FILLINGS OF UNIT COTANGENT BUNDLES OF NONORIENTABLE SURFACES

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ABSTRACT. We prove that any minimal weak symplectic filling of the canonical contact structure on the unit cotangent bundle of a nonorientable closed connected smooth surface other than the real projective plane is s-cobordant rel boundary to the disk cotangent bundle of the surface. If the nonorientable surface is the Klein bottle, then we show that the minimal weak symplectic filling is unique up to homeomorphism.

1. INTRODUCTION

Let M denote a closed and connected smooth surface which is not assumed to be orientable. The bundle of cooriented lines tangent to M has a projection onto M, which we denote by π. For a point q in M and a cooriented line u in T_qM, we denote by $\xi_{(q,u)}$ the cooriented plane described uniquely by the equation $\pi_*(\xi_{(q,u)}) = u \in T_qM$. The canonical contact structure ξ_{can} on the bundle of cooriented lines tangent to M consists of these planes (see, for example, [19]).

The bundle of cooriented lines tangent to M can be identified with the unit cotangent bundle ST^*M, once M is equipped with a Riemannian metric. Under this identification, the contact structure ξ_{can} is given by the kernel of the Liouville 1-form λ_{can}. It follows that the symplectic disk cotangent bundle $(DT^*M, d\lambda_{can})$ is a Weinstein and hence Stein filling of the contact 3-manifold (ST^*M, ξ_{can}) (cf. [3, Example 11.12 (2)]).

Let Σ_g denote the closed connected orientable smooth surface of genus $g \geq 0$. The unit cotangent bundle $ST^*\Sigma_0$ is diffeomorphic to the real projective space $\mathbb{RP}^3$, and ξ_{can} is the unique tight contact structure on $\mathbb{RP}^3$, up to isotopy (cf. [12]). McDuff [20] showed that any minimal symplectic filling of $(\mathbb{RP}^3, \xi_{can})$ is diffeomorphic to $DT^*\Sigma_0$.

2010 *Mathematics Subject Classification.* 57R17.

The first author was partially supported by grant no. 11471212 of the National Natural Science Foundation of China. The second author was partially supported by a BIDEP-2219 research grant of the Scientific and Technological Research Council of Turkey.

The unit cotangent bundle $ST^*\Sigma_1$ is diffeomorphic to the 3-torus T^3 and Eliashberg [5] showed that ξ_{can} is the unique strongly symplectically fillable contact structure on T^3, up to contactomorphism. Moreover, Stipsicz [26] proved that any Stein filling of (T^3, ξ_{can}) is homeomorphic to the disk cotangent bundle $DT^*\Sigma_1 \cong T^2 \times D^2$. This result was improved by Wendl [28], who showed that, in fact, any minimal strong symplectic filling of (T^3, ξ_{can}) is symplectic deformation equivalent to $DT^*\Sigma_1$ equipped with its canonical symplectic structure.

Recently, Sivek and Van Horn-Morris [25] proved that, for $g \geq 2$, any Stein filling of the contact 3-manifold $(ST^*\Sigma_g, \xi_{can})$ is s-cobordant rel boundary to the disk cotangent bundle $DT^*\Sigma_g$.

Moreover, Li, Mak and Yasui proved that, for $g \geq 2$, $(ST^*\Sigma_g, \xi_{can})$ admits minimal strong symplectic fillings with arbitrarily large b_2^+ (see [17, Proof of Corollary 1.6])) despite the fact that any exact filling of $(ST^*\Sigma_g, \xi_{can})$ has the same integral homology and intersection form as $DT^*\Sigma_g$ [17, Theorem 1.4].

In this paper, we study the topology of the symplectic fillings of the canonical contact structure ξ_{can} on the unit cotangent bundle of any *nonorientable* closed surface. A significant feature in the nonorientable surface case is that the unit cotangent bundle equipped with ξ_{can} is a planar contact 3-manifold, i.e., supported by a planar open book [22]. In contrast, for $g \geq 1$, $(ST^*\Sigma_g, \xi_{can})$ is not a planar contact 3-manifold (cf. [6]). Therefore, the topology of the symplectic fillings of ξ_{can} in the nonorientable surface case is greatly restricted by the results in [6], [21], [27], and [28].

According to a theorem of Niederkrüger and Wendl [21], any weak symplectic filling of a planar contact 3-manifold is symplectic deformation equivalent to a blow up of one of its Stein fillings. Most importantly, every Stein filling of a planar contact 3-manifold admits an allowable Lefschetz fibration over the disk that fills the planar open book [28].

