GENERALIZED PLUMBINGS AND MURASUGI SUMS

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Abstract. We propose a generalization of the classical notions of plumbing and Murasugi summing operations to smooth manifolds of arbitrary dimensions, so that in this general context Gabai's credo "the Murasugi sum is a natural geometric operation" holds. In particular, we prove that the sum of the pages of two open books is again a page of an open book and that there is an associated summing operation of Morse maps. We conclude with several open questions relating this work with singularity theory and contact topology.

Contents

1. Introduction

Around 1960, Milnor and Mumford introduced independently particular cases of an operation which builds new manifolds with boundary from given ones: "*plumbing*". Milnor used this operation to construct exotic spheres in higher dimensions and Mumford in order to describe the boundaries of nice neighborhoods of isolated singular points on complex surfaces.

Around the same time, Murasugi defined an analogous operation on Seifert surfaces of links in the 3-sphere. This operation was done on embedded objects rather than abstract

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ones. Nevertheless, this operation agrees with (a slight generalization of) the plumbing operation on the embedded surfaces.

In the mid-seventies, Stallings introduced the name of "*Murasugi sum*" for the operation above, and he showed that *the Murasugi sum of two pages of open books is again the page of an open book*. Several years later, Gabai proved that Murasugi sum preserves other properties of surfaces embedded in 3-manifolds, and summarized the general philosophy behind such results by the credo "*Murasugi sum is a natural geometric operation*".

In the mid-eighties, Lines proved an analog of Stallings' theorem for special types of open books in higher dimensional spheres, after having extended to that context the operation of Murasugi sum.

Details about the previous historical facts may be found in Sections 2 and 3 of our paper.

The effect of the Murasugi sum on the hypersurfaces under scrutiny is to *plumb* them, that is, roughly speaking, to identify by a special diffeomorphism two balls embedded in them, in such a way that the result is again a manifold with boundary.

The aim of this paper is *to identify the most general operation of plumbing in arbitrary dimensions, which allows one to extend the classical operation of Murasugi sum, such that Gabai's credo still holds*.

Our main result (see Theorem 9.3) is that *an analog of Stallings' theorem holds if the plumbing operation is generalized by allowing the gluing of two manifolds with boundary through any diffeomorphism of compact full-dimensional submanifolds, provided that the result is again a manifold with boundary.*

In particular, we never impose orientability hypotheses. Instead, throughout the paper the crucial assumptions are about *coorientability* of hypersurfaces. Moreover, we work with fixed coorientations. As those coorientations are present in the absence of any orientations on the ambient manifold or on the hypersurface, we work in a slightly non-standard context. This obliges us to give careful definitions of all the objects we manipulate, by lack of a convenient source in the literature.

An important message of our work is that it is much easier to prove that generalized Murasugi sums conserve geometric properties (illustrating Gabai's credo) if the fundamental notion of *sum* is defined on special types of *cobordisms*. In fact, the most difficult result of our work from the technical viewpoint (Proposition 9.1) states that our generalization of the Murasugi sum to arbitrary dimensions coincides with another definition given in terms of cobordisms.

We believe that, combining our new operations with those explored in [25] and [35], one will get a better understanding of the differential topology of singularities.

Let us describe the structure of the paper.

In Section 2 we sketch the historical evolution of the notions of plumbing and Murasugi sum, through the works of Milnor, Mumford, Murasugi, Stallings, Gabai and Lines. We quote from the original sources, in order to allow the reader to compare easily those classical constructions to ours.

In Section 3 we explain Gabai's geometric proof of Stallings' theorem. We describe a variant of his proof given by Giroux and Goodman and give a second interpretation of it as explained by Etnyre.

In Section 4 we explain our basic conventions about *coorientations* of hypersurfaces in manifolds with boundary (see Definition 4.3), their *sides* and *collar neighborhoods* (see Definition 4.7).

In Section 5 we set up our notation for *cobordism of manifolds with boundary* (see Definition 5.1), which is essential for our approach, mainly through its special case of *cylindrical cobordisms* (see Definition 5.5). Cobordisms of manifolds with boundary may also be *composed*, just like usual cobordisms. In the following sections, for concision, we simply speak about *cobordisms* instead of *cobordism of manifolds with boundary*.

In Section 6 we describe the notions of *Seifert hypersurfaces* (see Definition 6.1) and *open books* (see Definition 6.14) and establish the equivalence of these notions with some special types of cobordisms.

In Section 7 we introduce our generalizations of the classical notions of *plumbing* and *Murasugi sum*. We call them *abstract* and *embedded summing* respectively (see Definitions 7.4 and 7.8). For the latter, the hypersurfaces to be summed are not assumed to be coorientable, but only the identified *patches* (see Definition 7.2) are assumed to be *cooriented*. We show that embedded summing is an associative but in general non-commutative operation (see Proposition 7.10).

In Section 8 we introduce a supplementary structure on cylindrical cobordisms, *stiffenings*, which exist and are unique up to isotopy, but which are not canonical. Such structures are essential for the proofs presented in Section 9. We also define a summing operation on stiffened cylindrical cobordisms (see Definition 8.6).

In Section 9 we show that, under the assumption that the hypersurfaces which are to be summed in an embedded way are *globally cooriented*, the operation of embedded summing may be reinterpreted as a summing operation on cylindrical cobordisms (see Proposition 9.1). Our generalization of Stallings' theorem (see Theorem 9.3) is obtained then easily by working with a stiffening adapted to the open books under scrutiny. We also formulate an extension of this theorem to what we call *Morse open books* (see Definition 9.5).

Finally, in Section 10 we list several open questions. Some of them concern the relations of open books with singularity theory and contact topology. For this reason, we begin that section by recalling briefly the basics of those relations. We hope that this work will be useful in particular to the researchers interested in the topology of singular spaces and to those interested in the topology of contact manifolds.

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2. Plumbing and Murasugi sums in the literature

In this section we recall the classical notions of plumbing, as defined by Milnor and by Mumford, as well as Murasugi's original construction, its extensions by Stallings and Gabai to more general 3-dimensional operations and by Lines to arbitrary dimension.

In [29, p.71], Milnor constructed for any $k \geq 1$ a $(2k-1)$-connected manifold-with-boundary M_k of dimension $4k$ whose intersection form in dimension $2k$ has the following matrix:

$$\begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & -1 & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & -1 & & 1 & 2 & 1 & & \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & 1 \\ & & & & & & 1 & 2 \end{pmatrix}$$

in an appropriate basis, where the missing entries are 0. The determinant of this matrix is 1, which ensures that the boundary of the constructed manifold is homeomorphic to a sphere. Milnor showed that this boundary generated the cyclic group of 7-dimensional homotopy spheres which bound parallelizable manifolds.

In order to construct M_k, Milnor started from two transversal copies of the sphere $\mathbb{S}^{2k}$ inside $\mathbb{S}^{2k} \times \mathbb{S}^{2k}$, intersecting in exactly two points, and having self-intersections $+2$: the diagonal and its image by the map $1 \times \alpha$, where $\alpha : \mathbb{S}^{2k} \to \mathbb{S}^{2k}$ denotes in his words the "twelve hour rotation which leaves the north pole fixed, and satisfies $\alpha(x) = -x$ for x on the equator".

He took the universal cover $\tilde{U}$ of a tubular neighborhood U of the union X of the two spheres, and looked at the total preimage $\tilde{X}$ of X inside $\tilde{U}$. He could easily find in $\tilde{U}$ a sequence:

$$T_1 \cup T_1' \cup T_2 \cup T_2' \cup T_3 \cup T_3' \cup T_4 \cup T_4'$$

of tubular neighborhoods of eight $(2k)$-dimensional spheres of $\tilde{X}$ intersecting in a chain, whose intersection matrix is isomorphic to the one given above, except that the two -1's are replaced by 0-s. Milnor explains at this point:

> "To correct this intersection matrix it is necessary to introduce an intersection between T_1' and T_3, so as to obtain an intersection number -1. Choose a rotation of $\mathbb{S}^{2k} \times \mathbb{S}^{2k}$ which carries a region of T' near the "equator" onto a region of T near the "equator", so as to obtain an intersection number of -1. Matching the corresponding regions of T_1' and T_3, we obtain a topological manifold W_2, with the required intersection matrix."

We note that W_2 is not the final manifold in Milnor's construction, but this is not so important for our purposes. It is this "matching" of regions which was later named "*plumbing*", following a denomination introduced for a related object by Mumford [32].

Mumford's problem in [32] was to study the topology of the boundary of a "tubular neighborhood" of a reducible compact complex curve in a smooth complex surface. He assumed that the irreducible components E_i of the curve are *smooth* and he described the boundary M of their union as the result of a cut-and-paste operation done on the boundaries M_i of tubular neighborhoods of the individual E_i's. One first has to cut some solid tori from the M_i's and then glue pairwise collar neighborhoods of the boundary components created in this way. He described those collar neighborhoods as "*standard plumbing fixtures*" (see [32, Page 8]). The term "*plumbing*" was brought to this context! Later, it was used as a name for two different but related operations:

- following Mumford, a cut-and-paste operation used to describe the *boundary* of a tubular neighborhood of a union of submanifolds of a smooth manifold, intersecting generically (see [34] and [36]);
- following Milnor, a purely pasting operation used to describe the tubular neighborhoods themselves.

One of the first definitions of this operation in a textbook is to be found in [24, Chapter 8]. Let us quote from it the definition of the plumbing of two n-disc bundles (see Figure 1, reproduced from the same book):

> **Definition 2.1.** "Let $\xi_1 = (E_1, p_1, \mathbb{S}^n_1)$ and $\xi_2 = (E_2, p_2, \mathbb{S}^n_2)$ be two oriented n-disc bundles over $\mathbb{S}^n$. Let $D^n_i \subset \mathbb{S}^n_i$ be embedded n-discs in the base spaces and let:
> $$f_i : D^n_i \times D^n \to E_i|D^n_i$$
> be trivializations of the restricted bundles $E_i|D^n_i$ for $i = 1, 2$. To **plumb** ξ_1 and ξ_2 we take the disjoint union of E_1 and E_2 and identify the points $f_1(x, y)$ and $f_2(y, x)$ for each $(x, y) \in D^n \times D^n$."

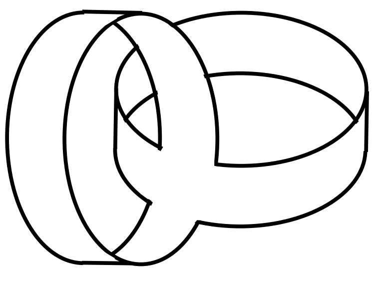

FIGURE 1. Plumbing of two n-disc bundles according to [24]

It was Hirzebruch [23] who related Milnor's and Mumford's constructions:

> "$M(E_8)$ was constructed by "plumbing" 8 copies of the circle bundle over $\mathbb{S}^{2k}$ with Euler number -2. By replacing this basic constituent by the tangent bundle of $\mathbb{S}^{2k}$ one obtains a manifold $M^{4k-1}(E_8)$ of dimension

$4k-1$. This carries a natural differentiable structure. For $k \geq 2$ it is homeomorphic to $\mathbb{S}^{4k-1}$, but not diffeomorphic (Milnor sphere)."

Here Hirzebruch proposed an alternative construction of a generator of the group of homotopy spheres of dimension 7, as the intersection matrix of the E_8 diagram is simpler than the one considered by Milnor in [29]. In fact, Milnor presented later in [30] Hirzebruch's "plumbing" construction according to the E_8 diagram rather than his initial construction.

The operation of "plumbing" was generalized from n-disc bundles over n-dimensional spheres to arbitrary n-dimensional manifolds as base spaces, the identifications of $f_1(x, y)$ and $f_2(-y, x)$ being also allowed (see, for example, Browder's book [2, Section V.2]). Nevertheless, what remained unchanged was the structure of the subbundles to be patched: products $D^n \times D^n$ of n-dimensional discs.

Now let us turn our attention to the related notion of *Murasugi sum*. We quote below the original construction by Murasugi [33, p.545], illustrating it in Figures 2 and 3 by drawings which are similar to Murasugi's original ones:

"Let us consider an orientable surface F in $\mathbb{S}^3$ [...] consisting of two disks D_1, D_2 to which n bands $B_1, B_2, ..., B_n$ are attached. All B_i are twisted once in the same direction, and the bands are pairwise disjoint and do not link one another. Let us call F a *primitive s-surface* of type (n, ϵ), where $\epsilon = \pm 1$ according as the twisting is right-handed or left-handed. [...]

Consider two primitive s-surfaces F and F' in $\mathbb{S}^3$ of type (n, ϵ) and (m, η). Take two disks, D_1 and D_1' say, from each F and F' and identify them so that the resulting orientable surface $\tilde{F} = F \cup F'$ spans a link, and that $\tilde{F} - F$ and $\tilde{F} - F'$ are *separated*, i.e. there exists a 2-sphere S in $\mathbb{S}^3$ such that $S \cap \tilde{F} = D_1 (= D_1')$ and each component of $\mathbb{S}^3 - S$ contains points of $\tilde{F} - D_1$. [...] $\tilde{F}$ will be called an *s-surface*. [...] In general, by an *s-surface* is meant an orientable surface obtained from a number of primitive s-surfaces by identifying their disks in this manner."

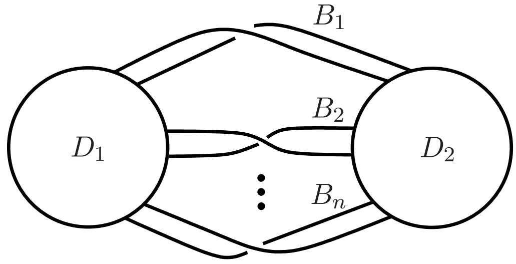

FIGURE 2. Primitive s-surface of type $(n, 1)$, whose boundary is the $(-2, n)$-torus link

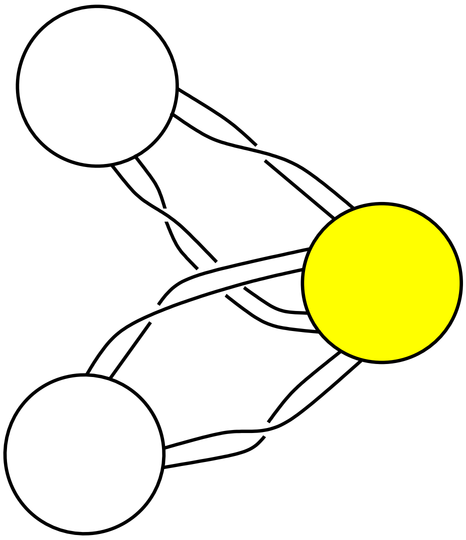

Figure 3. Disks in primitive s-surfaces of type $(2, 1)$ and of type $(2, -1)$ are identified to give a Seifert surface for a figure-eight knot.

The "*primitive s-surfaces*" used by Murasugi are *fiber-surfaces*, that is, they appear as the pages of some open books in $\mathbb{S}^3$ (see Definition 6.14 below). In [39, p.56], Stallings generalized Murasugi's construction to arbitrary fiber-surfaces as follows:

> "Consider two oriented fibre surfaces T_1 and T_2. On T_i let D_i be 2-cells, and let $h : D_1 \to D_2$ be an orientation-preserving homeomorphism such that the union of T_1 and T_2 identifying D_1 with D_2 by h is a 2-manifold T_3. That is to say:

$$h(D_1 \cap \text{Bd } T_1) \cup (D_2 \cap \text{Bd } T_2) = \text{Bd } D_2. \tag{2.1}$$

> [Here Bd X denotes the boundary of X].
>
> We can realize T_3 in $\mathbb{S}^3$ as follows: Thicken D_1 on the positive side of T_1, to get a 3-cell, whose complementary 3-cell E_1 contains T_1 with $T_1 \cap \text{Bd } E_1 = D_1$ and with negative side of T_1 contained in the interior of E_1. Likewise, there is a 3-cell E_2 containing T_2 with $T_1 \cap \text{Bd } E_1 = D_1$ and with the positive side of T_2 contained in the interior of E_2. The homeomorphism $h : D_1 \to D_2$ extends to $h : \text{Bd } E_1 \to \text{Bd } E_2$. The union of E_1 and E_2, identifying their boundaries by h - this is $\mathbb{S}^3$ - contains T_3 as $T_1 \cup T_2$. We say T_3 is obtained from T_1 and T_2 by *plumbing*."

The main result of Stallings' paper is:

Theorem 2.2. *If T_1 and T_2 are fibre surfaces, so is T_3.*

This shows in particular that the s-surfaces of Murasugi are fibre surfaces. Note that, Stallings' definition of (embedded) "plumbing" applies to any oriented surfaces in $\mathbb{S}^3$, not only to fibre surfaces.

In [11, p.132], Gabai coined the name "Murasugi sum" for a slightly restricted operation:

> "The oriented surface $R \subset \mathbb{S}^3$ is a Murasugi sum of compact oriented surfaces R_1 and R_2 in $\mathbb{S}^3$ if:
> (1) $R = R_1 \cup_D R_2$, $D = 2n$ gon
> (2) $R_1 \subset B_1, R_2 \subset B_2$ where $B_1 \cap B_2 = S, S$ a 2-sphere, $B_1 \cup B_2 = \mathbb{S}^3$ and $R_1 \cap S = R_2 \cap S = D$. "

As remarked by Gabai, this definition extends immediately to an operation on oriented surfaces in arbitrary oriented 3-manifolds.

Note that in the definition above, the way that D is embedded in $R_1 \cup_D R_2$ is not explicitly stated, but in Gabai's drawing [11, Figure 1] the edges on the boundary of the $2n$-gon D appear as arcs included *alternately* in the interior of R_1 and in the interior of R_2. Thus we may deduce that this slightly restricted operation is what Gabai had in mind from the fact that ∂D gets a structure of a polygon with an even number of edges from its embedding in both R_1 and R_2.

In [38] Rudolph called this second, more restrictive interpretation of the summing operation, "Murasugi sum" and reserved the name "Stallings plumbing" for Stallings' more general definition. Changing his notations to those of Stallings' paper, in order to be able to make reference to the identity (2.1), let us quote his comparison of the two definitions:

> "On its face, Stallings plumbing is a strict generalization of Murasugi sum, [...] its seemingly special case in which [...] (2.1) is supplemented by:

$$h(D_1 \cap \mathrm{Bd}\ T_1) \cap (D_2 \cap \mathrm{Bd}\ T_2) = \partial(D_2 \cap \mathrm{Bd}\ T_2). \tag{2.2}$$

> In fact, however, it is easy to see that (up to ambient isotopy) every Stallings plumbing is a Murasugi sum of the same plumbands. The distinction is nonetheless useful and will be maintained here."

The fact that the more general notion of "Stallings plumbing" is "nonetheless useful", even if it describes the same objects as the "Murasugi sum" may be seen already from the first application of Theorem 2.2 given by Stallings in his paper ([39, Theorem 2]):

Theorem 2.3. *The oriented link $\hat{\beta}$ obtained by closing a homogeneous braid β is fibered.*

A *homogeneous braid* is described by a word in the standard presentation of the braid groups, such that each generator appears always with exponents of the same sign. In the special case in which the generators are always positive, one obtains the usual notion of *positive braid*. Stallings' proof considers the Seifert algorithm for constructing a Seifert surface applied to the diagram of the link $\hat{\beta}$ associated to the given word. The Seifert surface appears constructed as a finite sequence of disks situated in parallel planes, successive disks being connected by twisted bands. The condition of homogeneity says that *all* the bands between two given successive disks are twisted in the same sense (see Figure 4). One recognizes therefore an s-surface of Murasugi, which is in general a "Stallings plumbing" in Rudolph's sense, but not a "Murasugi sum" in Gabai's sense.

