ON THE HEEGAARD GENUS OF CONTACT 3-MANIFOLDS

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ABSTRACT. It is well-known that Heegaard genus is additive under connected sum of 3-manifolds. We show that Heegaard genus of contact 3-manifolds is not necessarily additive under *contact* connected sum. We also prove some basic properties of the contact genus (a.k.a. open book genus [8]) of 3-manifolds, and compute this invariant for some 3-manifolds.

1. INTRODUCTION

We assume that all 3-manifolds are closed, connected and oriented and all contact structures are co-oriented and positive throughout this paper. Let (B, π) denote an open book on a 3-manifold Y, where B is the binding, and π is the fibration of $Y \setminus B$ over S^1. It follows that $(\pi^{-1}([0, 1/2]) \cup B)$ and $(\pi^{-1}([1/2, 1]) \cup B)$ are handlebodies which induce a Heegaard splitting of Y. Therefore an open book can be viewed as a special Heegaard splitting. Note that a stabilization of an open book at hand corresponds to a stabilization of the induced Heegaard splitting.

We define the Heegaard genus $\mathrm{Hg}(Y, \xi)$ of a contact 3-manifold (Y, ξ) as the minimal genus of a Heegaard surface in any Heegaard splitting of Y induced from an open book supporting ξ. Equivalently, $\mathrm{Hg}(Y, \xi) = 1 + \mathrm{sn}(\xi) = \min\{1 - \chi(\Sigma) \mid \Sigma$ is a page of an open book supporting $\xi\}$, where $\mathrm{sn}(\xi)$ denotes the support norm of ξ (cf. [4]) and $\chi(\Sigma)$ denotes the Euler characteristic of Σ. This is certainly an adaptation of the usual definition of Heegaard genus to contact 3-manifolds. It is well-known that Heegaard genus is additive under connected sum of 3-manifolds. Here we show that Heegaard genus is sub-additive but not necessarily additive under connected sum of *contact* 3-manifolds.

Moreover we define the contact genus $\mathrm{cg}(Y)$ of a 3-manifold Y as the minimal Heegaard genus over all contact structures, i.e., $\mathrm{cg}(Y) = \min\{\mathrm{Hg}(Y, \xi) \mid \xi$ is a contact structure on $Y\}$ which, by Giroux's correspondence [5], is the minimal genus of a Heegaard surface in any Heegaard splitting of Y induced from an open book. In other words, the contact genus of a 3-manifold is a topological invariant obtained by taking the minimum of the sum $2g + r - 1$ over all open books, where g and r denote the genus of the page and the number of binding components of the open book, respectively. We show that contact genus is sub-additive (and conjecture that it is additive) under connected sum of 3-manifolds.

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We would like to point out that the contact invariant was first studied by Rubinstein who named it the open book genus of Y (cf. [8]). We prefer to call it the contact genus to emphasize its connection with contact topology. It is clear by definition that for any contact structure ξ on Y we have

$$\mathrm{hg}(Y) \leq \mathrm{cg}(Y) \leq \mathrm{Hg}(Y, \xi),$$

where $\mathrm{hg}(Y)$ denotes the usual Heegaard genus of Y. In [8], it was shown that "most" 3-manifolds of Heegaard genus 2 have contact genus > 2, which implies the existence of 3-manifolds where the first inequality above is strict. In particular, it follows that not every Heegaard splitting of a 3-manifold comes from an open book.

Here we show that "most" 3-manifolds of Heegaard genus 1 have contact genus > 1. Namely we show that a lens space which is not diffeomorphic to an oriented circle bundle over S^2 have contact genus ≥ 2. On the other hand, the contact genus of any oriented circle bundle over S^2 is equal to its Heegaard genus. We also show that there are many small Seifert fibered 3-manifolds (which are not lens spaces) which have this property. Examples of such 3-manifolds were constructed in [8], but our examples are much simpler. We refer the reader to [3] and [7] for more on open books and contact structures.