Here we show that the canonical contact structure on the unit cotangent bundle of any nonorientable closed surface other than the real projective plane admits a unique minimal weak symplectic filling, up to s-cobordism rel boundary. More precisely, we prove the following.

Theorem 1.1. *Let N_k denote the nonorientable closed smooth surface obtained by the connected sum of $k \geq 1$ copies of the real projective plane $\mathbb{RP}^2$. Then, for $k \geq 2$, any minimal weak symplectic filling of the canonical contact structure ξ_{can} on the unit cotangent bundle ST^*N_k is s-cobordant rel boundary to the disk cotangent bundle DT^*N_k.*

Suppose that X_0 and X_1 are compact (topological) 4-manifolds such that ∂X_0 is homeomorphic to ∂X_1. Then X_0 and X_1 are said to be *s-cobordant rel boundary* (cf. [7, page 89]) if there exists a compact 5-manifold Z such that (i) X_0 and X_1 are disjoint submanifolds of ∂Z, (ii) $\partial Z \setminus int(X_0 \cup X_1)$ is homeomorphic to $\partial X_0 \times [0, 1]$ and (iii) for each $i = 0, 1$, the inclusion $X_i \to Z$ is a simple homotopy equivalence. The 5-manifold Z is called an s-cobordism between X_0 and X_1. It follows that if X_0 and X_1 are s-cobordant rel boundary, then they are simple homotopy equivalent, by definition. The reader can turn to [4] for more on simple homotopy equivalences. If $\pi_1(Z)$ is polycyclic for an s-cobordism Z, then according to [7, 7.1A Theorem], Z is homeomorphic to $X_0 \times [0, 1]$, and in particular X_0 is homeomorphic to X_1.

Since the fundamental group of the Klein bottle is polycyclic, we have the following corollary.

Corollary 1.2. *Any minimal weak symplectic filling of the canonical contact structure ξ_{can} on the unit cotangent bundle ST^*N_2 of the Klein bottle N_2 is homeomorphic to the disk cotangent bundle DT^*N_2.*

The unit cotangent bundle ST^*N_1 of the real projective plane $N_1 = \mathbb{RP}^2$ is diffeomorphic to the lens space $L(4, 1)$ and ξ_{can} is the unique universally tight contact structure on $L(4, 1)$, up to contactomorphism. McDuff [20] showed that $(L(4, 1), \xi_{can})$ has two minimal symplectic fillings up to diffeomorphism:

(i) The disk cotangent bundle DT^*N_1, which is a rational homology 4-ball, and

(ii) The disk bundle over the sphere with Euler number -4.

Both of these fillings are in fact Stein and clearly not homotopy equivalent (and hence not s-cobordant), since the latter is simply-connected while the former is not.

Notation. If α is a simple closed curve on an oriented surface Σ, we denote the positive (a.k.a. right-handed) Dehn twist along α by $D(\alpha)$. We use $Map(\Sigma, \partial\Sigma)$ for the surface mapping class group — the group of isotopy classes of orientation-preserving diffeomorphism of the surface Σ, where diffeomorphisms and isotopies are assumed to fix the boundary $\partial\Sigma$ pointwise. We use functional notation for the products in $Map(\Sigma, \partial\Sigma)$, i.e., $D(\beta)D(\alpha)$ means that we first apply $D(\alpha)$.

If (X, ω) is a symplectic 4-manifold and we are only interested in its diffeomorphism type, then we suppress ω from the notation. Similarly, for a Stein surface (W, J), we suppress the complex structure J from the notation if it is irrelevant for the discussion. The reader is advised to turn to [23] for the background material that we will use throughout the paper.

2. HOMOLOGY OF THE FILLINGS

Our goal in this section is to prove the following proposition.

Proposition 2.1. *Suppose that W is a minimal weak symplectic filling of (ST^*N_k, ξ_{can}). Then*

$$H_1(W;\mathbb{Z}) = H_1(DT^*N_k;\mathbb{Z}) = \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2,$$

and

$$H_2(W;\mathbb{Z}) = H_2(DT^*N_k;\mathbb{Z}) = 0$$

provided that $k \geq 2$.

Remark 2.2. Note that Proposition 2.1 does not hold for $k = 1$, since the contact 3-manifold $(ST^*N_1 = L(4,1), \xi_{can})$ has a *simply-connected* Stein filling, namely the disk bundle over the sphere with Euler number -4, whose second homology group is $\mathbb{Z}$.

We rely on the following results to prove Proposition 2.1.