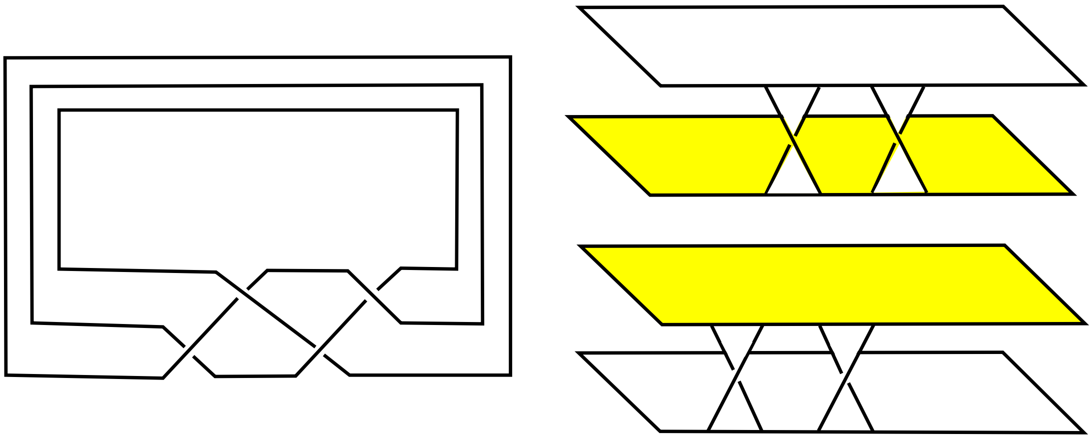

FIGURE 4. On the left: the figure eight knot $\hat{\beta}$ which is the closure of the homogeneous braid $\beta = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$. On the right: the top two disks with twisted bands connecting them form a primitive s-surface of type $(2, -1)$, while the lower two disks with twisted bands connecting them form a primitive s-surface of type $(2, 1)$. By gluing these primitive s-surfaces in the obvious way, we get a Seifert surface for $\hat{\beta}$. Compare with Figure 3.

For a special type of higher dimensional hypersurfaces in spheres, a generalization of Murasugi summing was studied by Lines in a series of papers [26, 27, 28]. Here are the definitions he used:

Definition 2.4. A **knot** $K \subset \mathbb{S}^{2k+1}$ is a $(k-2)$-connected oriented $(2k-1)$-dimensional submanifold. A **Seifert surface** for K is a compact oriented hypersurface of $\mathbb{S}^{2k+1}$ with boundary K. The knot K is called **simple** if it admits a $(k-1)$-connected Seifert surface.

The following definition appeared in [26, Section 2]:

Definition 2.5. Let K_1 and K_2 be two simple knots in $\mathbb{S}^{2k+1}$ bounding $(k-1)$-connected Seifert surfaces F_1 and F_2 respectively. Suppose that $\mathbb{S}^{2k+1}$ is the union of two balls B_1 and B_2 with a common boundary which is a $(2k)$-sphere S. Let $\psi : \mathbb{D}^k \times \mathbb{D}^k \to S$ be an embedding such that:

(1) $F_1 \subset B_1$, $F_2 \subset B_2$;
(2) $F_1 \cap S = F_2 \cap S = F_1 \cap F_2 = \psi(\mathbb{D}^k \times \mathbb{D}^k)$;
(3) $\psi(\partial\mathbb{D}^k \times \mathbb{D}^k) = \partial F_1 \cap \psi(\mathbb{D}^k \times \mathbb{D}^k)$ and $\psi(\partial\mathbb{D}^k \times \partial\mathbb{D}^k) = \partial F_2 \cap \psi(\mathbb{D}^k \times \mathbb{D}^k)$.

Then the submanifold $F := F_1 \cup F_2 \subset \mathbb{S}^{2k+1}$, after smoothing the corners, is said to be **obtained by plumbing together** the Seifert surfaces F_1 and F_2.

In [26, Proposition 2.1], Lines proved that Theorem 2.2 extends to this context. His proof is algebraic, not geometric. In the sequel, we will extend his definition, dropping any hypothesis on the topology of the pages and of the ambient manifold (see Definition

7.8), and we will show, through a geometric proof, that Theorem 2.2 extends also to this more general context (see Theorem 9.3).

3. A geometric proof of Stallings' Theorem

For the sake of completeness, we include here a geometric proof of Theorem 2.2, for the most frequently used case in the literature, where the plumbing region is just a rectangle ($n = 2$ in Gabai's Murasugi sum). The principle of the proof below is due to Gabai [11, p.139-141], although we will present here another formulation of his proof which appeared more recently in [17, p.101], using the language of open books (see Definition 6.14), rather than fibered surfaces or foliations.

First we prepare a local model of a neighborhood of a properly embedded arc in the page of an open book in an arbitrary 3-manifold as follows. Set:

$$\tilde{K} = \{(x, y, z) \in \mathbb{R}^3 \mid x = \pm 1, \ \ y = 0\}$$

and let $\tilde{\theta} : \mathbb{R}^3 \setminus \tilde{K} \to \mathbb{S}^1$ be the map defined by:

$$\tilde{\theta}(x, y, z) = \arg\big(\frac{1 + x + iy}{1 - x - iy}\big) = \arg(1 - x^2 - y^2 + 2iy).$$

As $\tilde{\theta}$ does not depend on the z-coordinate, for each $t \in \mathbb{S}^1$, the preimage $\tilde{\theta}^{-1}(t)$ can be described as the intersection $\tilde{\theta}^{-1}(t) \cap \{z = 0\}$, translated invariantly in the z-direction. Therefore, to visualize $\tilde{\theta}^{-1}(t)$, it suffices to understand $(\tilde{\theta}|_{\{z=0\}})^{-1}(t)$ which is the preimage of a ray starting from the origin in the complex plane under the homography defined by the equation $w = \dfrac{1+u}{1-u}$, where $u = x + iy$. Since homographies preserve the circles, each such preimage is included in some circle on the xy-plane. Using the last equality above, it is possible to see that for each $t \neq 0, \pi \in \mathbb{S}^1$, the preimage $(\tilde{\theta}|_{\{z=0\}})^{-1}(t)$ is an open arc of a circle passing through $(1, 0)$ and $(-1, 0)$, as depicted in Figure 5. For $t = 0$ and $t = \pi$, these preimages are given by the segment $(-1, 1)$ and $\mathbb{R} \setminus [-1, 1]$ on the x-axis, respectively.

It follows that, for $t \neq \pi \in \mathbb{S}^1$, the union $\tilde{\theta}^{-1}(t) \cup \tilde{K}$ is a connected infinite strip parallel to the z-axis, while $\tilde{\theta}^{-1}(\pi) \cup \tilde{K}$ consists of two connected components. Therefore, $\tilde{\theta}$ is not a locally trivial fibration over $\mathbb{S}^1$ (and hence it does not define an open book on $\mathbb{R}^3$), but nevertheless, $\tilde{\theta}^{-1}(t) \cup \tilde{K}$ is still called a "page" of $\tilde{\theta}$, since it gives a "piece" of an open book.

Let $E = \partial B$ denote the ellipsoid which is the boundary of the domain:

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 2y^2 + z^2 \leq 2\}.$$

Note that, for all $t \neq \pm\pi/2$, the pages of the "open book" $(\tilde{K}, \tilde{\theta})$ intersect E transversely inducing a foliation on $E \setminus \tilde{K}$, where $E \cap \tilde{K} = \{(1, 0, 1), (1, 0, -1), (-1, 0, -1), (-1, 0, 1)\}$. This foliation agrees with what Gabai depicted in [11, Fig.4]. It is invariant with respect

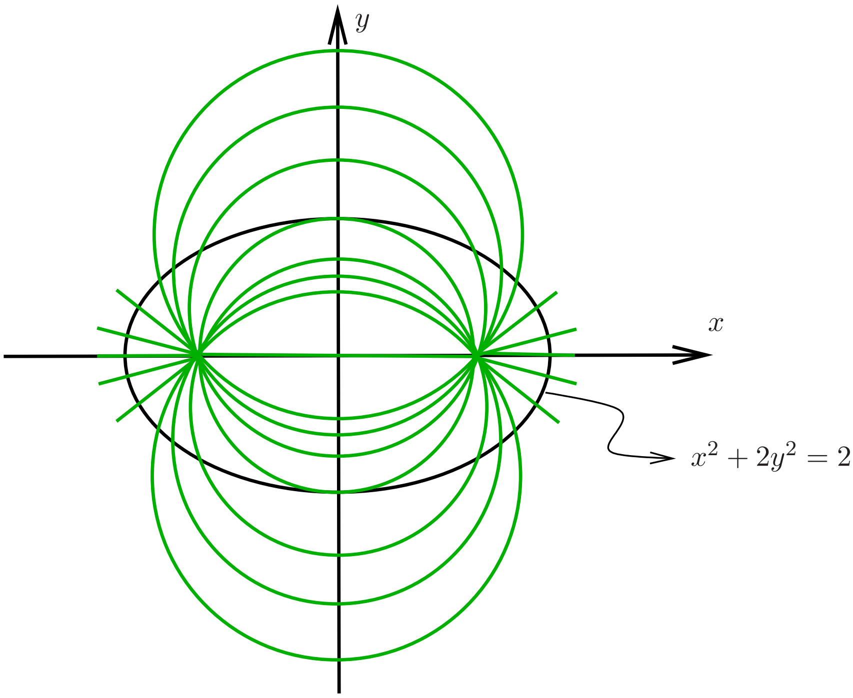

FIGURE 5. The intersection of the "pages" of $(\tilde{K}, \tilde{\theta})$ with the xy-plane, and the ellipse $x^2 + 2y^2 = 2$.

to the reflections along all three coordinate planes, and under a rotation of angle π about all three coordinate axes.

The four points in $E \cap \tilde{K}$ are the corners of a square inscribed in the circle of radius $\sqrt{2}$ on the xz-plane (see Figure 6). Moreover, the map:

$$\tilde{\rho} : E \to E, \;\; (x, y, z) \to (z, -y, -x)$$

cyclically permutes these four points, rotating the square (clockwise) in the xz-plane by an angle of $\pi/2$. Furthermore, $\tilde{\rho}$ is an orientation reversing self-diffeomorphism of E such that:

$$\tilde{\theta} \circ \tilde{\rho}(x, y, z) = \tilde{\theta}(x, y, z) + \pi \quad \text{for any } (x, y, z) \in E \setminus \tilde{K}.$$

Let M_i be an arbitrary closed oriented 3-manifold for $i = 1, 2$, and let (K_i, θ_i) be an open book in M_i. Our goal is to construct an open book (K, θ) in the connected sum of M_1 and M_2 such that the page of (K, θ) is obtained by plumbing the pages of (K_1, θ_1) and (K_2, θ_2). Suppose that C_i is a properly embedded arc in the page $\theta_i^{-1}(0) \cup K_i$. Then C_i has a neighborhood $U_i \subset M_i$ with an orientation-preserving diffeomorphism $\psi_i : U_i \to \mathbb{R}^3$, carrying $(K_i \cap U_i, \theta_i|_{U_i})$ to $(\tilde{K}, \tilde{\theta})$ and C_i to the segment $[-1, 1]$ on the x-axis. This last claim follows from two basic facts:

(i) any locally trivial fibration is trivial over an interval;

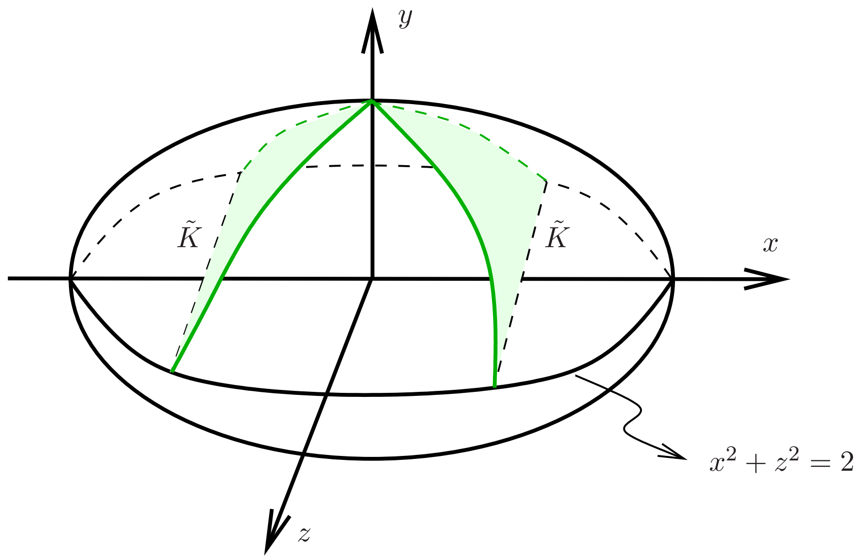

FIGURE 6. The intersection of the $(\pi/2)$-page of $(\tilde{K}, \tilde{\theta})$ with B

(ii) the geometric monodromy can be assumed to be the identity near the binding of an open book.

Consequently, the composition:

$$\rho = \psi_2^{-1} \circ \tilde{\rho} \circ \psi_1 : E_1 = \psi_1^{-1}(E) \to E_2 = \psi_2^{-1}(E)$$

is an orientation-reversing diffeomorphism which can be used to construct the connected sum:

$$M = M_1 \# M_2 = (M_1 \setminus \mathrm{int}(B_1)) \cup_\rho (M_2 \setminus \mathrm{int}(B_2)),$$

where $B_i = \psi_i^{-1}(B)$.

There is a natural open book (K, θ) on M which is defined as follows: Let K be the union of $K_1 \setminus \mathrm{int}(B_1)$ and $K_2 \setminus \mathrm{int}(B_2)$, which is a link in M because of the properties of the map $\tilde{\rho}$ discussed above. Since $\theta_2 \circ \rho(x, y, z) = \theta_1(x, y, z) + \pi$, the map θ defined as $\theta_i + (-1)^{i+1}\pi/2$ when restricted to $(M_i \setminus \mathrm{int}(B_i)) \setminus K_i$ induces a fibration on $M \setminus K$.

To understand the pages of the open book (K, θ) on M, consider the piece of (non-smooth) surface $(\tilde{\theta}^{-1}(\pi/2) \cup \tilde{K}) \setminus \mathrm{int}(B)$ depicted in Figure 7 (compare with Figure 6, but beware that we take the complement). Since $(\tilde{K}, \tilde{\theta})$ is a local model for both open books (K_1, θ_1) and (K_2, θ_2), we just need to understand how the pages in two copies of this local model fit together by the map $\tilde{\rho} : E \to E$. Because of the symmetry of the construction, $(\tilde{\theta}^{-1}(-\pi/2) \cup \tilde{K}) \setminus \mathrm{int}(B)$ is also homeomorphic to the surface depicted in Figure 7. These two *oriented* surfaces-with-boundary can be glued together along parts of their boundaries, dictated by the map $\tilde{\rho} : E \to E$, to give an oriented smooth surface with corners as we depicted on the left in Figure 8.

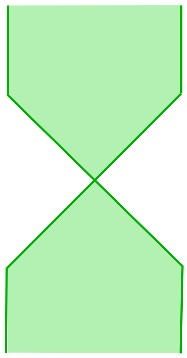

FIGURE 7. $(\tilde{\theta}^{-1}(\pi/2) \cup \tilde{K}) \setminus \mathrm{int}(B)$

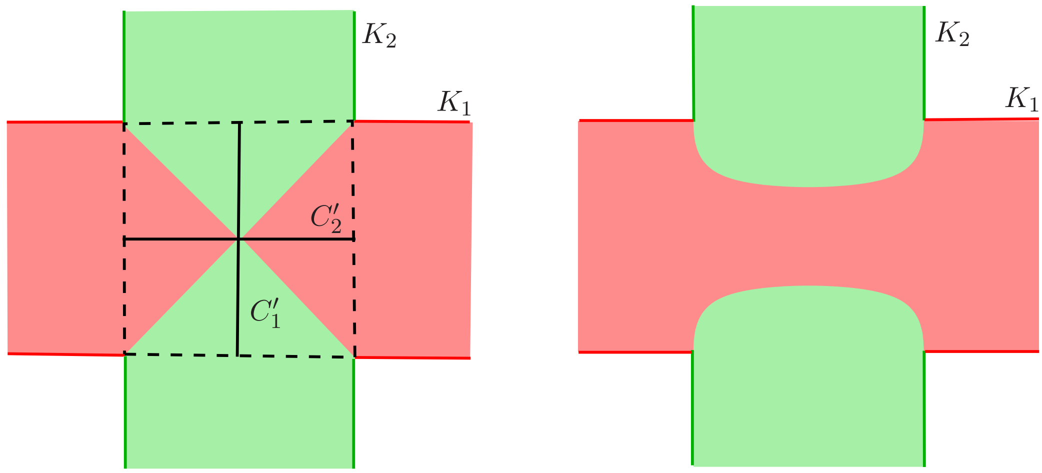

FIGURE 8. Local pictures of the pages of (K, θ): the 0-page on the left, other pages on the right

This shows that the 0-page of (K, θ) can be viewed as the plumbing of the $(-\pi/2)$-page of (K_1, θ_1) with the $(\pi/2)$-page of (K_2, θ_2) along the neighborhoods of the arcs C_1' and C_2' defined by:

$$C_1' = \psi_1^{-1}(C_1) \text{ and } C_2' = \psi_2^{-1}(C_2),$$

where:

$$C_1 = \{x^2 + y^2 = 1,\ y \leq 0,\ z = 0\} \text{ and } C_2 = \{x^2 + y^2 = 1,\ y \geq 0,\ z = 0\}.$$

Similarly, all the other pages of the open book (K, θ) will appear locally as drawn on the right in Figure 8, each of which is *globally* diffeomorphic to the 0-page, after smoothing the corners as usual. Hence $\theta : M \setminus K \to \mathbb{S}^1$ is a locally trivial fibration each of whose fibers is obtained by plumbing a page of (M_1, θ_1) with a page of (M_2, θ_2)—which finishes Gabai's proof of Stallings' Theorem 2.2.

The proof above can be described with another point of view which turns out to be more suitable for the generalizations we have in mind. One can interpret what is inside the domain B in the local model $(\mathbb{R}^3, \tilde{\theta})$ as the union of two (overlapping) pieces:

- a tubular neighborhood of the intersection $B \cap \tilde{K}$, which is nothing but two disjoint arcs of the binding $\tilde{K}$;
- a *thickening* of the plumbing region.

