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Surgery on contact 3-manifolds and Stein surfaces, by Burak Ozbagci and András I. Stipsicz, Springer-Verlag, Berlin; János Bolyai Mathematical Society, Budapest, 2004, 281 pp., US$89.95, ISBN 3-540-22944-2; ISBN 963-9453-03-X

The venerable subject of contact geometry has gone through a dramatic transformation over the last 10 to 15 years that has made it into a fundamental tool in low-dimensional topology as well as revealing itself as a field with a great deal of beauty and subtlety. The origins of contact geometry date back more than two centuries, to the work of Huygens, Hamilton and Jacobi on geometric optics, and it has been studied by many great mathematicians, such as Lie, Cartan and Darboux. Despite its ancient origins and its reoccurrence in physical and geometric context over the years it is only recently that its strongly topological flavor has surfaced and it has moved into the foreground of mathematics. The book under review provides an introduction to several of these recent advances.

Recall that a contact structure on an oriented 3-manifold M is a two-dimensional sub-bundle ξ of the tangent bundle TM that can be defined (at least locally) as the kernel of a 1-form α; that is, $\xi = \ker \alpha$ satisfying $\alpha \wedge d\alpha$ is a positive volume form on M. As a first example of a contact structure consider $\mathbb{R}^3$ with cylindrical coordinates; then $\xi_{std} = \ker(dz + r^2\, d\theta)$ is easily seen to be a contact structure. See Figure 1. A theorem of Darboux says that any contact structure on any 3-manifold

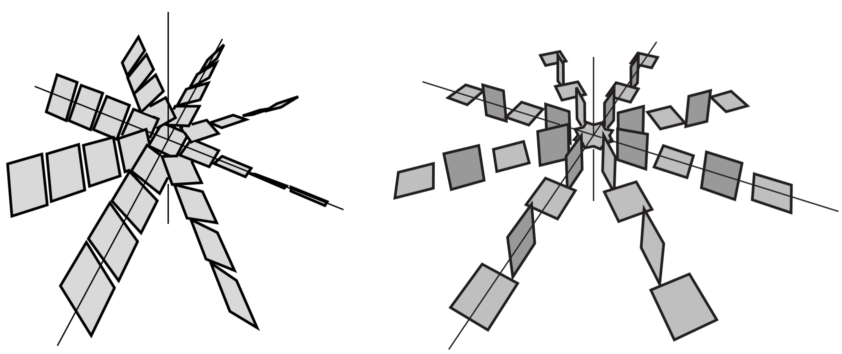

FIGURE 1. The standard contact structure ξ_{std}, left, and an overtwisted contact structure ξ_{ot}, right. (Pictures courtesy of Stephan Schönenberger.)

is locally equivalent to this one. That is, every point in a contact manifold has a neighborhood diffeomorphic to some open set in $\mathbb{R}^3$ by a diffeomorphism that takes the contact structure to ξ_{std}. Thus we could alternately define a contact structure on M to be a two dimensional sub-bundle ξ of TM that is locally equivalent to ξ_{std}. Another interesting contact structure we will discuss below is

$$\xi_{ot} = \ker(\cos r\, dz + r \sin r\, d\theta).$$

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When studying contact structures ξ on M, it will be useful to consider knots in them. A knot K in M is called Legendrian if K is always tangent to ξ.