Lemma 2.3 ([22])**.** *Let $V_0, V_1, \ldots, V_k, V_{k+1}$ be the simple closed curves shown in Figure 1 on the planar surface F_k with $2k+2$ boundary components, and let*

$$\phi_k := D(V_0)D(V_1)\cdots D(V_k)D(V_{k+1}) \in Map(F_k, \partial F_k).$$

Then, for all $k \geq 1$, the open book (F_k, ϕ_k) is adapted to (ST^*N_k, ξ_{can}).

Remark 2.4. For any $k \geq 1$, the disk cotangent bundle DT^*N_k is a Weinstein (and hence Stein) filling [3, Example 11.12 (2)] of its boundary (ST^*N_k, ξ_{can}). To see this with another point of view, one can directly check that the total space of the Lefschetz fibration over the disk whose boundary has the induced open book decomposition (F_k, ϕ_k) given in Lemma 2.3 is diffeomorphic to the disk cotangent bundle DT^*N_k (cf. [22, Appendix]).

Theorem 2.5 (Niederkrüger and Wendl [21])**.** *Every weak symplectic filling (X, ω) of a planar contact 3-manifold (Y, ξ) is symplectic deformation equivalent to a blow-up of a Stein filling of (Y, ξ).*

Note that the statement in Theorem 2.5 was proven for strong symplectic fillings in [28].

Theorem 2.6 (Wendl [28])**.** *Every Stein filling of a planar contact 3-manifold admits an allowable Lefschetz fibration over the disk that fills the planar open book.*

Remark 2.7. By combining Theorem 2.5 with Theorem 2.6, we conclude that any *minimal* weak symplectic filling of a planar contact 3-manifold admits an allowable Lefschetz fibration over the disk that fills the planar open book. Moreover, allowable Lefschetz fibrations over the disk filling an open book (not necessarily planar) is given by positive factorizations

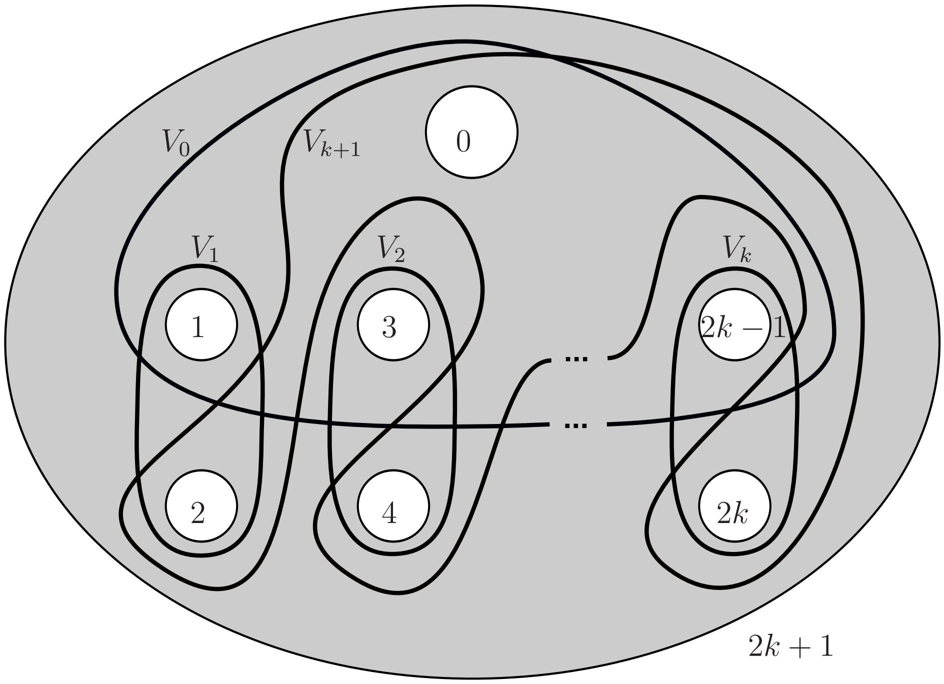

FIGURE 1. The curves $V_0, V_1, \ldots, V_k, V_{k+1}$ on the planar surface F_k. Each boundary component of F_k , denoted by c_j in the text, for $j = 0, 1, 2, \ldots, 2k+1$, is labeled by j in the figure.

of the monodromy of that open book into products of positive Dehn twists along homologically nontrivial curves (see, for example [1], [18]). Therefore, to classify minimal weak symplectic fillings of a planar contact 3-manifold, it suffices to study positive factorizations of the monodromy of the planar open book.