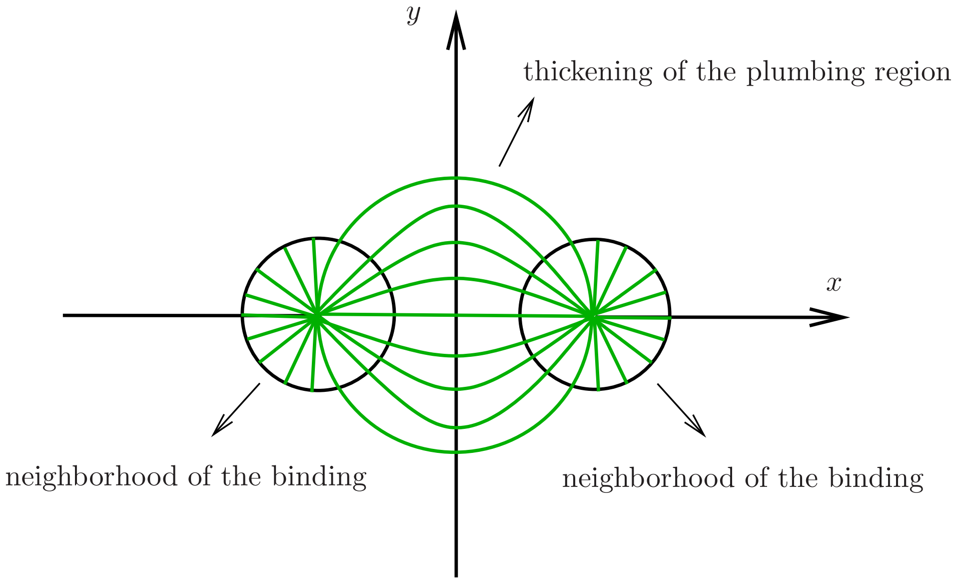

FIGURE 9. A model with truncated pages

The thickening (topologically a rectangle times an interval) consists of a rectangle from each page $\tilde{\theta}^{-1}(t) \cup \tilde{K}$ for $t \in [-\pi/2, \pi/2] \in \mathbb{S}^1$. To see this, we slightly truncate the pages of $\tilde{\theta}$ in B corresponding to the arc $[-\pi, -\pi/2] \cup [\pi/2, \pi]$ on $\mathbb{S}^1$ such that the pages intersect the xy-plane as shown in Figure 9. In other words, we slightly deform the domain B keeping all of its symmetries needed in the previous discussion. Therefore, by removing B, we remove the plumbing region from half of the pages of the open book corresponding to one "half" of $\mathbb{S}^1$, along with tubular neighborhood of the two arcs of the binding.

For the Murasugi sum of two open books, we remove the plumbing regions from half of the pages in both open books *but these halves correspond to complementary oriented arcs* on $\mathbb{S}^1$. (This fact reveals itself in the above proof by the appearance of the difference π in the parametrization of the fibrations to be glued.) So, when we glue the ambient manifolds after removing diffeomorphic copies of B from each one of them, the fibrations in the complements of the respective bindings will glue together so that the hole created as a result of removing a rectangle (the plumbing region) from any page will be sewn back up by the rectangle in the corresponding page of the complementary fibration. The way that these rectangles are identified is equivalent to plumbing, so that the resulting pages are smooth manifolds. One can also see that the aforementioned tubular neighborhoods of the arcs on the bindings will indeed disappear in the process, whereas the rest of the bindings will glue together to give the new binding in the glued up manifold.

There is yet another interpretation of the proof using abstract open books (see Etnyre [10, Theorem 2.17]). Given two abstract open books (Σ_i, ϕ_i), $i = 1, 2$ (see Remark 6.15 (1) below), let C_i be an arc properly embedded in Σ_i and $R_i = C_i \times [-1, 1] \subset \Sigma_i$ a rectangular neighborhood of C_i. The idea of the proof is to perform a Murasugi sum of the mapping tori $\mathcal{M}(\Sigma_1, \phi_1)$ and $\mathcal{M}(\Sigma_2, \phi_2)$ leaving the bindings out of the picture at first and then to complete the resulting mapping torus into an open book of the connected sum of the ambient manifolds.

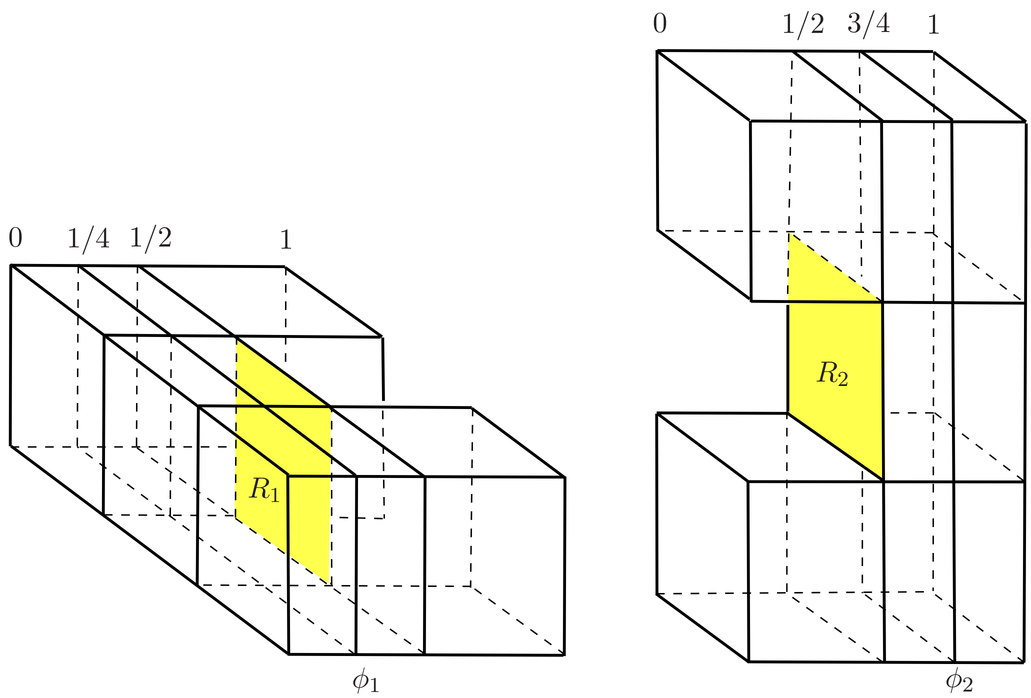

Figure 10. Local pictures of $\mathcal{M}(\Sigma_1, \phi_1) \setminus B_1$ (on the left) and $\mathcal{M}(\Sigma_2, \phi_2) \setminus B_2$ (on the right)

Note that $B_1 = R_1 \times [1/2, 1]$ is a 3-ball in $\mathcal{M}(\Sigma_1, \phi_1)$ and similarly $B_2 = R_2 \times [0, 1/2]$ is a 3-ball in $\mathcal{M}(\Sigma_2, \phi_2)$. We view the mapping torus $\mathcal{M}(\Sigma_1, \phi_1)$ obtained as gluing $\Sigma_1 \times \{0\}$ to $\Sigma_1 \times \{1\}$ using the identity and then cutting the resulting $\Sigma_1 \times \mathbb{S}^1$ along $\Sigma_1 \times \{1/4\}$ and regluing using ϕ_1 (see Figure 10). Similarly we view $\mathcal{M}(\Sigma_2, \phi_2)$ obtained as gluing $\Sigma_2 \times \{0\}$ to $\Sigma_2 \times \{1\}$ using the identity and then cutting the resulting $\Sigma_2 \times \mathbb{S}^1$ along $\Sigma_2 \times \{3/4\}$ and regluing using ϕ_2.

Let $\Sigma = \Sigma_1 + \Sigma_2$ denote the Murasugi sum of Σ_1 and Σ_2 along the rectangles R_1 and R_2. Then $\mathcal{M}(\Sigma_1, \phi_1) \setminus B_1$ and $\mathcal{M}(\Sigma_2, \phi_2) \setminus B_2$ can be glued together, as illustrated in

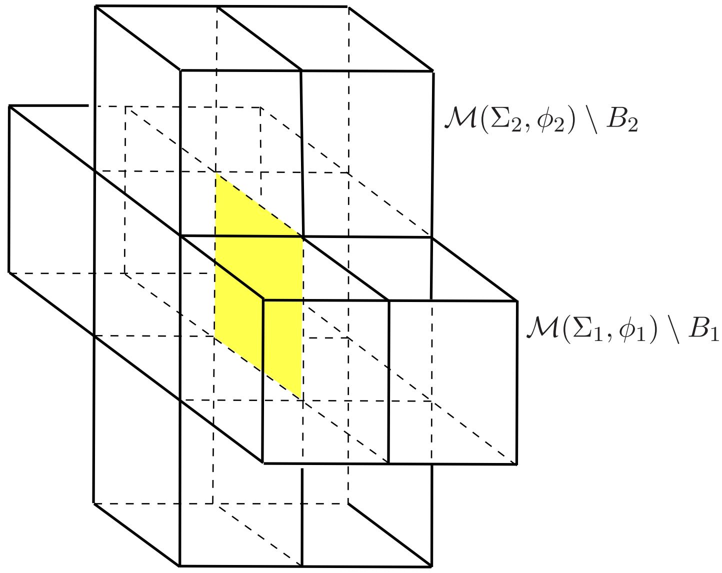

FIGURE 11. Two "lego" pieces of Figure 10 fitting together

Figure 11, to induce a mapping torus with page Σ. Therefore we conclude that:

$$\big(\mathcal{M}(\Sigma_1, \phi_1) \setminus B_1\big) \cup \big(\mathcal{M}(\Sigma_2, \phi_2) \setminus B_2\big) = \mathcal{M}(\Sigma, \phi),$$

where $\Sigma = \Sigma_1 + \Sigma_2$, and $\phi = \phi_1 \circ \phi_2$. Here we extend ϕ_i ($i = 1, 2$) from Σ_i to Σ by the identity map and then compose these extended diffeomorphisms, which we still denote by ϕ_i on Σ. As a matter of fact, from this monodromical viewpoint "Murasugi sum" appears more like a *composition* than a sum.

To show that the mapping torus $\mathcal{M}(\Sigma, \phi)$ extends to an open book of the connected sum $M_1 \# M_2$ we proceed as follows (see Goodman's Thesis [19, pages 9-10]). First of all, we view Σ_i as a submanifold of Σ and identify $R = R_i$, for $i = 1, 2$. Then $s_i =: R \cap \partial\Sigma_i$ is the disjoint union of two properly embedded arcs in Σ such that the set of four points $\partial s_1 = \partial s_2$ belongs to $\partial\Sigma$.

In the following we present the separating sphere S in the connected sum $M = M_1 \# M_2$. Let $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$. For each $i = 1, 2$, consider the disjoint union of two disks $s_i \times I_i \subset \Sigma \times I \subset \mathcal{M}(\Sigma, \phi)$. Let S' be the surface obtained as the following union of six disks:

$$(s_1 \times I_1) \cup (s_2 \times I_2) \cup (R \times \{0\}) \cup (R \times \{1/2\})$$

in $\mathcal{M}(\Sigma, \phi)$. Observe that $\partial S' = \mathbb{S}^1 \times \partial s_1$. We can cap off S' with the four disks $\mathbb{D}^2 \times \partial s_1$ to construct the desired sphere S as illustrated in Figure 12.

Now we claim that $M \setminus S$ decomposes into $M_1 \setminus B_1$ and $M_2 \setminus B_2$ for some 3-dimensional balls B_1 and B_2. To prove our claim, we note that $\mathcal{M}(\Sigma, \phi) = (\Sigma \times I_1) \cup (\Sigma \times I_2)$, where we identify $\Sigma \times \{1/2\}$ in the first product with $\Sigma \times \{1/2\}$ in the second product via ϕ_1 and $\Sigma \times \{1\}$ with $\Sigma \times \{0\}$ via ϕ_2. It follows that, by removing S, we have $(\Sigma_1 \sqcup (\Sigma_2 \setminus R)) \times I_1$

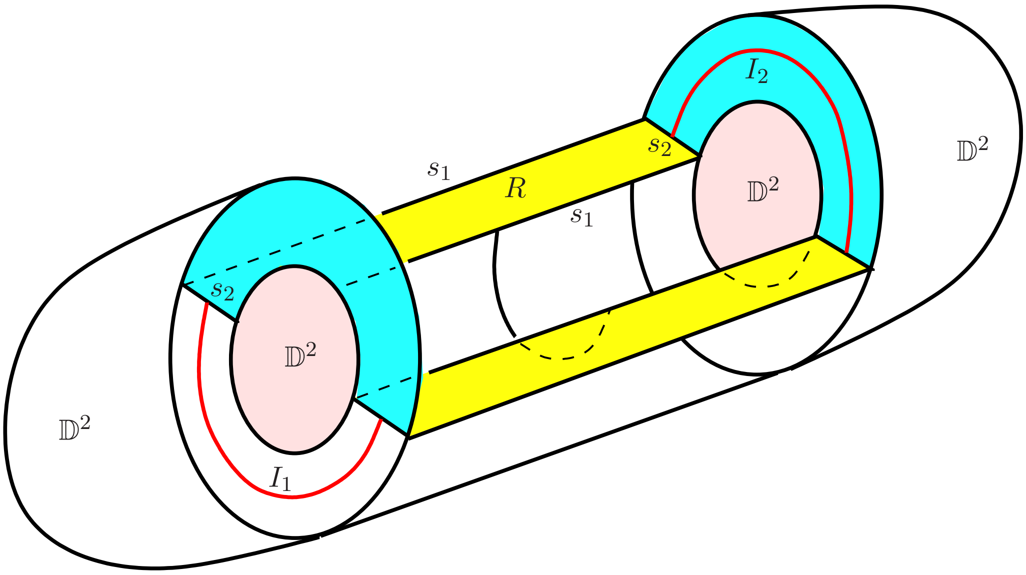

FIGURE 12. The four disks used to cap off S' in order to get the sphere S

glued to $(\Sigma_2 \sqcup (\Sigma_1 \setminus R)) \times I_2$. But since ϕ_1 is the identity on Σ_2 and ϕ_2 is the identity on Σ_2, the result can also be viewed as a union of two pieces $\mathcal{M}(\Sigma_1, \phi_1) \setminus (R \times I_1)$ and $\mathcal{M}(\Sigma_2, \phi_2) \setminus (R \times I_2)$.

Finally, we insert the binding as follows. Since ∂s_i is a set of four points in $\partial\Sigma$, the solid torus $\mathbb{D}^2 \times \partial\Sigma$ is cut into four pieces along $\mathbb{D}^2 \times \partial s_i$. Thus by gluing in the binding, we see that M decomposes into two pieces along the sphere S:

$$\big(\mathcal{M}(\Sigma_1, \phi_1) \cup (\mathbb{D}^2 \times \partial\Sigma_1)\big) \setminus \big((R \times I_1) \cup (\mathbb{D}^2 \times s_1)\big) = M_1 \setminus \big((R \times I_1) \cup (\mathbb{D}^2 \times s_1)\big)$$

and:

$$\big(\mathcal{M}(\Sigma_2, \phi_2) \cup (\mathbb{D}^2 \times \partial\Sigma_2)\big) \setminus \big((R \times I_2) \cup (\mathbb{D}^2 \times s_2)\big) = M_2 \setminus \big((R \times I_2) \cup (\mathbb{D}^2 \times s_2)\big).$$

Observe that each $B_i := (R \times I_i) \cup (\mathbb{D}^2 \times s_i)$ is a 3-ball with boundary S.

Our paper is motivated by the search of the most general operation of Murasugi-type sum (that is, embedded Milnor-style plumbing) for which one has an analog of Theorem 2.2. We figured out that we do not need to restrict in any way the full-dimensional submanifolds which are to be identified in the plumbing operation. That is why we define a general operation of "summing" of manifolds (see Definition 7.4), which reduces to the classical operation of Definition 2.1 when the identified submanifolds have product structures $\mathbb{D}^n \times \mathbb{D}^n$.

The greater level of generality obliged us to discard the special model used in the previous proof. The principle of the proof of our generalization 9.3 of Gabai's theorem is instead inspired by Etnyre's interpretation. In this respect, Figure 11 is to be compared with Figure 30.

4. Conventions and basic definitions

In this section we explain our conventions about manifolds, orientations and coorientations of hypersurfaces. We give rather detailed explanations because throughout the paper we work without any assumptions about orientability of the manifolds: the only important issues are about *coorientations*, which makes the setting rather non-standard when compared with the usual literature in differential topology.

In this paper, the manifolds are assumed to be smooth and *pure dimensional*, but not necessarily orientable or connected. If a manifold is endowed with an orientation, we explicitly say that it is an "*oriented manifold*". We use the expression "*manifold-with-boundary*" for a smooth manifold with possibly empty boundary. We denote by ∂W the boundary of the manifold-with-boundary W and by:

$$\mathrm{int}(W) := W \setminus \partial W$$

its *interior*.

In the sequel, we will implicitly use the facts that *the corners of a manifold with corners can be smoothed*, and that the resulting smooth manifold-with-boundary is well-defined up to isotopy as a zero-codimensional submanifold of the initial manifold with corners. A standard reference for these folklore facts is the Appendix of Milnor's paper [29]. We will also use the folklore fact that *two manifolds-with-boundary can be glued along compact zero-codimensional submanifolds of their boundaries, once a diffeomorphism between these submanifolds is fixed*, and that the result is well-defined up to diffeomorphism. A standard reference for this is Hirsch's book [22, Chapter 8.2]. All these facts are also treated in a detailed way by Douady in his contributions [6], [7], [8] to the Seminar Cartan.

Remark 4.1. In the sequel, the only gluings to be done will be special cases of identifications of submanifolds of two manifolds-with-boundary by diffeomorphisms. In order to simplify the notations, instead of giving different names to those submanifolds and labeling also the diffeomorphism used for the gluing, we will assume that the two submanifolds were identified using that diffeomorphism, which implies that the gluing diffeomorphism is the identity. For instance, we will not write "glue M_1 to M_2 using the diffeomorphism $\phi : P_1 \to P_2$ of $P_i \hookrightarrow M_i$", but "glue M_1 to M_2 along $P \hookrightarrow M_i$".

If V is a submanifold-with-boundary embedded in W, then we use the notation $V \hookrightarrow W$. We say that V is **properly embedded** into W if $V \cap \partial W = \partial V$ and if V and ∂W are transversal in W everywhere along ∂V. When $\partial V = \emptyset$, this means simply that $V \subset \mathrm{int}(W)$. In this paper, the submanifolds of interest are not necessarily properly embedded (for instance, the pages of an arbitrary open book). If $M \hookrightarrow W$ is a submanifold, we denote by $\mathrm{codim}_W(M)$ its *codimension* in the ambient manifold W.

If $V \hookrightarrow W$ is properly embedded, we denote by $\mathcal{U}_W(V)$ (or simply $\mathcal{U}(V)$ if W is clear from the context) a closed **tubular neighborhood** of V in W such that $\mathcal{U}_W(V) \cap \partial W$ is a tubular neighborhood of ∂V in ∂W. Moreover, we assume that $\mathcal{U}_W(V)$ is endowed with a structure of smooth fiber bundle over V, whose fibers are diffeomorphic to compact balls of dimension $\mathrm{codim}_W(V)$.

Let us examine the special case of properly embedded *hypersurfaces*. One has the following well-known proposition:

Proposition 4.2. *Let $M \hookrightarrow W$ be a compact hypersurface-with-boundary properly embedded inside the manifold W. The following conditions are equivalent:*

(1) *the normal bundle $\mathcal{N}_{M|W}$ of M in W is orientable;*
(2) *M admits a tubular neighborhood diffeomorphic to $[-1,1] \times M$, where $M \hookrightarrow W$ is identified with $\{0\} \times M$;*
(3) *each connected component U_i of an arbitrary regular neighborhood $\mathcal{U}_W(M)$ is disconnected by $U_i \cap M$.*

Moreover, if any of the conditions above is satisfied, then the following choices are equivalent:

(1') *an orientation of the normal bundle $\mathcal{N}_{M|W}$ of M in W;*
(2') *an embedding $[-1,1] \times M \hookrightarrow W$ which sends $\{0\} \times M$ to M by $\{0\} \times m \to m$ for any $m \in M$, up to isotopy;*
(3') *a choice of connected component of $U_i \setminus M$ for each connected component U_i of a tubular neighborhood $\mathcal{U}_W(M)$.*

More precisely, the normal vectors pointing towards the positive side for the chosen orientation of the normal bundle are tangent to the curves entering into $(0,1] \times M$, which defines the choice of connected component of each U_i.