Modern contact geometry was born in Bennequin's wonderful paper "Entrelacements et équations de Pfaff" [1]. In this paper Bennequin established the existence of two distinct contact structures on $\mathbb{R}^3$. The basic idea is quite simple. Note that the disk D of radius $\frac{\pi}{2}$ in the xy-plane is tangent to the contact planes of ξ_{ot} along the boundary of D. (Said another way, the framing of ∂D given by ξ_{ot} and by D agree.) Bennequin showed that no such disk exists in ξ_{std}. Not only did this result demonstrate that $\mathbb{R}^3$ can support more than one contact structure, it ushered in a low-dimensional topological approach to studying problems in contact geometry. To elaborate on this point, note that ∂D is a Legendrian knot in $\mathbb{R}^3$ and that as the knot is traversed the contact planes do not "twist" relative to the disk. Given any knot K in $\mathbb{R}^3$ it bounds an orientable surface Σ embedded in $\mathbb{R}^3$. If K is Legendrian one can ask how many times the contact planes twist relative to the surface Σ. This number is called the Thurston-Bennequin invariant of the knot K and is denoted $tb(K)$. From the discussion above it is clear that there is a Legendrian knot K in $\mathbb{R}^3$ with contact structure ξ_{ot} that bounds a disk and has $tb(K) = 0$. Bennequin showed that if K is a Legendrian knot bounding a disk in $\mathbb{R}^3$ with contact structure ξ_{std}, then $tb(K) \leq -1$. Bennequin proved this result by using a creative combination of braid theory, normal forms for Seifert surfaces and the study of singular foliations on surfaces.

After Bennequin's work, Eliashberg saw that there was a fundamental difference between contact structures containing embedded disks as above and ones that do not [2]. The former type of contact structures he called overtwisted and the latter tight. The terminology overtwisted is clear: the planes defining a contact structure must "twist" so that they are not tangent to a surface in an open neighborhood (this follows from the definition of contact structure and the Frobenious integrability criterion); but if they twist too much, the contact structure is called overtwisted. Indicating that the tight vs. overtwisted dichotomy was a useful one, Eliashberg showed that overtwisted contact structures are determined by algebraic topological information. Specifically, he showed that in every homotopy class of plane field there was an overtwisted contact structure (actually Lutz showed this, but he did not use the terminology overtwisted). Moreover, Eliashberg showed that two overtwisted contact structures were the same if and only if their underlying plane fields were homotopic. Thus studying the classification of overtwisted contact structures is reduced to studying homotopy types of plane fields, which is a purely algebro-topological question. (Subsequently, the slogan "overtwisted contact structures are topological, while tight contact structures are geometric" has been borne out in other settings.) Eliashberg showed, on the other hand, that there were strong restrictions on the existence of tight contact structures. For example, a contact structure ξ on a 3-manifold M is, among other things, an oriented two-dimensional vector bundle and thus has an Euler class $e(\xi) \in H^2(M;\mathbb{Z})$. If the contact structure is tight, then Eliashberg showed that

$$|e(\xi)[\Sigma]| \leq -\chi(\Sigma),$$

where Σ is an oriented surface, is not equal to S^2, is embedded in M, $[\Sigma]$ is its homology class, and χ is the Euler characteristic. This inequality implies that there are only a finite number of elements in $H^2(M;\mathbb{Z})$ that can be realized as the Euler class of a tight contact structure. This is in contrast to the fact that any even class can be realized as the Euler class of an overtwisted contact structure. This inequality also points out that tight contact structures can see subtle topological properties of the underlying manifold. In particular, the inequality gives a non-trivial lower bound on the genus of surfaces representing various homology classes in M.

While it is fairly easy to construct overtwisted contact structures, it is a much more difficult matter to construct tight contact structures. Bennequin proved that the standard contact structure on $\mathbb{R}^3$ is tight (and from this it is not hard to see that the standard contact structure on S^3 is tight). Apart from these results, examples were hard to come by until Eliashberg and Gromov proved that contact structures filled by a symplectic manifold are tight [4]. To be more precise, if X is a 4-manifold and ω is a symplectic form on it (recall this means ω is a closed 2-form and $\omega \wedge \omega$ is a volume form on X), then one says (X, ω) symplectically fills the contact manifold (M, ξ) if $\omega(v, w) > 0$ for any positively oriented basis v, w of ξ at any point in M. This result gave a second proof that the standard contact structures on S^3 and $\mathbb{R}^3$ are tight. It also allowed for the construction of other tight contact structures. Many of these contact structures were constructed via Stein fillings.