Lemma 2.8. *Assume that $k \geq 3$. Then any positive factorization of ϕ_k in Lemma 2.3 consists of Dehn twists $D(V_0'), D(V_1'), \ldots, D(V_{k+1}')$, where each curve V_j' encloses the same holes as V_j, for $j = 0, 1, \ldots, k+1$.*

Proof. In order to study positive factorizations of the given monodromy of a planar open book, we make use of a technique due to Plamenevskaya and Van Horn-Morris [24] (see also [14], [15]). Suppose that the monodromy ϕ_k in Lemma 2.3 has a factorization into positive Dehn twists along some curves. In the following we will refer to any such curve as a *monodromy* curve.

Recall that c_j's are the boundary components of the planar surface F_k depicted in Figure 1. For each $j \in \{0, 1, 2, \ldots, 2k\}$, the *multiplicity* $M_j(\phi_k)$ is defined as the number of monodromy curves enclosing c_j, and similarly, for each $i \neq j \in \{0, 1, 2, \ldots, 2k\}$, the *joint multiplicity* $M_{i,j}(\phi_k)$ is defined as the number of monodromy curves enclosing c_i and c_j, in any positive factorization of ϕ_k. The point is that these multipilicities are independent of the positive factorizations of ϕ_k (see [14], [15]).

It is easy to compute the multiplicities and joint multiplicities of ϕ_k using the positive factorization given in Lemma 2.3.

(1) For any $j \in \{0, 1, 2, \ldots, 2k\}$, $M_j = 2$.

(2) For any $j \in \{1, 2, \ldots, 2k\}$, $M_{0,j} = 1$.

(3) For any $h \in \{1, 3, \ldots, 2k-1\}$ and $l \in \{2, 4, \ldots, 2k\}$, $M_{h,l} = 0$ if $l \neq h+1$.

(4) For any $h \in \{1, 3, \ldots, 2k-1\}$, $M_{h,h+1} = 1$.

(5) For any $h_1, h_2 \in \{1, 3, \ldots, 2k-1\}$, $M_{h_1,h_2} = 1$.

(6) For any $l_1, l_2 \in \{2, 4, \ldots, 2k\}$, $M_{l_1,l_2} = 1$.

We first claim that there is no boundary parallel monodromy curve in any positive factorization of ϕ_k, a proof of which is spelled out in the next four paragraphs.

Suppose that there is a monodromy curve which is parallel to c_0. Then since $M_0 = 2$, by (1), there is another monodromy curve which encloses c_0 as well. Since, by (2), $M_{0,j} = 1$ for any $j \in \{1, 2, \ldots, 2k\}$, the second curve enclosing c_0 must enclose all c_j for $j \in \{1, 2, \ldots, 2k\}$. This contradicts to (3) by taking $h = 1$ and $l = 4$.

Let $i \in \{1, 3, \ldots, 2k-1\}$. Suppose that there is a monodromy curve which is parallel to c_i. Then there is another monodromy curve enclosing c_i, since by (1), $M_i = 2$. But the second curve enclosing c_i must enclose c_{i+1} by (4) and all c_j for $j \in \{0\} \cup \{1, 3, \ldots, 2k-1\}$, by (2) and (5). This contradicts to (3), by taking $l = i+1$ and $h \in \{1, 3, \ldots, 2k-1\} \setminus \{i\}$.

Let $i \in \{2, 4, \ldots, 2k\}$. Suppose that there is a monodromy curve which is parallel to c_i. Then there is another monodromy curve enclosing c_i, since by (1), $M_i = 2$. But the second curve enclosing c_i must enclose c_{i-1} by (4) and all c_j for $j \in \{0, 2, 4, \ldots, 2k\}$, by (2) and (6). This contradicts to (3), by taking $h = i-1$ and $l \in \{2, 4, \ldots, 2k\} \setminus \{i\}$.

Finally, we observe that by (3), there is no monodromy curve which is parallel to the outer boundary component c_{2k+1}. We only need to assume that $k \geq 2$ so far in the proof, since (3) is vacuous for $k = 1$ (see Remark 2.9 (1)).