The previous proposition allows us to define:

Definition 4.3. Let $M \hookrightarrow W$ be a properly embedded compact hypersurface-with-boundary. It is called **coorientable** if it satisfies any one of the equivalent conditions (1)–(3) of the previous proposition. A **coorientation** of M in W is an orientation of the normal bundle $\mathcal{N}_{M|W}$ of M in W.

Example 4.4. Consider a Möbius band W seen as a non-trivial segment-bundle over a circle. Any fiber is coorientable, but no section of it is coorientable.

Suppose that W is a manifold with nonempty boundary ∂W. Recall that we do not assume orientability of either W or ∂W. Even though ∂W is not properly embedded in W, it has an orientable normal bundle in W and hence we say that ∂W is coorientable by adapting Definition 4.3 to this case. Since ∂W is coorientable, then any codimension zero submanifold of ∂W is coorientable and for each connected component of such a submanifold of ∂W, the two coorientations may be distinguished as:

- **incoming**, if the corresponding normal vectors point inside W;
- **outgoing**, if the corresponding normal vectors point outside W.

Remark 4.5. In the sequel (see for instance Definition 5.1) we will not necessarily coorient a whole boundary component uniformly, but we might have to break it up by inserting "corners" as in Figure 14. For this reason, we also speak about the coorientation of any full-dimensional submanifold of the boundary.

If a manifold-with-boundary W is *oriented*, then for each connected component of its boundary ∂W we define the **outgoing orientation** by the rule known as "outside pointing normal vector comes first": a normal vector to ∂W pointing outside of W, followed by a positive basis of the tangent space to ∂W, gives a positive basis to the tangent space of W. It is customary to take the outgoing orientation as the *canonical orientation* induced on ∂W. The opposite orientation of the boundary is the **incoming orientation**.

Example 4.6. For each $n \geq 1$, we denote by $\mathbb{D}^n \subset \mathbb{R}^n$ the compact unit ball endowed with the restriction of the canonical orientation of $\mathbb{R}^n$ and by $\mathbb{S}^{n-1}$ its boundary sphere, endowed with the associated outgoing orientation.

If W is an oriented manifold-with-boundary and ∂W is *independently* oriented, then:

- its **outgoing boundary** $\partial_+ W$ is the union of the connected components of ∂W which are endowed with the outgoing orientation;
- its **incoming boundary** $\partial_- W$ is the union of the connected components of ∂W which are endowed with the incoming orientation.

We clearly have:

$$\partial W = \partial_+ W \bigsqcup \partial_- W.$$

In this case, we see W as a **cobordism from** $\partial_- W$ **to** $\partial_+ W$ (see Figure 13).

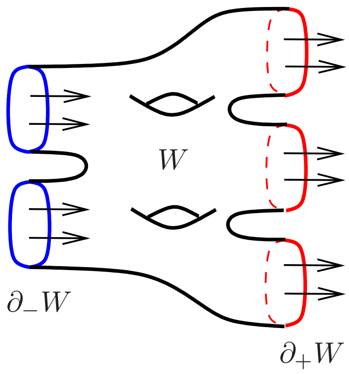

FIGURE 13. W is a cobordism from $\partial_- W$ to $\partial_+ W$.

For instance, the interval $[0, 1]$ endowed with its canonical orientation is a cobordism from the positively oriented point $\{0\} = \partial_-[0, 1]$ to the positively oriented point $\{1\} = \partial_+[0, 1]$. Note that to orient a point means to choose one of the signs $\pm$ attached to it, which allows us to speak in this case about positive/negative orientations.

More generally, we will denote by I or I_j (j varying inside some index set) an **oriented compact interval**, that is, an oriented compact manifold-with-boundary, diffeomorphic to $[0,1] \subset \mathbb{R}$. Its two boundary components will be endowed with their canonical orientations, therefore we may speak without ambiguity of the **outgoing point** $\partial_+ I$ and the **incoming point** $\partial_- I$ of I.

Definition 4.7. Let $M \hookrightarrow W$ be a *properly embedded* and *cooriented* compact hypersurface-with-boundary. A **positive side** of M is an embedding $I^+ \times M \hookrightarrow W$ such that $M \hookrightarrow W$ is identified with $\partial_- I^+ \times M$ and the positive normal vectors of M point into $I^+ \times M$. A **negative side** of M is an embedding $I^- \times M \hookrightarrow W$ such that $M \hookrightarrow W$ is identified with $\partial_+ I^- \times M$ and the positive normal vectors of M point outside it. Here both I^+ and I^- denote oriented compact intervals. A **collar neighborhood** of $M \hookrightarrow W$ is the union of a negative and of a positive side of M whose intersection is M.

In the sequel we will have to work with a more general notion of cobordism, which is described in the next section.

5. Cobordisms of manifolds-with-boundary

In this section we set up the notation for *cobordisms of manifolds with boundary*, without the assumption of orientability. We also introduce *cylinders*, *cylindrical cobordisms* and *endobordisms* as particular cases of cobordisms of manifolds with boundary. Moreover, we explain in which sense the notions of endobordism and properly embedded cooriented hypersurface in a manifold-with-boundary are equivalent.

In the next definition we extend the notion of cobordism to situations where:

- the total manifold is not necessarily orientable;
- the incoming and outgoing boundaries are not necessarily closed manifolds;
- the total manifold may have boundary components which are not labeled as incoming or outgoing.

What we keep instead from the situation described in the previous section is the disjointness of the two types of boundary regions and the fact that they are of codimension zero in the boundary of the cobordism.

Definition 5.1. Let M^- and M^+ be manifolds-with-boundary. A **cobordism** W from M^- to M^+ is a manifold-with-boundary W, whose boundary is decomposed as:

$$\partial W = Y \cup M^- \cup M^+,$$

where Y is a nonempty submanifold-with-boundary of ∂W such that $M^- \cap M^+ = \emptyset$, $Y \cap M^- = \partial M^-$, $Y \cap M^+ = \partial M^+$ and:

- M^- is endowed with the incoming coorientation, and
- M^+ is endowed with the outgoing coorientation.

We say that $M^\mp$ is the **incoming/outgoing boundary region** of the cobordism W and set $\partial_\mp W = M^\mp$. We denote this cobordism (of manifolds-with-boundary) by:

$$W : \partial_- W \Longmapsto \partial_+ W.$$

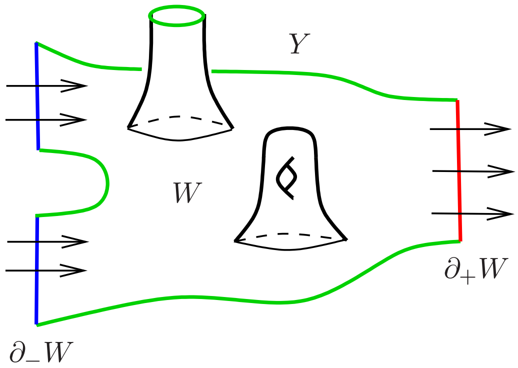

FIGURE 14. Cobordism of manifolds-with-boundary $W : \partial_- W \Longmapsto \partial_+ W$, where $\partial_- W$ is blue, $\partial_+ W$ is red and Y (the rest of ∂W) is green.

Remark 5.2. The definition above is not new (see, for example, [1]) except for the orientability assumptions. Strictly speaking, W is a manifold with corners (for this reason, we called them "*cobordisms with corners*" in a previous version of this paper), but nevertheless, corners along $\partial(\partial_- W) \sqcup \partial(\partial_+ W)$ may be smoothed. Note that $\partial_- W$ and $\partial_+ W$ may belong to the same connected component of ∂W after smoothing the corners and also, the boundary of W may have connected components disjoint from $\partial_- W \sqcup \partial_+ W$, as illustrated in Figure 14.

More generally, if one has two manifolds M^- and M^+ (possibly with boundaries) and fixed diffeomorphisms $M^\pm \simeq \partial_\pm W$, we simply say that W is a **cobordism from** M^- **to** M^+ and write $W : M^- \Longmapsto M^+$. Note that cobordisms can be *composed*: if $W_1 : M_1 \Longmapsto M_2$ and $W_2 : M_2 \Longmapsto M_3$ are two cobordisms then their **composition** $W_2 \circ W_1 : M_1 \Longmapsto M_3$ is a cobordism obtained by gluing W_1 and W_2 along M_2.

Remark 5.3. The notion of cobordism of manifolds-with-boundary weakens and extends to arbitrary dimension the notion of *sutured manifold* introduced in dimension 3 by Gabai [12, Definition 2.6]:

> "A sutured manifold (M, γ) is a compact oriented 3-manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. Furthermore, the interior of each component of $A(\gamma)$ contains a *suture*, i.e. a homologically nontrivial oriented simple closed curve. We denote the set of sutures by $s(\gamma)$. Finally every component of $R(\gamma) = \partial M - \mathrm{int}(\gamma)$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be those components of $\partial M - \mathrm{int}(\gamma)$ whose normal vectors point out of (into) M. The orientations on $R(\gamma)$ must be coherent with respect to $s(\gamma)$, i.e., if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma)$ as some suture."

A sutured manifold (M, γ) as in Gabai's definition is a cobordism of manifolds-with-boundary from $R_-(\gamma)$ to $R_+(\gamma)$ according to our definition. We drop any constraints on the structure of the complement of the union of outgoing and incoming boundary regions inside the full boundary. Moreover, we do not assume that the ambient manifold is oriented, or even orientable. Our definition is also more general than the extension to arbitrary dimensions of the notion of sutured manifold, given by Colin-Ghiggini-Honda-Hutchings in [5].

In the sequel, we will simply write "cobordisms" instead of "cobordisms of manifolds-with-boundary".

Definition 5.4. If the incoming and the outgoing boundaries M^- and M^+ of a cobordism $W : M^- \Longmapsto M^+$ are diffeomorphic and a diffeomorphism between them is fixed, then we say that W is an **endobordism** of $M \simeq M^- \simeq M^+$. The **mapping torus** of the endobordism $W : M^- \Longmapsto M^+$ is the manifold-with-boundary $T(W)$ obtained by gluing M^- and M^+ using this diffeomorphism. The mapping torus comes equipped with a cooriented proper embedding $M \hookrightarrow T(W)$, which is the image inside $T(W)$ of the boundary manifolds M^- and M^+ which are identified (see Figure 15).

In the notation "$T(W)$", we suppress for simplicity the diffeomorphism which identifies the incoming and outgoing boundaries. Note that it is nevertheless of fundamental importance for the construction. The reason we chose the name "mapping torus" is explained in Remark 6.15 (2) below.

We will be mainly concerned with the following types of endobordisms:

Definition 5.5. Let M be a manifold-with-boundary. A **cylinder** with **base** M is a trivial cobordism $W = I \times M$, the incoming boundary being $\partial_- I \times M$ and the outgoing one being $\partial_+ I \times M$. A **cylindrical cobordism** with **base** M is a cobordism W from a copy M^- of M to another copy M^+ such that the union of connected components of ∂W which intersect $M^- \cup M^+$—the **cylindrical boundary** $\partial_{cyl} W$—is endowed with a diffeomorphism (respecting the incoming and outgoing boundary regions) to the boundary $\partial(I \times M) = (\partial I \times M) \cup (I \times \partial M)$ of a cylinder with base M. The segment I is called the **directing segment** of the cylindrical cobordism.

Note that cylinders with base M are special cases of cylindrical cobordisms with base M, which are special cases of endobordisms of M.

The composition of two cylinders/cylindrical cobordisms with the same base M is a cylinder/cylindrical cobordisms with base M. More generally, the composition of two endobordisms of M is again an endobordism of M.

To any *cooriented* and *properly embedded* hypersurface M of a (not necessarily oriented or even orientable) manifold-with-boundary is associated canonically (up to diffeomorphisms) an endobordism with base M.

Definition 5.6. Let W be a compact manifold-with-boundary and let $M \hookrightarrow W$ be a *cooriented* and *properly embedded* compact hypersurface-with-boundary. We view a collar neighborhood $[-1, 1] \times M \hookrightarrow W$ of M as the cylinder $Z_{[-1,1]} : \{-1\} \times M \Longmapsto \{+1\} \times M$. Denote by $Z_{[-1,0]}$ and $Z_{[0,1]}$ the analogous cylinders corresponding to the intervals $[-1, 0]$

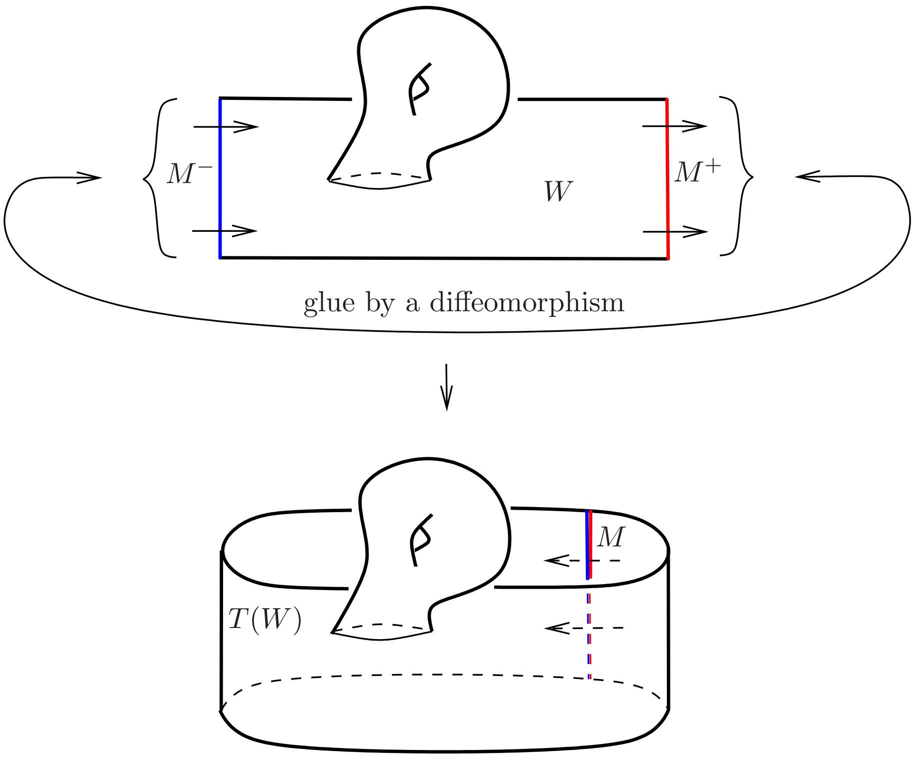

FIGURE 15. Mapping torus of an endobordism

and $[0,1]$, which implies that $Z_{[-1,1]} \simeq Z_{[0,1]} \circ Z_{[-1,0]}$. Let W_M be the closure inside W of the complement $W \setminus ([-1,1] \times M)$. We see it as an endobordism $W_M : \{1\} \times M \Longmapsto \{-1\} \times M$, hence the composition $Z_{[-1,0]} \circ W_M \circ Z_{[0,1]}$ is also an endobordism of M. We call this endobordism the **splitting of W along M** and denote it by:

$$\Sigma_M(W) : M^- \Longmapsto M^+$$

(see Figure 16), where $M^{\mp}$ are two copies of M. The natural map:

$$\sigma_M : \Sigma_M(W) \to W$$

is called the **splitting map** of W along M or of $M \hookrightarrow W$.

Intuitively, one modifies W replacing each point of M by the set of two orientations of the normal line to M at that point.

Remark 5.7. (1) The splitting map σ_M is a diffeomorphism above $W \setminus M$, the preimage of M by σ_M being the disjoint union $M^+ \sqcup M^-$ of two copies of M, distinguished canonically as the incoming and the outgoing boundaries of the cobordism $\Sigma_M(W) : M^- \Longmapsto M^+$. Both figures 15 and 16 may be seen as graphical representations of the splitting map σ_M. In the first case one starts from the source and in the second case from the target, before constructing the map σ_M.

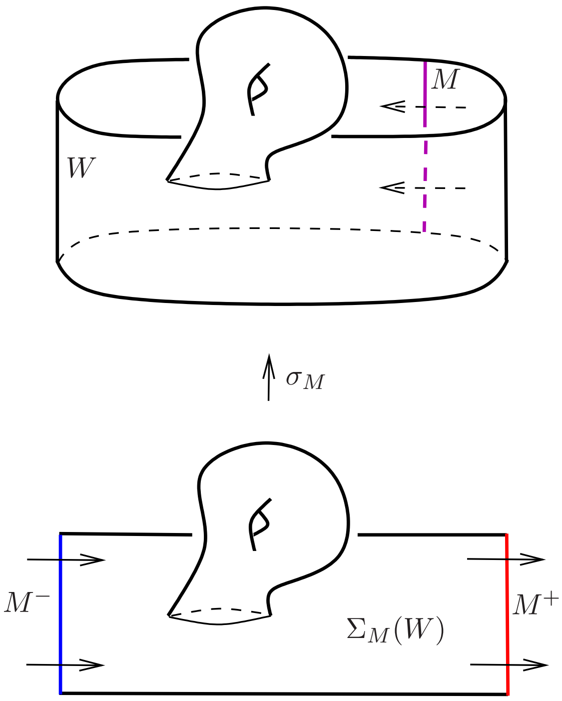

FIGURE 16. Splitting of W along a cooriented properly embedded hypersurface M

(2) The splitting map σ_M allows one to prove that the splitting of W along M is unique up to a unique diffeomorphism above W (that is, any two such splittings are related by a unique diffeomorphism compatible with their splitting maps). One may see $\Sigma_M(W)$ as a generalization of the surface obtained by cutting a given surface along a properly embedded arc, an operation fundamental in Riemann's approach of [37] to the topology of surfaces. Another way to model this splitting operation is to remove a collar neighborhood of M. We preferred the previous definition because of its canonical nature.

(3) One could also define a splitting map along non-coorientable hypersurfaces. In this case one would not obtain a cobordism, because above M the map would restrict to a non-trivial covering of degree 2. We did not define such splittings because we do not use them in this paper.

We have the following immediate relation between the operations of taking the mapping torus and of splitting:

Proposition 5.8. *The operations of taking the mapping torus of an endobordism and of splitting along a cooriented properly embedded hypersurface are inverse to each other.*

6. Seifert hypersurfaces and open books

In this section we introduce a notion of *Seifert hypersurface* and we explain in which sense it is equivalent to the notion of cylindrical cobordism introduced in the previous section. We conclude by treating the special case of Seifert hypersurfaces which are pages of open books.

Assume that M is still a cooriented compact hypersurface-with-boundary in W, but which is *not* properly embedded. Instead, we require *M to be contained in the interior of W*. In order to write more concisely, we introduce a special name for such hypersurfaces:

Definition 6.1. Let W be a manifold-with-boundary. A compact hypersurface-with-boundary $M \hookrightarrow W$ is a **Seifert hypersurface** if:

- the boundary of each connected component of M is non-empty;
- $M \hookrightarrow \text{int}(W)$;
- M is cooriented.

Remark 6.2. Traditionally, a *Seifert surface* is defined as an oriented surface embedded in $\mathbb{S}^3$, whose boundary is an oriented link L which one wants to study. Seifert surfaces are often used algebraically through their associated *Seifert forms*. To define the Seifert form, one needs to choose a positive side of the Seifert surface, to push some 1-cycles off the surface towards that direction and to compute some linking numbers. An important ingredient in this construction is the *coorientation* of the Seifert surface, which is canonically determined by the orientation of L and $\mathbb{S}^3$. For this reason, we have decided to extend this aspect of Seifert surfaces in $\mathbb{S}^3$ to a general definition, that also subsumes Lines' Definition 2.4.