2. Heegaard genus and contact connected sum

Let (Y_1, ξ_1) and (Y_2, ξ_2) denote arbitrary contact 3-manifolds. By removing a Darboux ball from each of these contact 3-manifolds and gluing them along their convex boundaries by an orientation reversing map carrying respective characteristic foliations onto each other we get a well defined contact structure $\xi_1 \# \xi_2$ on the connected sum $Y_1 \# Y_2$. The contact 3-manifold $(Y_1 \# Y_2, \xi_1 \# \xi_2)$ is called the contact connected sum of (Y_1, ξ_1) and (Y_2, ξ_2). It is well-known that Heegaard genus is additive under connected sum of smooth 3-manifolds, which follows from Haken's Lemma. Here we show that

Theorem 1. *The Heegaard genus is sub-additive but not necessarily additive under connected sum of contact* 3*-manifolds.*

Proof. Let $\mathcal{OB}_i$ denote the open book realizing $\mathrm{Hg}(Y_i, \xi_i)$, for $i = 1, 2$. Then the contact structure $\xi_1 \# \xi_2$ on $Y_1 \# Y_2$ is supported by the open book $\mathcal{OB}$ obtained by plumbing the pages of the open books $\mathcal{OB}_1$ and $\mathcal{OB}_2$ by Torisu [9]. Denote a page of the open book $\mathcal{OB}_i$ by Σ_i. It follows that

$$-\chi(\Sigma) = -\chi(\Sigma_1) - \chi(\Sigma_2) + 1,$$

where Σ denotes the page of the open book $\mathcal{OB}$. Thus we have

$$\mathrm{Hg}(Y_1 \# Y_2, \xi_1 \# \xi_2) \leq \mathrm{Hg}(Y_1, \xi_1) + \mathrm{Hg}(Y_2, \xi_2),$$

which implies that Hg is sub-additive under contact connected sum.

Next we show that Hg is not necessarily additive under contact connected sum. Let ξ_d denote the overtwisted contact structure in S^3 whose d_3 invariant (cf. [6]) is equal to the half integer d. The following result was obtained in [1]: If (Y, ξ) is a contact structure with $c_1(\xi)$ torsion, then

$$d_3(Y, \xi \# \xi_d) = d_3(Y, \xi) + d_3(S^3, \xi_d) + 1/2.$$

Now suppose that Y is an integral homology sphere. It follows that $c_1(\xi) = 0$ for every contact structure ξ on Y, and Y carries a unique spin^c structure. Thus for an arbitrary contact structure ξ on Y we have

$$d_3(Y, \xi \# \xi_{-\frac{1}{2}}) = d_3(Y, \xi) + d_3(S^3, \xi_{-\frac{1}{2}}) + \frac{1}{2} = d_3(Y, \xi),$$

which implies that the connected sum $\xi \# \xi_{-\frac{1}{2}}$ is homotopic to ξ as oriented plane fields (cf. [6]). In fact, $\xi \# \xi_{-\frac{1}{2}}$ is isotopic to ξ by the classification of overtwisted contact structures due to Eliashberg [2]. As a consequence we have

$$\mathrm{Hg}(Y, \xi \# \xi_{-\frac{1}{2}}) = \mathrm{Hg}(Y, \xi).$$

On the other hand, in ([4], Lemma 5.5), it was proved that $\mathrm{Hg}(S^3, \xi_{-\frac{1}{2}}) = 2$. Note that an open book realizing $\mathrm{Hg}(S^3, \xi_{-\frac{1}{2}})$ can be described by taking a pair of pants as a page and $t_1 t_2^{-2} t_3^{-3}$ as the monodromy, where t_i denotes a right-handed Dehn twist along a boundary component. Consequently we have

$$\mathrm{Hg}(Y \# S^3, \xi \# \xi_{-\frac{1}{2}}) < \mathrm{Hg}(Y, \xi) + \mathrm{Hg}(S^3, \xi_{-\frac{1}{2}}).$$

□

3. Contact genus of three dimensional manifolds

Here we provide some basic properties of the contact genus of 3-manifolds, and compute this invariant for some 3-manifolds.

Proposition 2. *Let Y denote a 3-manifold. Then we have*

(a) $\mathrm{cg}(Y) \geq 0$ *($= 0$ if and only if $Y \cong S^3$),*
(b) $\mathrm{cg}(Y) = 1$ *if and only if Y is an oriented circle bundle over S^2 (which is not diffeomorphic to S^3).*

Proof. For a 3-manifold Y, $\mathrm{cg}(Y)$ is obtained by taking the minimum of the sum $2g + r - 1$ over all open books, where g and r denote the genus of the page and the number of binding components of an open book, respectively. Hence we have $0 \leq \mathrm{cg}(Y)$ for an arbitrary 3-manifold Y, since $g \geq 0$ and $r \geq 1$. It is clear that the absolute minimum of the expression $2g + r - 1$ is realized when $g = 0$ and $r = 1$ and the open book with disk pages and trivial monodromy supports the unique tight contact structure on S^3, which proves (a).