Next we note that, by (4), for any $i \in \{1, 3, \ldots, 2k-1\}$, there must be a monodromy curve in the factorization enclosing c_i and c_{i+1}. We claim that this curve cannot enclose any other boundary components. This monodromy curve cannot enclose c_j for any $j \in \{1, 2, \ldots, 2k\} \setminus \{i, i+1\}$, since by (3), $M_{i,j} = 0$ for all $j \in \{2, 4, \ldots, 2k\} \setminus \{i+1\}$ and $M_{i+1,j} = 0$ for all $j \in \{1, 3, \ldots, 2k-1\} \setminus \{i\}$. Suppose that c_0 is enclosed by this monodromy curve. Then, by (1) and (2), there must be another monodromy curve enclosing c_0 and all c_j for $j \in \{1, 2, \ldots, 2k\} \setminus \{i, i+1\}$. This contradicts to (3), provided that $k \geq 3$. For each $i \in \{1, 3, \ldots, 2k-1\}$, we denote by V'_h, the monodromy curve enclosing only c_i and c_{i+1}, where $h = (i+1)/2$. Note that, for $k = 2$, there is another possibility of configuration of monodromy curves, as explained in Remark 2.9 (2) below.

By (1) and (2), there must be two more monodromy curves in the factorization, in addition to the ones described in the previous paragraph, both enclosing c_0. We describe these in the next two paragraphs.

One of these, which we denote by V'_0, must enclose c_0 and c_1 by (2). We claim that V'_0 encloses c_j if and only if $j \in \{0\} \cup \{1, 3, \ldots, 2k-1\}$. By (3), V'_0 cannot enclose c_l for $l \in \{4, 6, \ldots, 2k\}$. If we assume that it encloses c_2, then it cannot enclose c_j for any $j \in \{3, 5, \ldots, 2k-1\}$ by (3), and therefore, by (1) and (2), there is another monodromy curve enclosing c_0 and c_j for all $j \in \{3, 4, 5, \ldots, 2k-1, 2k\}$, which again contradicts to (3), provided that $k \geq 3$. Suppose now that V'_0 does not enclose c_h for some $h \in \{3, 5, \ldots, 2k-1\}$. Then there must be another monodromy curve enclosing c_0, c_h and c_l for all $l \in \{2, 4, \ldots, 2k\}$, by (1) and (2). This contradicts to (3) for $l \neq h+1$.

The last monodromy curve, which we denote by V'_{k+1}, must enclose c_0 and c_2. By an argument similar to the above paragraph, V'_{k+1} encloses c_j if and only if $j \in \{0, 2, 4, \ldots, 2k\}$, provided that $k \geq 3$. There cannot be any further additional monodromy curves, which is immediate from $(1), (3)-(6)$. □

Remark 2.9. (1) In order to rule out the existence of boundary parallel monodromy curves in the factorization, we used (3), which is vacuous for $k = 1$. As a matter of fact, ϕ_1 has two positive factorizations,

$$D(V_0)D(V_1)D(V_2) = D(c_0)D(c_1)D(c_2)D(c_3),$$

the latter consisting of only boundary parallel curves. This equality in the mapping class group is the well-known lantern relation.

(2) For $k = 2$, there is another possibility of configuration of monodromy curves which we can not rule out by the above argument. Namely, the four monodromy curves in the factorization may enclose the following set of boundary curves

$$\{c_0, c_1, c_2\}, \{c_0, c_3, c_4\}, \{c_1, c_3\}, \{c_2, c_4\},$$

respectively.

Proof of Proposition 2.1. We apply Remark 2.7 to the monodromy ϕ_k of the planar open book described in Lemma 2.3. Each positive factorization of ϕ_k yields a Lefschetz fibration over the disk whose boundary has the induced open book decomposition (F_k, ϕ_k). The regular fiber of this Lefschetz fibration is F_k and the vanishing cycles are exactly the monodromy curves in the given factorization. Therefore, one obtains a handlebody decomposition and the corresponding Kirby diagram of the total space of the fibration as follows.

First of all, a neighborhood of the regular fiber F_k is diffeomorphic to $F_k \times D^2$ which is obtained by attaching $2k+1$ 1-handles to the unique 0-handle. In the corresponding Kirby diagram, one can visualize the fiber F_k as depicted in Figure 1, whose orientation is induced from the standard orientation of $\mathbb{R}^2$. Note that for each $j \in \{0, 1, 2, \ldots, 2k\}$, the corresponding 1-handle can be conveniently depicted as a dotted circle passing through the hole labeled by j, and linking once the outer boundary component labeled by $2k+1$. In addition, a 2-handle with framing -1 is attached along each monodromy curve in the factorization, which can be visualized on the fiber F_k.

By Remark 2.7, any *minimal* weak symplectic filling W of (ST^*N_k, ξ_{can}) is diffeomorphic to a Lefschetz fibration over the disk which corresponds to some positive factorization of ϕ_k. Moreover, by Lemma 2.8 and Remark 2.9 (2), each positive factorization of ϕ_k has $k+2$ Dehn twists. Thus, the total space of the correponding Lefschetz fibration has a handle decomposition consisting of a 0-handle, $2k+1$ 1-handles and $k+2$ 2-handles.