There is a canonical way to associate to a Seifert hypersurface M of W a *cooriented* and *properly embedded* hypersurface-with-boundary in a new manifold (see Definition 6.8). But in order to achieve this, one has first to "pierce" W along ∂M. We will define this piercing procedure using special trivialized tubular neighborhoods of $\partial M \hookrightarrow W$:

Definition 6.3. Let W be a manifold-with-boundary and let $M \hookrightarrow W$ be a Seifert hypersurface. A tubular neighborhood $\mathcal{U}_W(\partial M)$ of $\partial M \hookrightarrow W$ is called **adapted to** M if it is endowed with a product structure $\mathbb{D}^2 \times \partial M$ such that M intersects it along $[0,1] \times \partial M$ (where $[0,1] \hookrightarrow \mathbb{D}^2$ is the canonical embedding) and if the canonical orientation of $\partial \mathbb{D}^2$ coincides with the given coorientation of M in W. The composition of the first projection $\mathcal{U}_W(\partial M) \setminus \partial M \simeq (\mathbb{D}^2 \setminus \{0\}) \times \partial M \to \mathbb{D}^2 \setminus \{0\}$ with the angular coordinate $\theta : \mathbb{D}^2 \setminus \{0\} \to \mathbb{S}^1$ is called an **angular coordinate of ∂M adapted to M** (see Figure 17).

An adapted tubular neighborhood of the boundary of a Seifert hypersurface always exists and is unique up to isotopy. The reason is that the normal bundle $\mathcal{N}_{\partial M | W}$ of ∂M in W is canonically trivialized up to homotopy, by taking as a first section a nowhere vanishing incoming vector field on M along ∂M and as an independent section a positively normal vector field of M along ∂M (recall the fundamental hypothesis that M is *cooriented*).

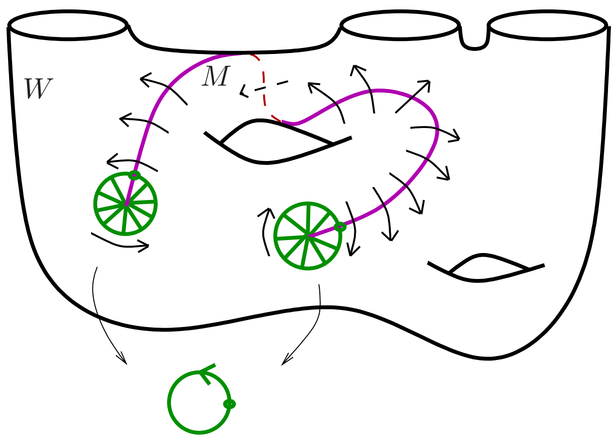

FIGURE 17. Angular coordinate of ∂M adapted to M

We want to pierce or blow-up W in an oriented way along ∂M. We will define this operation using the following local model to be used in each fiber of an adapted tubular neighborhood:

Definition 6.4. The **radial blow-up** of $\mathbb{D}^2$ is the map $\pi_0 : [0,1] \times \mathbb{S}^1 \to \mathbb{D}^2$ which expresses the cartesian coordinates on $\mathbb{D}^2$ in terms of polar ones:

$$(r, \theta) \mapsto (r \cos\theta, r \sin\theta).$$

One may perform the radial blow-up operation fiberwise in an adapted tubular neighborhood of a Seifert hypersurface:

Definition 6.5. Let W be a manifold-with-boundary and let $M \hookrightarrow W$ be a Seifert hypersurface. Let $\mathbb{D}^2 \times \partial M \hookrightarrow W$ be a tubular neighborhood of ∂M adapted to M. Let $\Pi_{\partial M}(W)$ be the manifold obtained as the union of $W \setminus \partial M$ and $[0,1] \times \mathbb{S}^1 \times \partial M$, where $(\mathbb{D}^2 \setminus \{0\}) \times \partial M$ in $W \setminus \partial M$ is identified with $(0,1] \times \mathbb{S}^1 \times \partial M$ in $[0,1] \times \mathbb{S}^1 \times \partial M$ through the diffeomorphism $\pi_0 \times \mathrm{id}_{\partial M}$. The **radial blow-up of W along ∂M** is the map:

$$\pi_{\partial M} : \Pi_{\partial M}(W) \to W$$

described as follows: $\pi_{\partial M}$ is just the the inclusion map on $W \setminus \partial M$ and is given by $\pi_0 \times \mathrm{id}_{\partial M}$ on $[0,1] \times \mathbb{S}^1 \times \partial M$. We also say that $\Pi_{\partial M}(W)$ is obtained by **piercing W along ∂M**. The **strict transform** M' of M by $\pi_{\partial M}$ is the closure of $(\pi_{\partial M})^{-1}(\mathrm{int}(M))$ inside $\Pi_{\partial M}(W)$.

The operation of radial blow-up is also called *oriented blow-up* in the literature, but under that name it is in general used in the semi-algebraic category. Intuitively, W is modified by replacing each point of ∂M by the circle of oriented lines passing through the origin of the normal plane to ∂M at that point. We have the following easy lemma:

Lemma 6.6. *The radial blow-up map $\pi_{\partial M}$ is proper and a diffeomorphism above $W \setminus M$. The restriction $\pi_{\partial M}|_{M'} : M' \to M$ is a diffeomorphism.*

In the sequel, we will identify M and M' using this diffeomorphism, which will allow us to speak about the embedding $M \hookrightarrow \Pi_{\partial M}(W)$. This embedding is cooriented (by the lift of the coorientation of M in W) and proper.

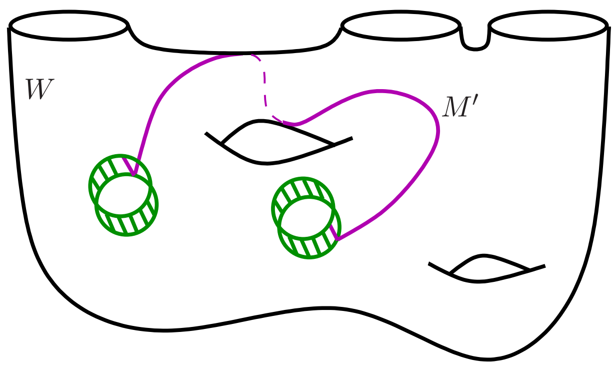

FIGURE 18. The radial blow-up of the surface W of Figure 17 along the end points of the arc M, and the strict transform M' of M. The two green cylinders are attached transversely to W (in the usual 3-dimensional space), where we first remove two disks from W.

Remark 6.7. (1) The radial blow-up allows us to pass from a Seifert hypersurface to a properly embedded cooriented hypersurface in the pierced manifold.
(2) This remark is to be compared with Remark 5.7 (3). One could define an analogous operation of radial blow-up along an arbitrary submanifold of codimension 2, as one does not need to have a globally trivial fibered tubular neighborhood in order to do fiberwise radial blow-ups of the centers of the discs. Nevertheless, we introduced this more restricted definition, as the only one which is needed in the paper.

As M is cooriented and properly embedded in $\Pi_{\partial M}(W)$, one may consider the splitting $\Sigma_M(\Pi_{\partial M}(W))$, as introduced in Definition 5.6:

Definition 6.8. Let W be a manifold-with-boundary and let $M \hookrightarrow W$ be a Seifert hypersurface. The **splitting of W along M**, denoted $\Sigma_M(W)$, is defined as the splitting $\Sigma_M(\Pi_{\partial M}(W))$ of the properly embedded hypersurface $M \hookrightarrow \Pi_{\partial M}(W)$. It is therefore an endobordism of M. The composition $\pi_{\partial M} \circ \sigma_M : \Sigma_M(W) \to W$ is called the **splitting map** of W along M.

Remark 6.9. This remark is a continuation of Remark 5.7 (2). Riemann explained that one has to cut a surface along a curve which goes from the boundary to the boundary. This is the operation we modeled in arbitrary dimensions by Definition 5.6. He added that if the surface has no boundary, then one has first to pierce it, creating like this an infinitely small boundary, and then one may cut it along a curve going from this boundary to itself. This is the operation we modeled in arbitrary dimensions in Definition 6.8. We

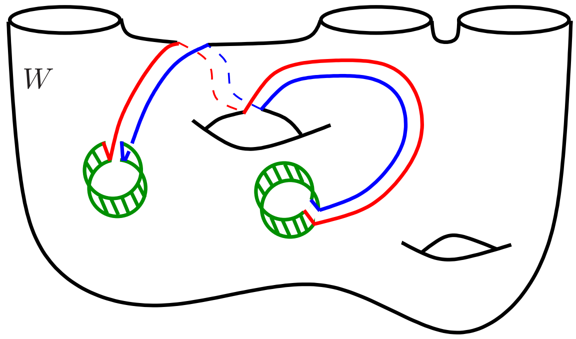

FIGURE 19. The splitting of W along M for the pair (W, M) of Figure 17. Recall that the intermediate radial blow-up is drawn in Figure 18.

first "pierced" W along the boundary of M (Definition 6.5), and then we were able to apply Definition 5.6.

One has the following immediate observation, consequence of the fact that one gets a segment by splitting a circle at a point. This observation is nevertheless very important for the sequel. Recall that both notions of cylindrical cobordisms and of their cylindrical boundaries were introduced in Definition 5.5:

Lemma 6.10. *If $M \hookrightarrow W$ is a Seifert hypersurface, then the splitting $\Sigma_M(W)$ is a cylindrical cobordism whose cylindrical boundary is given by:*

$$\sigma_M^{-1}(M \cup \pi_{\partial M}^{-1}(\partial M)).$$

Assume conversely that $W : M^- \Longmapsto M^+$ is a cylindrical cobordism with base M, its cylindrical boundary being identified with $\partial(I \times M)$. Fix an orientation-preserving identification of $\mathbb{S}^1$ with the circle obtained from I by gluing $\partial_- I$ and $\partial_+ I$. One identifies therefore to $\mathbb{S}^1 \times \partial M$ the image of the cylindrical boundary inside the mapping torus $T(W)$. This allows us to define:

Definition 6.11. Let $W : M^- \Longmapsto M^+$ be a cylindrical cobordism with base M. Its **circle-collapsed mapping torus** $T_c(W)$ is obtained from the mapping torus $T(W)$ by collapsing the circle $\mathbb{S}^1 \times \{m\}$ to $\{0\} \times \{m\}$, for all $m \in \partial M$. The **Seifert hypersurface associated to the cylindrical cobordism** W is the natural embedding $M \hookrightarrow T_c(W)$.

We have the following analog of Proposition 5.8:

Proposition 6.12. *The operations of taking the circle-collapsed mapping torus of a cylindrical cobordism and of splitting along a Seifert hypersurface are inverse to each other.*

This shows that, in differential-topological constructions, one may use interchangeably either cylindrical cobordisms or Seifert hypersurfaces.

One may describe the construction of the circle-collapsed mapping torus of a cylindrical cobordism in a slightly different way, by filling the boundary of the mapping torus with a product manifold, instead of collapsing the circles contained in it:

Lemma 6.13. *Let $W : M^- \Longmapsto M^+$ be a cylindrical cobordism with base M. The manifold obtained by gluing the mapping torus $T(W)$ to the product $\mathbb{D}^2 \times \partial M$ through the canonical identification of their boundaries is diffeomorphic to the circle-collapsed mapping torus $T_c(W)$ through a diffeomorphism which is the identity on the complement of an arbitrary neighborhood of $\mathbb{D}^2 \times \partial M$ and which sends $0 \times \partial M$ onto ∂M.*

We will use this second description in the proof of Proposition 9.1.

We apply now the previous considerations to the special situation where $M \hookrightarrow W$ is a page of an *open book*. Let us recall first this notion:

Definition 6.14. An **open book** in a closed manifold W is a pair (K, θ) consisting of:

(1) a codimension 2 submanifold $K \subset W$, called the **binding**, with a trivialized normal bundle;
(2) a fibration $\theta : W \setminus K \to \mathbb{S}^1$ which, in a tubular neighborhood $\mathbb{D}^2 \times K$ of K is the normal angular coordinate (that is, the composition of the first projection $\mathbb{D}^2 \times K \to \mathbb{D}^2$ with the angular coordinate $\mathbb{D}^2 \setminus \{0\} \to \mathbb{S}^1$).

It follows that for each $\theta_0 \in \mathbb{S}^1$, the closure in W of $\theta^{-1}(\theta_0)$—called a **page** of the open book—is a Seifert hypersurface whose boundary is the binding K. Its coorientation is defined by turning the pages in the positive sense of $\mathbb{S}^1$. If v is a vector field which is transverse to the pages, meridional near K and such that its vectors project to positive vectors on $\mathbb{S}^1$, then the first return map of v on an arbitrary page is called the **geometric monodromy** of the open book. As in the 3-dimensional case, such a geometric monodromy is well-defined up to isotopies relative to the boundary and conjugations by diffeomorphisms which are the identity on the boundary. No conjugation appears if the initial page is fixed.

One may describe the previous monodromical considerations in a slightly different way, using the splitting of the ambient manifold along a page (see Definition 6.8). Let $M \hookrightarrow W$ be an arbitrary page of the open book. The splitting of W along M is a cylindrical cobordism $\Sigma_M(W) : M \Longmapsto M$. Consider the same vector field as before. Its flow realizes a diffeomorphism from the incoming boundary (a copy of M) to the outgoing boundary (another copy of M). Therefore it gives a diffeomorphism of M, which is moreover fixed on the boundary of M. It is the geometric monodromy diffeomorphism!

This geometric monodromy is isotopic to the identity if and only if $\Sigma_M(W)$ is isomorphic to the cylinder $I \times M$ by an isomorphism which is the identity on the boundary and respects the fibrations over the interval I. Note that $\Sigma_M(W)$ is always isomorphic to that cylinder, if we do not impose constraints on the boundary.

Conversely, for any self-diffeomorphism ϕ of a compact manifold-with-boundary M which is the identity on ∂M, one can construct as follows a closed manifold equipped with an open book with page M and monodromy ϕ:

- take the cylinder $[0, 1] \times M$;

- consider it as a cylindrical cobordism $[0,1] \times M : M_0 \Longmapsto M_1$ where M_0, M_1 are two copies of M, that M_0 is identified to $\{0\} \times M$ using the identity of M and M_1 is identified to $\{1\} \times M$ using $\phi : M \simeq M_1$;
- take the circle-collapsed mapping torus associated to this cylindrical cobordism (see Definition 6.11). The fibers of the first projection $[0,1] \times M \to [0,1]$ induce the pages of an open book structure on it.

Remark 6.15. (1) The pair (M, ϕ) is sometimes called an **abstract open book**.

(2) The mapping torus of the previous cylindrical cobordism (according to Definition 5.4) coincides with the classical mapping torus $\mathcal{M}(M, \phi)$ of the diffeomorphism ϕ. This is the reason why we chose the name "mapping torus" for the object introduced in Definition 5.4.
(3) A codimension 2 closed submanifold $K \hookrightarrow V$ of a closed manifold is called a **fibered knot** if it is the binding of some open book (K, θ). In this case, the map θ is not part of the structure.
(4) One may consult [45] for a survey of the use of open books till 1998. Since then, Giroux's paper [15] started a new direction of applications of open books, into contact topology. The expression "open book" appeared for the first time in 1973 in the work of Winkelnkemper [44]. Before, equivalent notions of "fibered knots" and "spinnable structures" were introduced in 1972 by Durfee and Lawson [9] and Tamura [40] respectively. All those papers were partly inspired by Milnor's discovery in [31] of such structures — without using any name for them — associated to any germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ of polynomial with an isolated singularity at 0. In [3] was introduced the name "Milnor open book" for the open books associated more generally to holomorphic functions on germs of complex spaces, when both have isolated singularities.

7. Abstract and embedded summing

In this section we define a notion of *sum* of manifolds-with-boundary of the same dimension (see Definition 7.4), which generalizes the usual notion of *plumbing* recalled in Definition 2.1. The sum is done along identified *patches* and extends to a commutative and associative operation on *patched manifolds* with identified patches. Then we define an embedded version of this sum (see Definition 7.8). Unlike the abstract sum, this operation is in general non-commutative, but it is still associative (see Proposition 7.10). It generalizes both Stallings' and Lines' plumbing operations recalled in Section 2.

In the sequel, we will consider embedded submanifolds-with-boundary in other manifolds-with-boundary of the same dimension, where part of the boundary of the submanifold belongs to the interior, and part to the boundary of the ambient manifold. The next two definitions will allow us to speak shortly about such embeddings:

Definition 7.1. Let P be a compact manifold-with-boundary. An **attaching region** $A \hookrightarrow \partial P$ is a compact manifold-with-boundary of the same dimension as ∂P. The closure $B := \overline{\partial P \setminus A}$ of the complement of the attaching region is the **non-attaching region**. We say that (P, A) is an **attaching structure** on P. The **complementary attaching structure** of (P, A) is (P, B).

Definition 7.2. Let M be an n-dimensional compact manifold-with-boundary. A **patch** of M is the datum of an attaching structure (P, A) on another n-dimensional compact manifold-with-boundary and of an embedding $P \hookrightarrow M$ such that $P \cap \partial M = B$, where B is the non-attaching region of (P, A) (see Figure 20). That is, the attaching region A is the closure of $\partial P \cap \mathrm{int}(M)$. A manifold endowed with a patch is a **patched manifold**. We denote it either as a pair (M, P) or as an embedding $P \hookrightarrow M$.

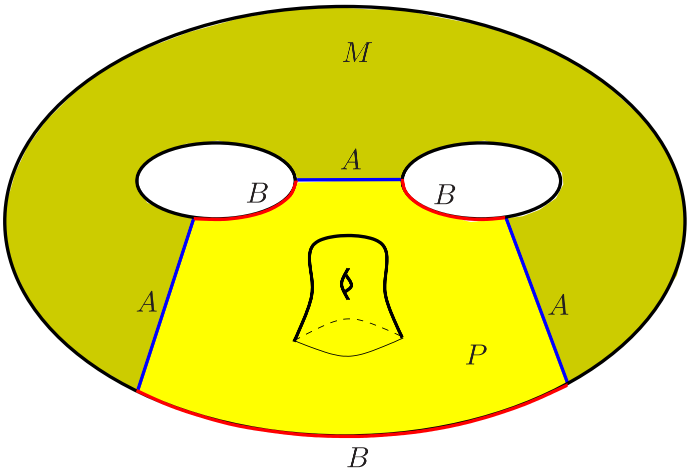

FIGURE 20. A patched manifold (M, P) with patch (P, A)

Remark 7.3.
(1) The condition $P \cap \partial M = B$ is equivalent to the condition that the attaching region A is the closure of $\partial P \cap \mathrm{int}(M)$. Therefore, the attaching region is determined by the embedding $P \hookrightarrow M$. We chose the name "*attaching region*" thinking to the fact that P is attached to $\overline{M \setminus P}$ along it.
(2) As represented in Figure 20, a patch (P, A) is best thought as a manifold with corners. When we speak about P as a manifold-with-boundary, we again use implicitly the fact, recalled at the beginning of Section 4, that the corners may be smoothed.

