as well so that the homeomorphism descends to the Stein fillings. $\square$

Corollary 5.5. *For each $h \geq 7$, the Seifert fibered singularity link $Y_{h,(2)}$ with its canonical contact structure $\xi_{h,(2)}$ admits*

- *an infinite family of exotic simply-connected Stein fillings,*
- *for each positive integer n, an infinite family of exotic Stein fillings whose fundamental group is $\mathbb{Z} \oplus \mathbb{Z}_n$, and*
- *for each positive integer n, a Stein filling whose first homology group is $\mathbb{Z}^{h-2} \oplus \mathbb{Z}_n$.*

In particular, none of the Stein fillings in the last two items are homeomorphic to a Milnor fiber of the singularity.

Proof. Recall that, with respect to our notation in Section 4, $Y_{h,(2)}$ denotes the plumbing of $\Sigma_h \times D^2$ with an oriented circle bundle over S^2 whose Euler number is 2. It follows that $Y_{h,(2)}$ is a Seifert fibered 3-manifold over a genus h surface with a unique singular fiber of multiplicity 2.

For any $h \geq 6$, an infinite family of simply connected, homeomorphic but pairwise non-diffeomorphic Stein fillings of the singularity link $(Y_{h,(2)}, \xi_{h,(2)})$ is given in Theorem 4.4. Similarly, according to Theorem 5.3, for any $h = g + 2k \geq 7$, and for each positive integer n, $(Y_{h,(2)}, \xi_{h,(2)})$ admits an infinite family of homeomorphic but pairwise non-diffeomorphic Stein fillings with fundamental group $\mathbb{Z} \oplus \mathbb{Z}_n$. The third family of Stein fillings is given in [33]. In addition, none of the Stein fillings in the last two items are homeomorphic to any Milnor fiber of the singularity, since a Milnor fiber of a normal surface singularity has vanishing first Betti number [16]. $\square$

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