Suppose $k \geq 3$. Then by Lemma 2.8, for any $j \in \{0, 1, 2, \ldots, k+1\}$, the monodromy curve V_j' which appears in a factorization of ϕ_k encloses the same holes as V_j on the planar surface F_k. It follows that the linking number of the circle V_j' with any dotted circle in the corresponding Kirby diagram, is the same as the linking number of V_j with that dotted circle. Hence we deduce (see, for example, [23, page 42]) that the homology groups of the total space of the Lefschetz fibration is independent of the positive factorization of ϕ_k. As a consequence, since we already know one positive factorization of ϕ_k which gives a Lefschetz fibration on the disk cotangent bundle DT^*N_k (see Remark 2.4), we conclude that

$$H_1(W;\mathbb{Z}) = H_1(DT^*N_k;\mathbb{Z}) = H_1(N_k;\mathbb{Z}) = \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2,$$

and

$$H_2(W;\mathbb{Z}) = H_2(DT^*N_k;\mathbb{Z}) = H_2(N_k;\mathbb{Z}) = 0.$$

For $k = 2$, there are two possible configurations of monodromy curves, the standard one as above and another one as described in Remark 2.9 (2). For the standard one, the proof above is valid. The second configuration is a relabelling of the boundaries, and hence induces the same Kirby diagram as the first one. Therefore, the homology groups are the same as in the standard one. □

3. HOMOTOPY TYPE OF THE FILLINGS

Our goal in this section is to prove Theorem 1.1 that we stated in the introduction. We begin with some basic observations.

The unit cotangent bundle ST^*N_k is a circle bundle over the nonorientable surface N_k whose Euler number is equal to $-\chi(N_k) = k-2$. The fundamental group of ST^*N_k is given (cf. [13, page 91]) as follows:

$$\pi_1(ST^*N_k) =< a_1, \dots, a_k, t \mid a_j t a_j^{-1} = t^{-1}, \prod_{j=1}^{k} a_j^2 = t^{k-2} > .$$

Here t represents the homotopy class of the circle fiber and it generates a cyclic normal subgroup of $\pi_1(ST^*N_k)$. After abelianization, we get

$$H_1(ST^*N_k;\mathbb{Z}) = \begin{cases} \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{if } k \text{ is even,} \\ \mathbb{Z}^{k-1} \oplus \mathbb{Z}_4, & \text{if } k \text{ is odd.} \end{cases}$$

Proposition 3.1. *Any Stein filling of* (ST^*N_k, ξ_{can}) *is aspherical, provided that* $k \geq 2$.

Proof. Suppose that W is a Stein filling of (ST^*N_k, ξ_{can}). Then, since W has a handlebody consisting of handles of index at most two, the inclusion map

$$i : ST^*N_k = \partial W \to W$$

induces a surjective homomorphism

$$i_* : \pi_1(ST^*N_k) \to \pi_1(W)$$

and hence we obtain the following commutative diagram,

$$\begin{array}{ccc} \pi_1(ST^*N_k) & \xrightarrow{\;i_*\;} & \pi_1(W) \\ \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f_2} \\ H_1(ST^*N_k;\mathbb{Z}) & \xrightarrow{\;i_*\;} & H_1(W;\mathbb{Z}) \end{array}$$

where f_1 and f_2 are Hurewicz maps.

Let $\alpha_j = i_*(a_j)$, for $j = 1, \dots, k$, and let $\tau = i_*(t)$. Then $f_2(\alpha_j)$, $j = 1, \dots, k$, and $f_2(\tau)$ generate $H_1(W;\mathbb{Z})$. If k is even, then both $\sum_{j=1}^{k} f_1(a_j)$ and $f_1(t)$ are torsion of order 2 in $H_1(ST^*N_k;\mathbb{Z})$. If k is odd, then $\sum_{j=1}^{k} f_1(a_j)$ is torsion of order 4 and $f_1(t) = 2\sum_{j=1}^{k} f_1(a_j)$ in $H_1(ST^*N_k;\mathbb{Z})$. We know that $H_1(W;\mathbb{Z}) = \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$, by Proposition 2.1. Thus

$\sum_{j=1}^{k} f_2(\alpha_j) \in H_1(W;\mathbb{Z})$ is either trivial or torsion of order 2 if k is even, and is torsion of order 2 if k is odd. Moreover, $f_2(\alpha_j)$ is non-torsion in $H_1(W;\mathbb{Z})$ for $j = 1, \ldots, k$. Therefore, we can define a surjective homomorphism $\psi : H_1(W;\mathbb{Z}) \to \mathbb{Z}_2$ such that $\psi(f_2(\alpha_j)) = 1$ for all $j = 1, \ldots, k$ and $\psi(f_2(\tau)) = 0$.