Now we are ready to give the main definition of this section, that of an operation of summing of two patched manifolds with identified patches:

Definition 7.4. Let M_1 and M_2 be two compact manifolds-with-boundary of the same dimension. Assume that a manifold P is a patch of both M_1 and M_2, with the corresponding attaching regions A_1 and A_2, such that $A_1 \cap A_2 = \emptyset$. Then we say that the two patched manifolds (M_1, P) and (M_2, P) are **summable**. The **(abstract) sum of** M_1 **and** M_2 **along** P, denoted by:

$$M_1 \overset{P}{\biguplus} M_2,$$

is the compact manifold-with-boundary obtained from the disjoint union $M_1 \bigsqcup M_2$ by gluing the points of both copies of P through the identity map. Its **associated patch** is the canonical embedding $P \hookrightarrow M_1 \overset{P}{\biguplus} M_2$, obtained by identifying the two given patches with attaching region $A_1 \cup A_2$ (see Figure 21).

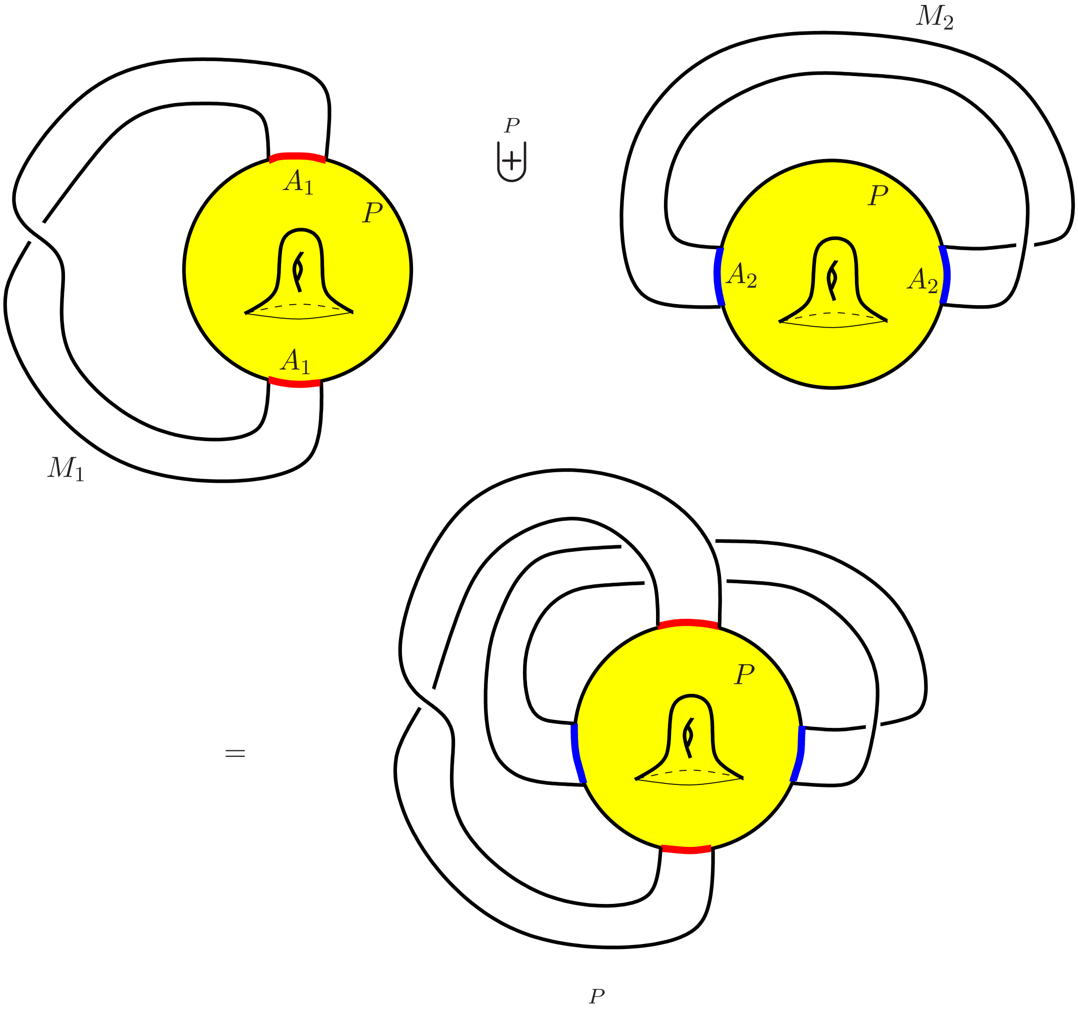

FIGURE 21. The abstract sum $M_1 \overset{P}{\biguplus} M_2$ of M_1 and M_2 along P

Note that Definition 7.4 respects our convention explained in Remark 4.1. It may be immediately extended to the case where the patches are distinct, and are identified by a given diffeomorphism, such that after the identification the attaching regions are disjoint.

Remark 7.5. (1) The attaching region of $P \hookrightarrow M_1 \overset{P}{\biguplus} M_2$ is the union of the attaching regions of $P \hookrightarrow M_1$ and $P \hookrightarrow M_2$.

(2) One may also present the construction of $M_1 \overset{P}{\biguplus} M_2$ in the following way (see Figure 22): glue $\overline{M_1 \setminus P}$ to M_2 by the canonical identification of $A_1 \hookrightarrow \partial(\overline{M_1 \setminus P})$ and $A_1 \hookrightarrow \partial M_2$. One has this last inclusion because the hypothesis $A_1 \cap A_2 = \emptyset$ implies that $A_1 \subset B_2 \subset \partial M_2$, where B_2 denotes the non-attaching region of (P, A_2). This second description shows that, indeed, the sum $M_1 \overset{P}{\biguplus} M_2$ has a structure of manifold-with-boundary. One has of course a symmetric description obtained by permuting the indices 1 and 2.

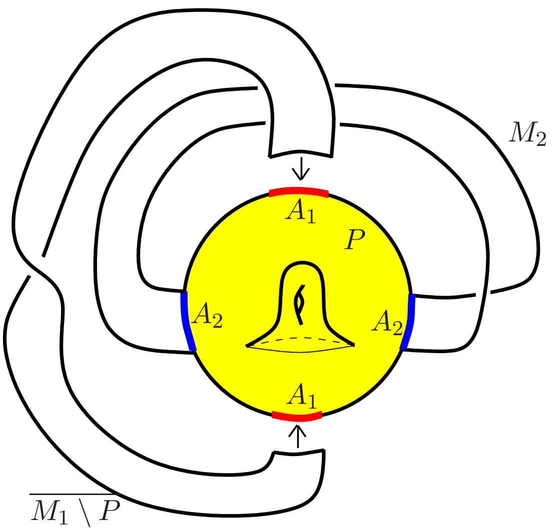

FIGURE 22. An alternative description of the abstract sum $M_1 \overset{P}{\biguplus} M_2$

(3) If $M_1 \overset{P}{\biguplus} M_2$ is viewed as described in the previous remark, one can see that a diffeomorphic manifold is obtained by allowing isotopies of A_1 inside the non-attaching region $B_2 := \overline{\partial P \setminus A_2}$. In other words, it is sufficient to require only that *the interiors of A_1 and A_2 are disjoint*. Note that, if $A_1 \cap A_2 \neq \emptyset$, then strictly speaking, P is not a patch inside $M_1 \overset{P}{\biguplus} M_2$. Nevertheless, in this case

one still gets a canonical realization of P as a patch, *up to isotopy*, in $M_1 \overset{P}{\uplus} M_2$, by isotoping A_1 inside itself so that the hypothesis $A_1 \cap A_2 = \emptyset$ is achieved. As explained in Section 4, the operations of gluing done here are defined up to smoothing of the corners.

(4) When the two patches used in the summing are the complementary patches $(\mathbb{D}^n \times \mathbb{D}^n, \mathbb{S}^{n-1} \times \mathbb{D}^n)$ and $(\mathbb{D}^n \times \mathbb{D}^n, \mathbb{D}^n \times \mathbb{S}^{n-1})$, one gets the classical notion of *plumbing* recalled in Definition 2.1. This is an example of a situation discussed in Remark 7.5 (3), in which only the interiors of the attaching regions are disjoint.

Remark 7.5 (3) shows that one may define the **abstract sum**:

$$\overset{P}{\biguplus_{i \in I}} M_i$$

whenever P appears as a patch of all the manifolds in a finite collection $(M_i)_{i \in I}$ of manifolds-with-boundary with *pairwise disjoint interiors of attaching regions*.

This sum is *commutative and associative* (up to unique isomorphisms), which motivates the absence of brackets in the notation. It is again endowed with a canonical patch $P \hookrightarrow \overset{P}{\biguplus_{i \in I}} M_i$ whenever the attaching regions themselves are pairwise disjoint. As explained in Remark 7.5 (3), if only the interiors of the initial patches are assumed to be disjoint, then there is still such a patch, but only well-defined up to isotopy.

We pass now to the definition of the *embedded sum*. Let us explain first which are the objects which may be summed in this way.

Definition 7.6. Let W be a compact manifold-with-boundary and $P \hookrightarrow M$ be a patched manifold. Assume that $M \hookrightarrow \text{int}(W)$ is an embedding of M as a hypersurface of $\text{int}(W)$. We say that the triple (W, M, P), also denoted $P \hookrightarrow M \hookrightarrow W$, is a **patch-cooriented triple** if:

- P is coorientable in W;
- a coorientation of P in W is chosen.

In the previous definition, M is not necessarily a Seifert hypersurface of W (see Definition 6.1). Indeed, we only assume that a coorientation was chosen *along* P. It is even possible that M is not coorientable inside W. To illustrate this, we depict in Figure 23 a cooriented quadrilateral patch P of a Möbius band $M \hookrightarrow W := \mathbb{S}^3$.

Recall that the notion of *positive side* for a cooriented hypersurface was explained in Definition 4.7. Let I^+ and I^- denote oriented compact intervals.

Definition 7.7. Let (W, M, P) be a patch-cooriented triple. A **positive thick patch** of (W, M, P) is a choice of positive side $I^+ \times P \hookrightarrow W$ of $P \hookrightarrow W$ intersecting M only along P. If for example $I^+ = [0, 1]$, then this means that $\{0\} \times P$ maps to P in M, and the

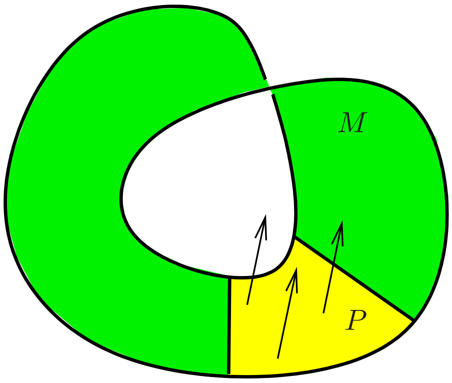

Figure 23. Cooriented quadrilateral patch P in a Möbius band M

positive tangents to I^+ point in the direction of co-orientation. Analogously, a **negative thick patch** of (W, M, P) is a choice of negative side $I^- \times P \hookrightarrow W$ of $P \hookrightarrow W$, also intersecting M only along P.

We may now describe a generalization of Stallings' (embedded) plumbing operation recalled in Section 2 (see the quotation containing equality (2.1)) and of Lines' higher dimensional plumbing operation (see Definition 2.5):

Definition 7.8. Let (W_1, M_1, P) and (W_2, M_2, P) be two patch-cooriented triples with identified patches, such that (M_1, P) and (M_2, P) are two summable patched manifolds (recall Definition 7.4). Then we say that the two triples are **summable** and their **(embedded) sum**, denoted by:

$$(W_1, M_1) \overset{P}{\uplus} (W_2, M_2),$$

is the compact manifold-with-boundary obtained by the following process (see Figure 24):

- choose a positive thick patch $I^+ \times P \hookrightarrow W_1$ of (W_1, M_1, P) and a negative thick patch $I^- \times P \hookrightarrow W_2$ of (W_2, M_2, P);
- consider the complements of their interiors $W_1' := W_1 \setminus \text{int}(I^+ \times P)$ and $W_2' := W_2 \setminus \text{int}(I^- \times P)$;
- glue W_1' to W_2' by identifying $\partial(I^+ \times P) \hookrightarrow W_1'$ to $\partial(I^- \times P) \hookrightarrow W_2'$ through the restriction of the map $\sigma \times \text{id}_P : I^+ \times P \to I^- \times P$. Here $\sigma : I^+ \to I^-$ denotes any diffeomorphism which reverses the orientations (that is, such that $\sigma(\partial_\pm I^+) = \partial_\mp I^-$).

It follows that:

$$\Big((W_1, M_1) \overset{P}{\uplus} (W_2, M_2), M_1 \overset{P}{\uplus} M_2, P\Big)$$

is a patch-cooriented triple through the canonical embeddings:

$$P \hookrightarrow M_1 \overset{P}{\uplus} M_2 \hookrightarrow (W_1, M_1) \overset{P}{\uplus} (W_2, M_2)$$

and the gluing of the coorientations of P in W_1 and in W_2.

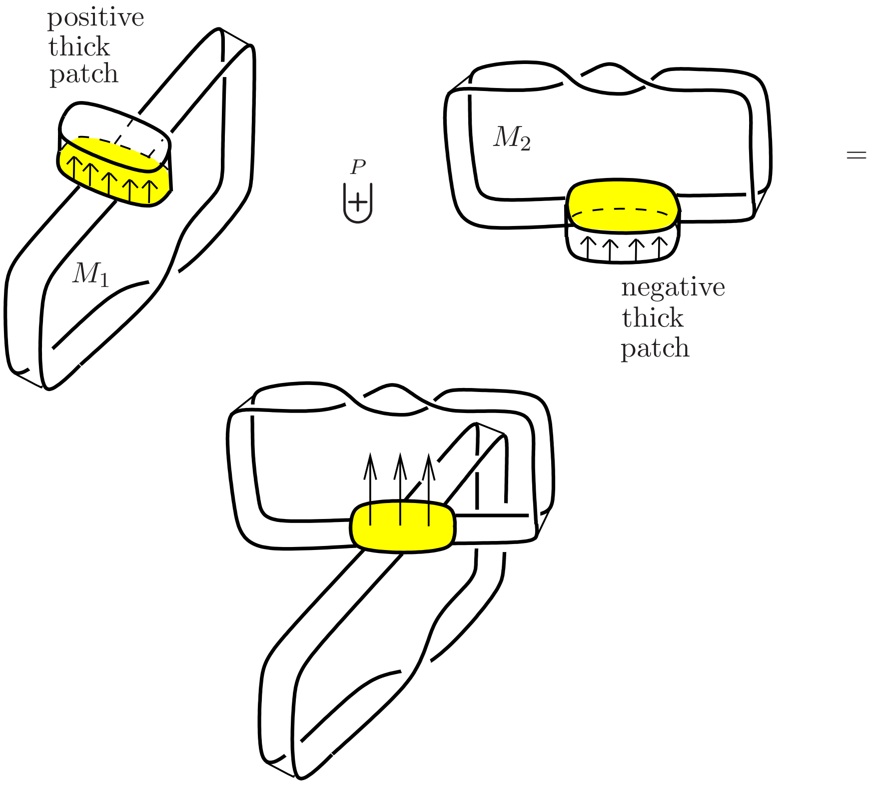

FIGURE 24. Embedded sum $(W_1, M_1) \overset{P}{\uplus} (W_2, M_2)$ of two patch-cooriented triples

Remark 7.9. (1) The manifold $(W_1, M_1) \overset{P}{\uplus} (W_2, M_2)$ has non-empty boundary if and only if either W_1 or W_2 has a non-empty boundary.

(2) The abstract sum $M_1 \overset{P}{\uplus} M_2$ is obtained inside $(W_1, M_1) \overset{P}{\uplus} (W_2, M_2)$ as the union of the images of $M_1 \hookrightarrow W_1'$ and of $M_2 \hookrightarrow W_2'$.

(3) We choose to take a positive thick patch for the first hypersurface and a negative one for the second hypersurface in order to respect Stallings' convention (see the citation containing formula (2.1)). If we choose the other way around, we get an alternative definition of the embedded sum of the triples (W_1, M_1, P), (W_2, M_2, P), which is diffeomorphic to $(W_2, M_2) \overset{P}{\uplus} (W_1, M_1)$ by a diffeomorphism which fixes $M_1 \overset{P}{\uplus} M_2$ and the coorientation of P. The operation of embedded sum being

in general non-commutative (see Proposition 7.10), this alternative definition is indeed different from Definition 7.8.

Proposition 7.10. *The patch being fixed, the operation of embedded sum of patch-cooriented triples is associative, but non-commutative in general.*

Proof. Let us prove first the *associativity* of the operation. Consider three summable patch-cooriented triples $(W_1, M_1, P), (W_2, M_2, P), (W_3, M_3, P)$, that is, assume that the attaching regions A_1, A_2, A_3 are pairwise disjoint. We want to prove that the two patch-cooriented triples:

$$\Big(\big((W_1, M_1) \overset{P}{\biguplus} (W_2, M_2)\big) \overset{P}{\biguplus} (W_3, M_3), M_1 \overset{P}{\biguplus} M_2 \overset{P}{\biguplus} M_3, P \Big),$$
$$\Big((W_1, M_1) \overset{P}{\biguplus} \big((W_2, M_2) \overset{P}{\biguplus} (W_3, M_3)\big), M_1 \overset{P}{\biguplus} M_2 \overset{P}{\biguplus} M_3, P \Big)$$

are isomorphic. But this is an immediate consequence of the fact (see Definition 7.8) that both may be obtained from the disjoint union $W_1 \sqcup W_2 \sqcup W_3$ by removing:

- the interior of a positive thick patch of (W_1, M_1, P);
- the interiors of a positive and of a negative thick patch of (W_2, M_2, P), which intersect only along P;
- the interior of a negative thick patch of (W_3, M_3, P);

and executing then the same gluings.

Let us show now that the operation is *non-commutative* in general. Consider the particular case where the triples to be summed are bands in 3-spheres, as in Figure 24, that is, M_1 and M_2 are either annuli or Möbius bands. Moreover, assume that the patches are disks disposed as in that figure, that is, such that one may choose core circles K_1, K_2 of the two bands such that they intersect transversally once inside P.

Denote by J_1 the arc of K_1 intercepted by P. Isotope K_1 inside both $(\mathbb{S}^3, M_1) \overset{P}{\biguplus} (\mathbb{S}^3, M_2)$ and $(\mathbb{S}^3, M_2) \overset{P}{\biguplus} (\mathbb{S}^3, M_1)$ by pushing the arc J_1 a little outside P *towards the positive side* of P, and keeping its complement in K_1 fixed. Denote by K_1^+ the new circle, contained either in $(\mathbb{S}^3, M_1) \overset{P}{\biguplus} (\mathbb{S}^3, M_2)$ or in $(\mathbb{S}^3, M_2) \overset{P}{\biguplus} (\mathbb{S}^3, M_1)$. Look then at the linking number (modulo 2) $\mathrm{lk}(K_1^+, K_2)$. It is equal to 1 in the first case and to 0 in the second case.

This shows that there is no isomorphism from $(\mathbb{S}^3, M_1) \overset{P}{\biguplus} (\mathbb{S}^3, M_2)$ to $(\mathbb{S}^3, M_2) \overset{P}{\biguplus} (\mathbb{S}^3, M_1)$ which is fixed on $M_1 \overset{P}{\biguplus} M_2$ and respects the coorientation of P. This is enough in order to deduce that the operation of embedded summing is in general non-commutative. □

In the next section we will consider carefully the special situation in which the hypersurfaces $M_i \hookrightarrow W_i$ are globally cooriented:

Definition 7.11. Let (W_1, M_1, P) and (W_2, M_2, P) be two patch-cooriented triples with identified patches. They are called **summable patched Seifert hypersurfaces** if both $M_1 \hookrightarrow W_1$ and $M_2 \hookrightarrow W_2$ are Seifert hypersurfaces whose coorientations extend those of the patches.