To prove (b), we note that $\mathrm{cg}(Y) = 1$ is realized if and only if $g = 0$ and $r = 2$. Any self-diffeomorphism of an annulus is given by t_c^m, for some $m \in \mathbb{Z}$, where c is the core of the annulus, and t_c denotes a right-handed Dehn twist along c. If $m \geq 0$, this open book supports the unique tight contact structure on the lens space $L(m, -1)$ which is an oriented circle bundle over S^2 with Euler number m. Otherwise (i.e., when $m < 0$) the induced contact structure is the overtwisted contact structure on $L(-m, 1)$ which is an oriented circle bundle over S^2 with Euler number m. Combining, we showed that $\mathrm{cg}(Y) = 1$ if and only if Y is an oriented circle bundle over S^2, which is not diffeomorphic to S^3.

□

Note that oriented circle bundles over S^2 are very special lens spaces and therefore we immediately conclude from Proposition 2 that

Corollary 3. *Most* 3*-manifolds of Heegaard genus* 1 *have contact genus* > 1.

For example, $\mathrm{cg}(L(5,3)) = 2$, since $L(5,3)$ is not a circle bundle over S^2 and it carries a (tight) contact structure which is supported by a planar open book with three binding components.

Lemma 4. *We have* $\mathrm{cg}(Y_{p,q,r}) \leq 2$ *, where* $Y_{p,q,r}$ *denotes the* 3*-manifold depicted in Figure 1, with* $p, q, r \in \mathbb{Z}$*. Moreover if* $|p| > 1$*,* $|q| > 1$ *and* $|r| > 1$ *then* $\mathrm{cg}(Y_{p,q,r}) = 2$.

Proof. It follows from [4] that $Y_{p,q,r}$ has a planar open book with at most three binding components, which indeed proves that $\mathrm{cg}(Y_{p,q,r}) \leq 2$. Moreover, under the assumption that $|p| > 1, |q| > 1$, and $|r| > 1$, the 3-manifold $Y_{p,q,r}$ is not diffeomorphic to any lens space and hence $\mathrm{cg}(Y_{p,q,r}) = 2$ by Proposition 2.

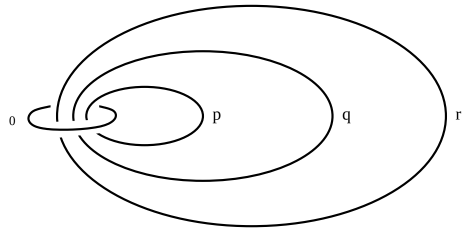

FIGURE 1. Integral surgery diagram for the small Seifert fibered 3-manifold $Y_{p,q,r}$

□

When we drop the assumption on p, q and r in Lemma 4, we observe that $Y_{p,q,r}$ is diffeomorphic to either S^3, $S^1 \times S^2$, a lens space, or certain connected sums of these for some values of the integers p, q and r.

Remark 5. *Note that Lemma 4 exhibits examples of* 3*-manifolds* $Y = Y_{p,q,r}$ *for which* $\mathrm{hg}(Y) = \mathrm{cg}(Y) = 2$*, although most* 3*-manifolds of Heegaard genus* 2 *have contact genus* > 2 *as was shown by Rubinstein* [8].

Lemma 6. *We have* $\mathrm{cg}(\#_k S^1 \times S^2) = k$*, for* $k \geq 1$.

Proof. Since $\mathrm{hg}(\#_k S^1 \times S^2) = k$, we know that $\mathrm{cg}(\#_k S^1 \times S^2) \geq k$. Hence to show that $\mathrm{cg}(\#_k S^1 \times S^2) = k$, we just need to realize this lower bound by a Heegaard splitting of $\#_k S^1 \times S^2$ induced from an open book. We use the fact that the unique tight contact structure on $\#_k S^1 \times S^2$ is supported by an planar open book with $k+1$ binding components, whose monodromy is the identity map.

□

The proof of the following result is similar to the proof of Theorem 1.

Proposition 7. *Let* Y_i *denote a* 3*-manifold, for* $i = 1, 2$*. Then we have*

$$\mathrm{cg}(Y_1 \# Y_2) \leq \mathrm{cg}(Y_1) + \mathrm{cg}(Y_2).$$

Conjecture 8. *Contact genus is additive under connected sum of* 3*-manifolds.*

Note that if $\mathrm{hg}(Y_i) = \mathrm{cg}(Y_i)$ for $i = 1, 2$, then $\mathrm{cg}(Y_1 \# Y_2) = \mathrm{cg}(Y_1) + \mathrm{cg}(Y_2)$.

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