Now consider the double covers of ST^*N_k and W induced by the surjective homomorphisms

$$\psi \circ f_2 \circ i_* : \pi_1(ST^*N_k) \to \mathbb{Z}_2 \text{ and } \psi \circ f_2 : \pi_1(W) \to \mathbb{Z}_2,$$

respectively. Since each generator a_j, for $j = 1, \ldots, k$, is mapped to $1 \in \mathbb{Z}_2$, and t is mapped to $0 \in \mathbb{Z}_2$, the double cover of ST^*N_k is the unit cotangent bundle $ST^*\Sigma_{k-1}$, where Σ_{k-1} is the closed orientable surface which is the double cover of the nonorientable surface N_k.

By [8], the canonical contact structure on ST^*N_k lifts to the canonical contact structure on $ST^*\Sigma_{k-1}$ under this double cover. On the other hand, since any finite cover of a Stein surface is Stein (cf. [10, page 436]), the corresponding double cover W' of W is a Stein filling of $(ST^*\Sigma_{k-1}, \xi_{can})$. By [25, Proposition 4.9], W' is aspherical, i.e., $\pi_n(W') = 0$ for all $n \geq 2$. Hence W is also aspherical, since it is well-known (see, for example, [11, Proposition 4.1]) that higher homotopy groups are preserved under coverings. □

Next we compute the fundamental group of the fillings.

Proposition 3.2. *If W is a Stein filling of (ST^*N_k, ξ_{can}), then $\pi_1(W) = \pi_1(N_k)$, provided that $k \geq 2$.*

Proof. Suppose that W is a Stein filling of (ST^*N_k, ξ_{can}). Since $\langle t \rangle$ is a normal subgroup in $\pi_1(ST^*N_k)$, its image $\langle \tau \rangle$ under the surjective homomorphism $i_* : \pi_1(ST^*N_k) \to \pi_1(W)$ is a normal subgroup of $\pi_1(W)$. Note that i_* induces a surjective homomorphism

$$p : \pi_1(N_k) \to \pi_1(W)/\langle \tau \rangle.$$

Let W' be the double cover of W as in the proof of Proposition 3.1. Then the surjective homomorphism $i_* : \pi_1(ST^*\Sigma_{k-1}) \to \pi_1(W')$ induces a surjective homomorphism

$$p' : \pi_1(\Sigma_{k-1}) \to \pi_1(W')/\langle \tau' \rangle,$$

where $\tau' = i_*(t')$, and t' represents the homotopy class of the circle fiber of $ST^*\Sigma_{k-1}$.

Using the same argument as in the proof of [25, Proposition 4.7], we have $\ker(p) = \ker(p')$, where $\pi_1(\Sigma_{k-1})$ is identified as a subgroup of $\pi_1(N_k)$. Now, since W' is a Stein filling of $(ST^*\Sigma_{k-1}, \xi_{can})$, we conclude that the surjective homomorphism p' above is also injective by [25, Proposition 4.8]. Hence we get $\ker(p) = \ker(p') = 0$. As a consequence, we have a short exact sequence

$$1 \to \langle\tau\rangle \to \pi_1(W) \to \pi_1(N_k) \to 1.$$

Since $\langle\tau\rangle$ is a cyclic subgroup of $\pi_1(W)$, isomorphic to $\mathbb{Z}_m$ for some non-negative integer m, where $\mathbb{Z}_0 = \mathbb{Z}$, it follows that the short exact sequence above can be expressed as

$$1 \to \mathbb{Z}_m \to \pi_1(W) \to \pi_1(N_k) \to 1.$$

Our goal is to show that $m = 1$. In order to achieve our goal, we first observe that $H_2(\pi_1(W);\mathbb{Z}) = H_2(W;\mathbb{Z})$, since W is aspherical by Proposition 3.1. But since we have $H_2(W;\mathbb{Z}) = 0$ by Proposition 2.1, we conclude that $H_2(\pi_1(W);\mathbb{Z}) = 0$.

Next we use the Lyndon/Hochschild-Serre spectral sequence (see, for example, [2])

$$E^2_{p,q} = H_p(\pi_1(N_k); H_q(\mathbb{Z}_m;\mathbb{Z})) \Longrightarrow H_{p+q}(\pi_1(W);\mathbb{Z})$$

to compute $H_2(\pi_1(W);\mathbb{Z})$, which a priori may depend on m.