8. The sum of stiffened cylindrical cobordisms

In Section 7 we defined an operation of embedded sum for (summable) *patch-cooriented* triples without assuming that the hypersurfaces endowed with the (identified) patches are themselves cooriented or even coorientable. In this section we will assume this supplementary condition and we give an alternative definition of the (embedded) sum based on the equivalence of Seifert hypersurfaces and cylindrical cobordisms stated in Proposition 6.12. In the next section we will show that this alternative definition gives the same result as Definition 7.8. This alternative definition will make the proof of a generalization of Stallings' Theorem 2.2 very easy (see Theorem 9.3).

In the following definition we enrich the structure of cylindrical cobordism of Definition 5.5:

Definition 8.1. A **stiffened cylindrical cobordism** (see Figure 25) is a cylindrical cobordism $W : M^- \Longmapsto M^+$ and a neighborhood V (the **stiffening**) of $M^- \bigsqcup M^+$ in W, endowed with a diffeomorphism to a neighborhood of $(\partial I) \times M$ in $I \times M$ of the form:

$$(I \setminus \mathrm{int}(C)) \times M,$$

which extends the restriction to V of the given diffeomorphism $\partial_{cyl} W \simeq \partial(I \times M)$. Here $C \hookrightarrow \mathrm{int}(I)$ denotes a compact subsegment, called the **core** of the stiffened cobordism. The pull-back to $V \cup \partial_{cyl} W$ of the first projection $I \times M \to I$ is called the **height function** of the stiffened cylindrical cobordism.

Remark 8.2. (1) Given a cylindrical cobordism, stiffenings exist and are unique up to isotopy.
(2) Our choice of name is motivated by the fact that we see this supplementary structure as a way to rigidify or stiffen the initial cobordism.

Recall from Lemma 6.10 that one obtains cylindrical cobordisms by splitting any manifold along a Seifert hypersurface. Moreover, the two notions are equivalent, as shown by Proposition 6.12. From this viewpoint, stiffenings correspond to tubular neighborhoods of the Seifert hypersurface:

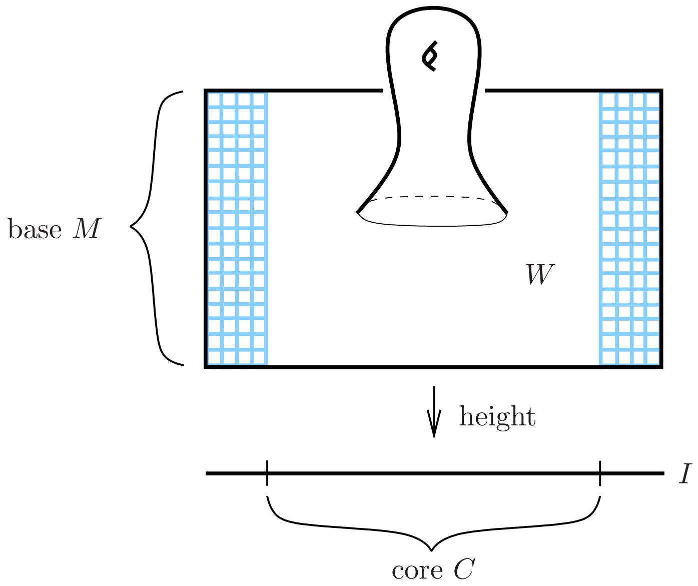

FIGURE 25. A stiffened cylindrical cobordism W with directing segment I

Lemma 8.3. *Let $M \hookrightarrow W$ be a Seifert hypersurface. Consider a collar neighborhood $[-\theta, \theta] \times M$ of the strict transform $M \hookrightarrow \Pi_{\partial M} W$ of M (see Definition 6.5), which intersects the boundary $\mathbb{S}^1 \times \partial M \hookrightarrow \Pi_{\partial M} W$ along $[-\theta, \theta] \times \partial M$. Here $\theta \in (0, \pi)$, therefore the segment $[-\theta, \theta]$ is seen as an arc of the circle $\mathbb{S}^1$. Then its image inside the splitting $\Sigma_M(W)$ is a stiffening of this cylindrical cobordism, with directing segment the splitting of $\mathbb{S}^1$ at the point of argument 0 and core segment $[\theta, 2\pi - \theta]$.*

A straightforward proof of this lemma easily follows by inspecting Figures 18 and 19.

In the following definition we extend to stiffened cylindrical cobordisms the notion of sum introduced for manifolds (see Definition 7.4) and for hypersurfaces (see Definition 7.8):

Definition 8.4. Consider two summable patched manifolds $(M_i, P)_{i=1,2}$, with attaching regions $(A_i)_{i=1,2}$. Let $(W_i : M_i^- \Longmapsto M_i^+, V_i)_{i=1,2}$ be two stiffened cylindrical cobordisms *with identified directing segment I*. They are called **summable** if their core intervals $(C_i)_{i=1,2}$ are disjoint and if C_1 is situated *after* C_2 with respect to the orientation of I. In this case, their **sum**, denoted by:

$$(W_1, V_1) \overset{P}{\biguplus} (W_2, V_2)$$

is obtained by performing the following operations using the stiffenings (see Figures 26 and 27):

- Over, $I \setminus (\mathrm{int}(C_1) \cup\ \mathrm{int}(C_2))$, sum fiberwise (M_1, P) to (M_2, P) (that is, one has to multiply the gluing map used to do this abstract sum by $\{t\}$, for any $t \in I \setminus (\mathrm{int}(C_1) \cup\ \mathrm{int}(C_2)))$.
- Over C_1, glue $C_1 \times \overline{M_2 \setminus P}$ to W_1 along $C_1 \times \partial M_1$ fiberwise (for each $t \in C_1$) by the canonical identification of $A_2 \hookrightarrow \partial(\overline{M_2 \setminus P})$ and $A_2 \hookrightarrow \partial M_1$.
- Over C_2, glue $C_2 \times \overline{M_1 \setminus P}$ to W_2 along $C_2 \times \partial M_2$ fiberwise (for each $t \in C_2$) by the canonical identification of $A_1 \hookrightarrow \partial(\overline{M_1 \setminus P})$ and $A_1 \hookrightarrow \partial M_2$.

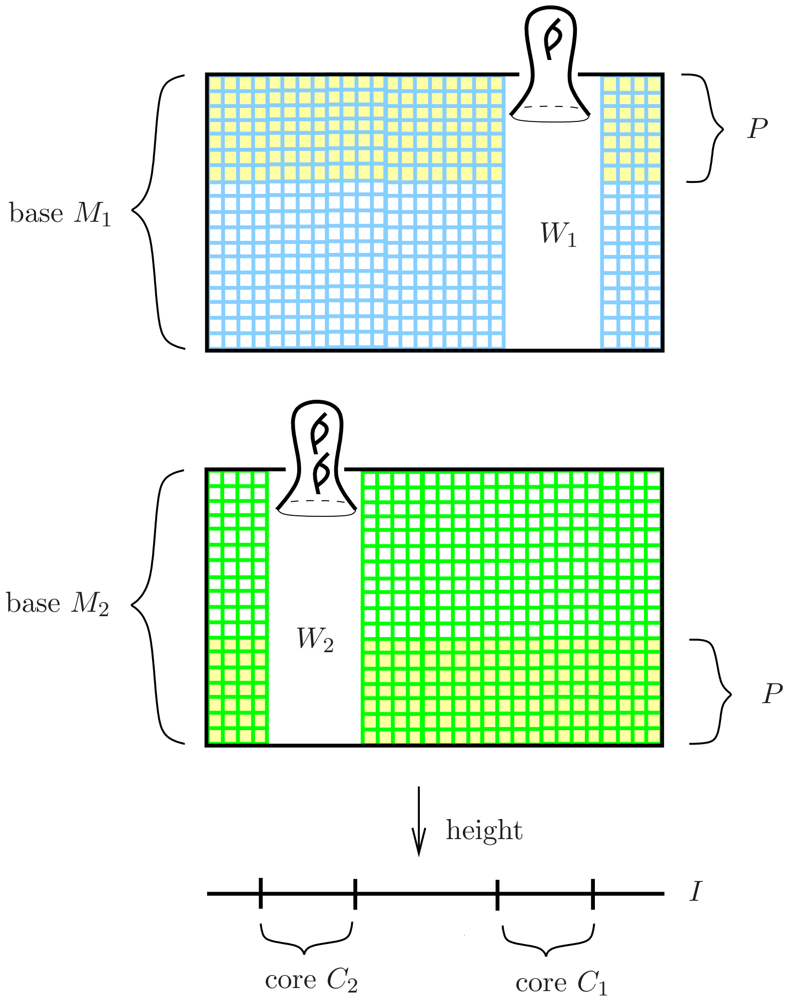

FIGURE 26. Two summable stiffened cylindrical cobordisms

Remark 8.5. (1) Let h_i denote the height function of the stiffened cylindrical cobordism W_i, for $i = 1, 2$. In Definition 8.4, we use the facts that for sufficiently small (and also sufficiently large) $t \in I$, the fiber $h_i^{-1}(t)$ is canonically identified with M_i

and that this identification extends to an identification of $h_i^{-1}(t) \cap \partial W_i$ with ∂M_i for all $t \in I$, by the definition of a stiffened cylindrical cobordism. All the gluings above fit together by Remark 7.5 (2).

(2) The sum $W_1 \overset{P}{\biguplus} W_2$ gets a natural structure of stiffened cylindrical cobordism with basis $M_1 \overset{P}{\biguplus} M_2$, directing segment I and core segment the convex hull inside I of the cores C_1 and C_2. The new stiffening is the image inside $W_1 \overset{P}{\biguplus} W_2$ of the union of the initial stiffenings, and the two initial height functions glue into the new height function.

Next, we extend the summing operation to cylindrical cobordisms whose directing segments are not identified, and which do not have fixed stiffenings. One has to make the following choices:

- Choose stiffenings. This choice is unique up to isotopy (see Remark 8.2 (1)).
- Identify their directing segments by an orientation-preserving diffeomorphism.

There are two ways to make such an identification, up to isotopy, in order to guarantee the disjointness of the cores, which is an essential hypothesis in Definition 8.4. Therefore, one gets an operation which is a priori non-commutative. The fact that it is indeed in general non-commutative results from the combination of propositions 7.10 and 9.1. More precisely, we use the fact, resulting from the proof of Proposition 7.10 using any kinds of bands, that the embedded summing operation is non-commutative even when the hypersurfaces are globally cooriented.

Definition 8.6. Consider two summable patched manifolds $(M_i, P)_{i=1,2}$, with attaching regions $(A_i)_{i=1,2}$. Let $(W_i : M_i^- \Longmapsto M_i^+)_{i=1,2}$ be two cylindrical cobordisms with directing segments $(I_i)_{i=1,2}$. Choose stiffenings for both of them. Let $\varphi : I_1 \to I_2$ be an orientation-preserving diffeomorphism which places the core segment of I_1 *after* the core segment of I_2. The **sum** of W_1 and W_2, denoted by:

$$W_1 \overset{P}{\biguplus} W_2$$

is obtained by applying Definition 8.4 after identifying the directing segments I_1 and I_2 using the diffeomorphism φ.

Remark 8.7. The diffeomorphism φ which places the core segment of I_1 *after* the core segment of I_2 being well-defined up to isotopy, as well as the stiffenings, we deduce that the sum is well-defined up to diffeomorphisms fixed on the cylindrical boundary of the cylindrical cobordism $W_1 \overset{P}{\biguplus} W_2$ (see Remark 8.5 (2)).

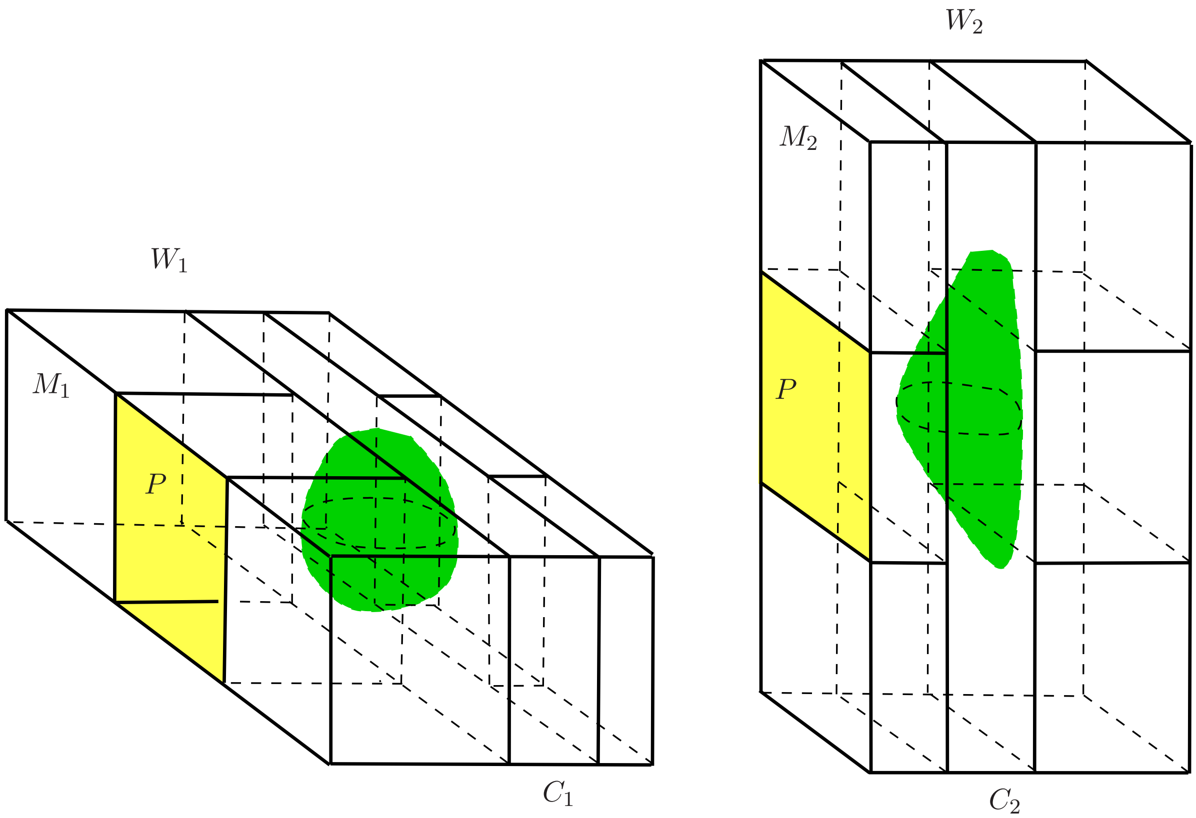

FIGURE 27. The stiffened cylindrical cobordism W_i (for $i = 1, 2$) is represented by the solid rectangular box where the solid green ball in the interior is removed. The sum $(W_1, V_1) \overset{P}{\biguplus} (W_2, V_2)$ will look like Figure 11, except that two disjoint solid balls have to be removed from the interior.

9. Embedded summing is a natural geometric operation

In this section we prove an extension of Stallings' Theorem 2.2 to arbitrary dimensions. Namely, we prove that the embedded sum of two pages of open books is again a page of an open book (see Theorem 9.3). We extend this result to pages of what we call *Morse open books* (see Theorem 9.7). A direct consequence of this theorem is a generalization to arbitrary dimensions of a theorem proved in dimension 3 by Goda. Both theorems illustrate Gabai's credo that "Murasugi sum is a natural geometric operation". Their proofs are parallel and are based on the fact that, in the case of Seifert hypersurfaces, the embedded sum as described in Definition 7.8 may be equivalently described using the operation of sum of cylindrical cobordisms described in Definition 8.6 (see Proposition 9.1). Technically speaking, this is the most difficult result of the paper.

The following proposition shows that in the case in which one works with summable patched Seifert hypersurfaces (see Definition 7.11), the previous notion of sum of cylindrical cobordisms gives the same result as the embedded sum of two patch-cooriented triples with identified patches:

Proposition 9.1. *Let (W_1, M_1, P) and (W_2, M_2, P) be two summable patched Seifert hypersurfaces. Then their embedded sum (see Definition 7.8):*

$$M_1 \overset{P}{\biguplus} M_2 \hookrightarrow (W_1, M_1) \overset{P}{\biguplus} (W_2, M_2)$$

is diffeomorphic, up to isotopy, to the Seifert hypersurface associated to the cylindrical cobordism (see definitions 6.11 and 8.4):

$$\Sigma_{M_1}(W_1) \overset{P}{\biguplus} \Sigma_{M_2}(W_2).$$

Proof. We start from the cylindrical cobordisms $\Sigma_{M_1}(W_1)$ and $\Sigma_{M_2}(W_2)$, to which we apply Definition 8.6. We want to show that the associated Seifert hypersurface is diffeomorphic to that obtained using Definition 7.8. In order to achieve this, we will show that the circle-collapsed mapping torus of $\Sigma_{M_1}(W_1) \overset{P}{\biguplus} \Sigma_{M_2}(W_2)$ may be obtained from the circle-collapsed mapping tori of the factors $\Sigma_{M_i}(W_i)$ by removing codimension 0 submanifolds which are diffeomorphic to $[0, 1] \times P$, and identifying the resulting boundaries appropriately.

The difficulty is that those submanifolds do not appear directly with the desired product structures, but as the unions of several codimension 0 submanifolds. It turns out that all of them are endowed with product structures and those structures are related in a way which allows us to achieve our aim.

Rather than working with the circle-collapsed mapping tori $T_c(\Sigma_{M_i}(W_i))$, we will use instead the manifolds obtained by filling the boundaries of the mapping tori $T(\Sigma_{M_i}(W_i))$ by the products $\mathbb{D}^2 \times \partial M_i$. As stated in Lemma 6.13, those are simply different models of the same Seifert hypersurfaces. Therefore, for $i = 1, 2$, we denote:

$$\Phi_{\partial M_i}(W_i) := \Pi_{\partial M_i}(W_i) \cup_{\mathbb{S}^1 \times \partial M_i} (\mathbb{D}^2 \times \partial M_i),$$

where $\Pi_{\partial M_i}(W_i)$ is the result of piercing W_i along ∂M_i (see Definition 6.5) and the two manifolds-with-boundary on the right-hand-side are glued through the canonical identifications of their boundaries with $\mathbb{S}^1 \times \partial M_i$. Similarly, we will fill by a product the boundary of $\Sigma_{M_1}(W_1) \overset{P}{\biguplus} \Sigma_{M_2}(W_2)$.

We choose stiffenings V_i of $\Sigma_{M_i}(W_i)$ and identifications of their directing segments that allow us to perform the sum as in Definition 8.4.

We may now apply the gluing operations described in the Definition 8.4 of the sum of stiffened cylindrical cobordisms with identified directing segments. Recall that over

$I \setminus (\mathrm{int}(C_1) \cup \mathrm{int}(C_2))$ those gluings may be described in several ways. The point here is to choose the description which is best adapted to our aim.

Denote $\alpha_\pm := \partial_\pm I$ and choose a point $\beta \in I$ which lies strictly between the two cores C_1 and C_2. Denote (see Figure 28):

$$I_1 := [\alpha_-, \beta], \quad I_2 := [\beta, \alpha_+].$$

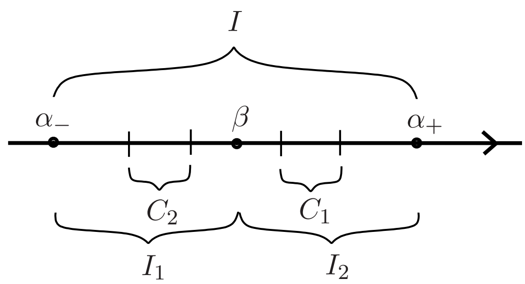

FIGURE 28. The interval I

We will do the gluings of Definition 8.4 by removing P fiberwise from $\Sigma_{M_i}(W_i)$ over I_i, for each $i \in \{1, 2\}$. But we interpret the gluing operations directly on the mapping torus of $\Sigma_{M_i}(W_i)$. A simple schematic representation of the operation of summing (M_1, P) and (M_2, P) is depicted abstractly in Figure 29, in order to help the reader following easily Figure 30. We denote by E_i the closure in ∂M_i of $\partial M_i \setminus B_i$, where B_i is the non-attaching region of (M_i, P) (see Definition 7.1), and by K the closure of $\partial P \setminus (A_1 \cup A_2)$.

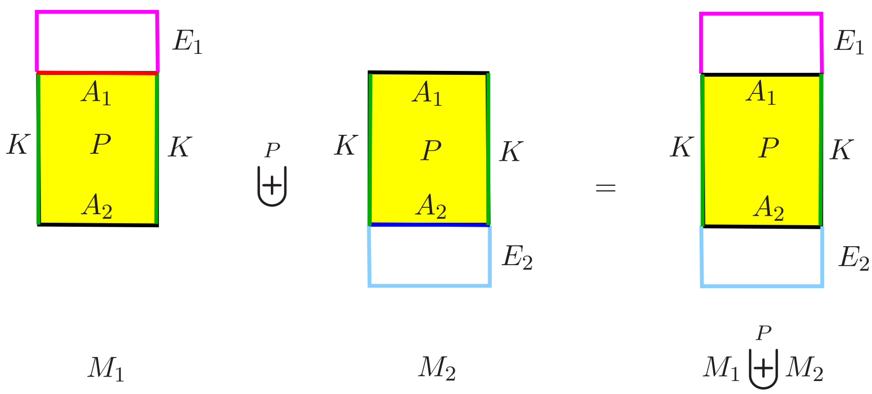

FIGURE 29. The schematic representation of E_i and K

The steps of the construction, interpreted using our filled models $\Phi_{\partial M_i}(W_i)$ of (W_i, M_i), are:

- For each $i \in \{1, 2\}$, remove $(I_i \times P) \cup (\mathbb{D}^2 \times \partial M_i)$ from $\Phi_{\partial M_i}(W_i)$, then take the closure.

- Glue through the canonical identification the portions of the resulting boundaries which are isomorphic to (see Figure 30):
$$(I_1 \times A_1) \cup (I_2 \times A_2) \cup (\alpha_\pm \times P) \cup (\beta \times P).$$
- Fill then the resulting boundary by:
$$\mathbb{D}^2 \times \partial(M_1 \overset{P}{\uplus} M_2) = (\mathbb{D}^2 \times E_1) \cup (\mathbb{D}^2 \times K) \cup (\mathbb{D}^2 \times E_2).$$

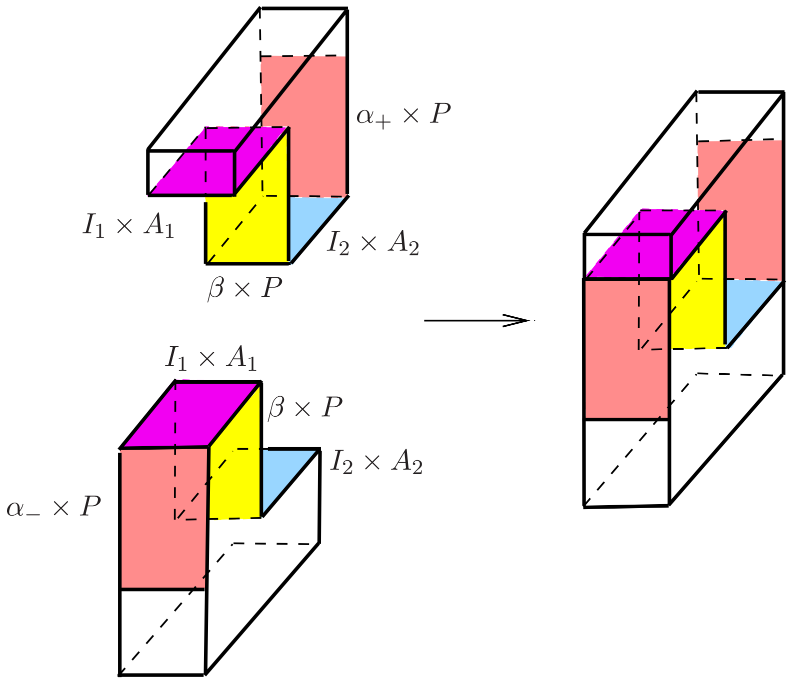

FIGURE 30. This Figure is to be compared with Figure 11

Note that the pieces $\mathbb{D}^2 \times E_i \hookrightarrow \mathbb{D}^2 \times \partial M_i$ are first removed, then inserted back into the same position (that is, we glue exactly as before to the adjacent pieces). Therefore, we obtain the same final result without touching them.

Instead, the piece $\mathbb{D}^2 \times K$ is removed *twice* and put back only *once*. One may obtain the same result by cutting the disc $\mathbb{D}^2$ into two half-discs $\mathbb{D}_1$ and $\mathbb{D}_2$, as represented in Figure 31, and only removing two conveniently chosen complementary half-discs. Namely, we will remove $\mathbb{D}_i \times K$ from $\Phi_{\partial M_i}(W_i)$.

The reinterpreted construction is:

- Remove $(I_1 \times P) \cup (\mathbb{D}^2 \times A_2) \cup (\mathbb{D}_1 \times K)$ from $\Phi_{\partial M_1}(W_1)$, then take the closure. Symmetrically, remove $(I_2 \times P) \cup (\mathbb{D}^2 \times A_1) \cup (\mathbb{D}_2 \times K)$ from $\Phi_{\partial M_2}(W_2)$, then take the closure.
- Glue the resulting boundaries through the canonical identification.

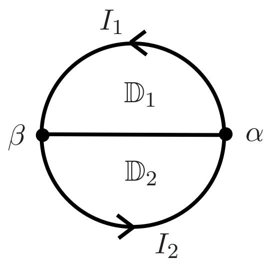

FIGURE 31. Splitting the unit disk

Notice now that $(I_1 \times P) \cup (\mathbb{D}^2 \times A_2) \cup (\mathbb{D}_1 \times K)$ is isomorphic to $I_1 \times P$, and similarly $(I_2 \times P) \cup (\mathbb{D}^2 \times A_1) \cup (\mathbb{D}_2 \times K)$ is isomorphic to $I_2 \times P$. Indeed, in each case we may apply Lemma 9.2 twice to end up with a description as in Definition 7.8. $\square$

We leave the proof of the following intuitively clear lemma to the reader:

Lemma 9.2. *Let Q be a manifold-with-boundary and $B \hookrightarrow \partial Q$ be a full-dimensional submanifold-with-boundary of the boundary. Then the result of gluing $[0,1] \times B$ to Q through the canonical identification of $0 \times B$ with B is isomorphic to Q through an isomorphism which is the identity outside an arbitrarily small neighborhood of B in Q.*

Here is our generalization of Stallings' Theorem 2.2 (recall that the notion of open book was explained in Definition 6.14):

Theorem 9.3. *Let $(W_i, M_i, P)_{i=1,2}$ be two summable patched Seifert hypersurfaces which are pages of open books on the closed manifolds W_i. Then the Seifert hypersurface associated to the sum $(W_1, M_1) \overset{P}{\biguplus} (W_2, M_2)$ is again a page of an open book. Moreover, the geometric monodromy of the resulting open book is the composition $\phi_1 \circ \phi_2$ of the monodromies of the initial open books. Here $\phi_i : M_i \to M_i$ is extended to $M_1 \overset{P}{\biguplus} M_2$ by the identity on $(M_1 \overset{P}{\biguplus} M_2) \setminus M_i$.*

Proof. Consider the splittings $\Sigma_{M_1}(W_1)$ and $\Sigma_{M_2}(W_2)$ of W_1, W_2 along the two pages. Let $(\partial M_i, \theta_i)$ be an open book on W_i such that $M_i = \theta_i^{-1}(0)$ (that is, such that M_i is the page of argument 0). The map $\theta_i : W_i \setminus \partial M_i \to \mathbb{S}^1$ lifts to an everywhere defined map $\tilde{\theta}_i : \Pi_{\partial M_i} W_i \to \mathbb{S}^1$ which is moreover a locally trivial fiber bundle projection. Therefore, it lifts to another fiber bundle projection:

$$\Sigma(\tilde{\theta}_i) : \Sigma_{M_i} W_i \to [0, 2\pi]$$

where the interval $[0, 2\pi]$ is obtained by splitting the circle $\mathbb{S}^1$ at the point of argument 0.

One may choose as stiffening of $\Sigma_{M_i} W_i$ a preimage $\Sigma(\tilde{\theta}_i)^{-1}([0, 2\pi] \setminus \mathrm{int}(C_i))$, where $C_i \subset (0, 2\pi)$ is an arbitrary compact segment with non-empty interior. Moreover, in order to get the hypothesis of Definition 8.4, we assume that C_2 and C_1 are disjoint and situated in this order on the segment $[0, 2\pi]$ endowed with its usual orientation. One may take as height functions the projections $\Sigma(\tilde{\theta}_i)$ themselves.

Definition 8.4 shows that the two height functions glue into a new globally defined height function:

$$h : \Sigma_{M_1}(W_1) \overset{P}{\biguplus} \Sigma_{M_2}(W_2) \to [0, 2\pi]$$

which is again a fiber bundle projection. Its generic fiber is isomorphic to $M_1 \overset{P}{\biguplus} M_2$. Therefore, the associated Seifert hypersurface is again an open book, with page isomorphic to $M_1 \overset{P}{\biguplus} M_2$.

But, by Proposition 9.1, this Seifert hypersurface is isomorphic to:

$$M_1 \overset{P}{\biguplus} M_2 \hookrightarrow (W_1, M_1) \overset{P}{\biguplus} (W_2, M_2).$$

The proof of the last statement in the theorem is similar to the proof in the 3-dimensional case (see Section 3). $\square$

Up to diffeomorphisms, all the choices of pages in an open book are equivalent. Therefore, the previous theorem allows to define a notion of *sum* (generalized Murasugi sum) for open books:

Definition 9.4. Assume that $(K_i, \theta_i)_{i=1,2}$ are open book structures on the closed manifolds W_i of the same dimension. Let M_i be pages of them, and P a common patch of M_1 and M_2. Assume that (M_1, P) and (M_2, P) are summable. The **sum** of the two open books is the open book on $(W_1, M_1) \overset{P}{\biguplus} (W_2, M_2)$ constructed in the previous proof.

The previous theorem may be extended to structures which are analogous to open books, in the sense that they have bindings and are similar to open books near them, but which are allowed to have Morse singularities away from the bindings:

Definition 9.5. A **Morse open book** in a closed manifold W is a pair (K, θ) consisting of:

(1) a codimension 2 submanifold $K \subset W$, called the **binding**, with a trivialized normal bundle;
(2) a map $\theta : W \setminus K \to \mathbb{S}^1$ which, in a tubular neighborhood $\mathbb{D}^2 \times K$ of K is the normal angular coordinate, and which has only Morse critical points. The closure of any fiber $\theta^{-1}(\theta_0)$ is a **page** of the Morse open book. A page is called **regular** if θ_0 is a regular value of θ and **singular** otherwise.

Remark 9.6. (1) The previous definition extends to arbitrary dimensions the notion of "*regular Morse map*" introduced in dimension 3 by Weber, Pajitnov and Rudolph in [43].

(2) The regular pages of Morse open books are Seifert hypersurfaces. Conversely, any Seifert hypersurface is a regular page of a Morse open book. Therefore, the problem of defining and finding the minimal complexity of such a Morse open book arises naturally, which motivates the rest of this section.
(3) All the pages of a classical open book are diffeomorphic, but this is certainly not true for a Morse open book which has a singular page. Even if one considers only the regular pages of a Morse open book, we may be sure that they are diffeomorphic only if they are preimages of points which belong to the same connected component of the complement of the critical image of θ inside $\mathbb{S}^1$.

One has the following extension to this setting of Theorem 9.3:

Theorem 9.7. *Let $(W_i, M_i, P)_{i=1,2}$ be two summable patched Seifert hypersurfaces which are regular pages of Morse open books on the closed manifolds W_i. Then the Seifert hypersurface associated to the sum $(W_1, M_1) \overset{P}{\biguplus} (W_2, M_2)$ is again a regular page of a Morse open book, whose multigerm of singularities is isomorphic to the disjoint union of the multigerms of singularities of the initial Morse open books.*

Proof. One may reason along the same lines as in the proof of Theorem 9.3. The difference is that one has to choose now the core intervals C_i such that $\text{int}(C_i)$ contains the critical values of the maps $\Sigma(\tilde{\theta}_i)$. One does not touch the neighborhoods of the critical points of the two Morse maps, which ensures that the new set of singularities are the disjoint unions of the two initial sets of singularities. □

Inspired by the *Morse-Novikov number* attached to a Seifert surface in [43, Section 6], we introduce the following invariants in order to measure how far a Seifert hypersurface is to being a page of an open book:

Definition 9.8. Let $M \hookrightarrow W$ be a Seifert hypersurface in the closed manifold W of dimension $w \geq 1$. For each $k \in \{1, ..., w-1\}$, denote by $m_k(W, M)$ be the minimal number of critical points of index k of a map $\theta : W \setminus \partial M \to \mathbb{S}^1$ such that $(\partial M, \theta)$ is a Morse open book, and M is a regular page. We call it the k-th **Morse number** of (W, M).

As an immediate consequence of Theorem 9.7, we have:

Proposition 9.9. *Let $(W_i, M_i, P)_{i=1,2}$ be two summable patched Seifert hypersurfaces in the closed manifolds $(W_i)_{i=1,2}$ of the same dimension $w \geq 1$. Then:*

$$m_k((W_1, M_1) \overset{P}{\biguplus} (W_2, M_2)) \leq m_k(W_1, M_1) + m_k(W_2, M_2)$$

for each $k \in \{1, ..., w-1\}$.

As explained in the introduction of [21], this theorem was proved in dimension 3 by Goda [18], under a different but equivalent formulation.

10. Questions related to contact topology and singularity theory

We conclude this paper with a list of questions. Almost all of them concern the sum of open books and its relations with singularity theory and contact topology. That is why we recall briefly the basics of those relations, developing part of the information given in Remark 6.15 (4).

Consider a germ of polynomial function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ which has an isolated singularity at the origin. Let $\mathbb{S}^{2n-1}(r) \hookrightarrow \mathbb{C}^n$ be the Euclidean sphere of radius $r > 0$ centered at the origin. The argument of f is well-defined outside the 0-level of f. Look at the restrictions of both objects to the sphere $\mathbb{S}^{2n-1}(r)$:

$$K := f^{-1}(0) \cap \mathbb{S}^{2n-1}(r), \qquad \theta : \mathbb{S}^{2n-1}(r) \setminus \ K \to \mathbb{S}^1.$$

Milnor proved in [31] that (K, θ) is an open book on $\mathbb{S}^{2n-1}(r)$, whenever r is sufficiently small. This result was extended by Hamm [20] to *holomorphic* functions f with isolated singularity, defined on any germ of complex analytic space $(X, 0)$ which is non-singular in the complement of the base point 0. In this case, one replaces $\mathbb{S}^{2n-1}(r)$ by the intersection $M(r)$ of X with a sphere of sufficiently small radius r, centered at 0, once $(X, 0)$ was embedded in some affine space $(\mathbb{C}^N, 0)$. For $r > 0$ small enough, one gets in this way open books (K, θ) on $M(r)$. In [3], such open books originating in singularity theory were called *Milnor open books*.

In 2002 Giroux [15] launched a program of study of contact topology through open books. Namely, he described a particularly adapted mutual position of a contact structure and an open book on any closed 3-dimensional manifold, saying that, in that case, *the open book supports the contact structure*. In fact, in 1975 Thurston and Winkelnkemper [41] proved that any open book supports a contact structure. Conversely, Giroux showed that any contact structure is supported by some open book. Moreover, he proved that two open books which support the same contact structure are stably equivalent, that is, one may arrive at the same open book by executing finite sequences of Murasugi sums with positive Hopf bands, starting from each one of the initial open books.

In the same paper, Giroux sketched an extension of this theory to higher dimensions. In particular, he defined higher dimensional analogs of supporting open books. In this case, if one wants to construct a contact structure starting from an open book, one has to enrich it with symplectic-topological structures. Namely, the pages are to be *Weinstein manifolds* (see the recent monograph [4] for a detailed exploration of this notion), and there should exist a geometric monodromy respecting in some sense the Weinstein structure.

In 2006, the paper [3] of Caubel, Némethi and the second author related the two instances where open books appear naturally: singularity theory and contact topology. Note that there are canonical contact structures on the manifolds $M(r)$, as they are level sets of a strictly plurisubharmonic function (the square of the distance to 0) on the complex manifold $(X \setminus 0)$. In [3], it was proved that the Milnor open book of any function $f : (X, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0 supports the canonical contact structure, whenever the radius r is sufficiently small. This generalized an analogous result proved before by Giroux [16], for the case where X is smooth and where instead of round spheres, deformed ones are chosen adapted to a given holomorphic germ f with isolated singularity.

Here are our questions:

(1) An open book is considered to be *trivial* if its page is a smooth ball and its geometric monodromy is the identity. We call an open book *indecomposable* if it cannot be written in a non-trivial way as a sum of open books (see Definition 9.4). Find sufficient criteria of indecomposability.

(2) Find sufficient criteria on germs of holomorphic functions $f : (X, 0) \to (\mathbb{C}, 0)$ with isolated singularity to define indecomposable open books.

(3) Find natural situations leading to triples $(X_i, f_i)_{1 \leq 1 \leq 3}$ of isolated singularities and holomorphic functions with isolated singularities on them, such that the Milnor open book of (X_3, f_3) is a sum of the Milnor open books of (X_1, f_1) and (X_2, f_2).

(4) Consider an open book and a contact structure supported by this open book on a closed manifold. Describe an adapted position of a patch inside a page, relative to the contact structure, allowing to extend the operation of sum of open books to a sum of open books which support contact structures. Also, prove an analog of the following result using appropriate patches in higher dimensions:

Theorem 10.1 (Torisu [42])**.** *Let ξ_i be the contact structure on a 3-manifold M_i supported by the open book (Σ_i, ϕ_i), for $i = 1, 2$. Then the connected sum $(M, \xi) = (M_1, \xi_1)\#(M_2, \xi_2)$ is supported by the open book (Σ, ϕ), where Σ is the Murasugi sum of Σ_1 and Σ_2 and $\phi = \phi_1 \circ \phi_2$.*

Let us point out that Giroux proved a particular instance of Theorem 10.1 for stabilizations of open books in higher dimensions.

(5) In analogy with Goda's results of [18], find lower bounds for the following difference of Morse numbers (see Definition 9.8):

$$m_k((W_1, M_1) \overset{P}{\biguplus} (W_2, M_2)) - (m_k(W_1, M_1) + m_k(W_2, M_2))$$

whenever (W_i, M_i) are Seifert hypersurfaces in closed manifolds of the same dimension.

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