Based on the facts that $H_0(\mathbb{Z}_m;\mathbb{Z}) = \mathbb{Z}$, $H_1(\mathbb{Z}_m;\mathbb{Z}) = \mathbb{Z}_m$, $H_2(\mathbb{Z}_m;\mathbb{Z}) = 0$, for all $m \geq 0$, and that the nonorientable surface N_k is aspherical for $k \geq 2$, we obtain

$$E^2_{0,2} = H_0(\pi_1(N_k); H_2(\mathbb{Z}_m;\mathbb{Z})) = H_0(N_k; 0) = 0,$$

$$E^2_{2,0} = H_2(\pi_1(N_k); H_0(\mathbb{Z}_m;\mathbb{Z})) = H_2(N_k;\mathbb{Z}) = 0,$$

and

$$E^2_{1,1} = H_1(\pi_1(N_k); H_1(\mathbb{Z}_m;\mathbb{Z})) = H_1(N_k;\mathbb{Z}_m) = \begin{cases} (\mathbb{Z}_m)^{k-1} \oplus \mathbb{Z}_2, & m \text{ is even,} \\ (\mathbb{Z}_m)^{k-1}, & m \text{ is odd.} \end{cases}$$

Since the cohomological dimension of the surface group $\pi_1(N_k)$ is equal to 2 for $k > 1$ (cf. [2, Page 185]), E^2 page is supported by $p \in \{0, 1, 2\}$. It follows that

$$E^\infty_{0,2} = E^2_{0,2},\ E^\infty_{1,1} = E^2_{1,1},\ \text{and } E^\infty_{2,0} = E^2_{2,0}.$$

Hence we have

$$H_2(\pi_1(W);\mathbb{Z}) = E^\infty_{0,2} \oplus E^\infty_{1,1} \oplus E^\infty_{2,0} = \begin{cases} (\mathbb{Z}_m)^{k-1} \oplus \mathbb{Z}_2, & m \text{ is even,} \\ (\mathbb{Z}_m)^{k-1}, & m \text{ is odd.} \end{cases}$$

Therefore $m = 1$, since $H_2(\pi_1(W);\mathbb{Z}) = 0$, and thus $\pi_1(W) = \pi_1(N_k)$. □

We are now ready to prove our main result which is Theorem 1.1.

Proof of Theorem 1.1. Suppose that $k \geq 2$ and W is a Stein filling of the contact 3-manifold (ST^*N_k, ξ_{can}). We will show that W is s-cobordant rel boundary to the disk cotangent bundle DT^*N_k, using the same argument as in the proof of [25, Theorem 4.10]. For the sake of completeness, we outline the argument here. We first observe that W is aspherical by Proposition 3.1, and $\pi_1(W) = \pi_1(N_k)$, a surface group, by Proposition 3.2. According to [16, Corollary 1.23], such a compact manifold W is topologically s-rigid. This condition implies that it suffices to find a homotopy equivalence $\rho : DT^*N_k \to W$ which restricts to a homeomorphism $ST^*N_k \to \partial W$ in order to prove that DT^*N_k is s-cobordant to W.

Now consider the standard handlebody decomposition of DT^*N_k consisting of a 0-handle, k 1-handles and a 2-handle. By turning it upside down, we can construct DT^*N_k by attaching a 2-handle, k 3-handles and a 4-handle to a thickened ST^*N_k. Next we define a homeomorphism $\rho : ST^*N_k \to \partial W$ sending a circle fiber to a circle fiber. Note that the attaching curve of the (upside down) 2-handle is a circle fiber in ∂W which is nullhomotopic in W. Therefore, ρ extends over the 2-handle of DT^*N_k. Moreover, ρ extends over the handles of index greater than 2, since W is aspherical. Furthermore, the map induced by ρ takes the normal subgroup of $\pi_1(ST^*N_k)$ generated by the circle fiber to the normal subgroup of $\pi_1(\partial W)$ generated by the circle fiber, and hence descends to an isomorphism between the quotients $\pi_1(DT^*N_k)$ and $\pi_1(W)$. It follows that $\rho : DT^*N_k \to W$ is a homotopy equivalence by Whitehead's Theroem.

To finish the proof, we simply observe that any *minimal* weak symplectic filling of the planar contact 3-manifold (ST^*N_k, ξ_{can}) is deformation equivalent to a Stein filling by Theorem 2.5. □

Acknowledgement: This work was mainly carried out at the Department of Mathematics at UCLA. The authors would like to express their gratitude to Ko Honda for his hospitality.

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