A *Stein manifold* X is a complex manifold (we denote its complex structure by an automorphism J of the tangent bundle such that $J^2 = -\mathrm{id}$) that admits a proper biholomorphic embedding in $\mathbb{C}^4$. Such a manifold will not be compact and will always have a proper Morse function $f\colon X \to [0, \infty)$ such that non-critical level-sets $f^{-1}(t)$ are 3-manifolds whose complex tangencies form a contact structure that is symplectically filled by $f^{-1}([0, t])$ with symplectic form $-d(J \circ df)$. If f has only finitely many critical values and t is larger than them all, then one frequently thinks of $f^{-1}([0, t])$ as a "Stein filling" of its boundary or calls it a "Stein manifold with boundary". This is an abuse of terminology but has become standard in contact geometry. The question now becomes how to construct Stein manifolds. This question was answered by Eliashberg (and further elaborated on by Gompf). First, a 4-manifold built out of 0- and 1-handles has a Stein structure. Thus its boundary has a contact structure. Moreover, given any Stein manifold with boundary Y and any Legendrian simple closed curve γ in its boundary, there is a Stein structure on Y union a 2-handle attached to γ if it is attached with framing one less than the framing the contact planes give to γ. If Y' is the Stein manifold obtained by attaching a 2-handle to Y as above, then one says that the contact manifold $\partial Y'$ is obtained by *Legendrian surgery* from ∂Y.

A few of the early triumphs of contact topology were Eliashberg's proof that there is a unique tight contact structure on S^3 and Gromov's proof that the unit ball in $\mathbb{C}^2$ is the unique Stein filling of this contact structure. These two beautiful results (and their proofs) can be used to give a "simple" proof of a deep theorem of Cerf: *Any diffeomorphism of S^3 extends over the 4-ball.* This result is one of the indications that contact/symplectic geometry could be used to probe subtle topological properties of 3- and 4-manifolds.

By the mid 1990's many tools in contact geometry had been developed by many people, but predominantly by Eliashberg and Giroux. These tools lead to the classification of contact structures on the 3-torus, by Kanda and, independently, Giroux, and some partial classification results on lens spaces, by the reviewer. Shortly thereafter Honda and, independently, Giroux were able to classify contact structures on all lens spaces, torus bundles over the circle and on circle bundles over surfaces.

Since that time there have been countless advances by many mathematicians. Two highlights are the surprising fact that there are 3-manifolds that do not admit tight contact structures (by Honda and the reviewer with further examples by Lisca and Stipsicz) and there are tight contact structures that do not admit any symplectic fillings (by Honda and the reviewer with many further examples given by Lisca and Stipsicz).

Many of the recent advances in contact geometry have been made possible by Giroux's correspondence between open book decompositions and contact structures [3]. An open book decomposition of a 3-manifold is a link L in the manifold such that the complement of the link fibers over the circle in such a way that the closure of the fibers have boundary L. It was observed in the 1970's by Thurston and Winkelnkemper that one could "perturb" an open book decomposition of a 3-manifold into a contact structure. Amazingly, all contact structures come from this construction. This was proven by Giroux in 2000. Moreover, Giroux gave a simple equivalence relation on open book decompositions such that the equivalence classes were in one-to-one correspondence with contact structures. This correspondence has been of immense importance over the last few years. The results that have followed from it are too numerous to discuss in this article, so we restrict ourselves to just two applications of Giroux's correspondence.

The first application involves Heegaard Floer homology. This is a powerful invariant of 3-manifolds defined by Ozsváth and Szabó in 2000 [5] that assigns to a closed oriented 3-manifold M (and spin^c structure on M) a graded group $HF(M)$. (Actually, there are many flavors of Heegaard Floer homology, but this is a point that is safe to suppress in this article.) These graded groups are conjectured to be related to Seiberg-Witten Floer homology, but despite the many parallel results this has not been established yet. There is also a version of this theory that gives an invariant of a knot K in M. Ozsváth and Szabó have used this knot invariant and Giroux's correspondence to define an element $c(\xi)$ in $HF(-M)$ for any contact structure ξ. (We denote M with its orientation reversed by $-M$.) This invariant vanishes for overtwisted contact structures, but is non-zero for fillable contact structures. Many results about contact structures have been proven using this new invariant. For example, just a few months ago Ghiggini settled a much-studied question by showing that a symplectically fillable contact structure need not be Stein fillable by using properties of the invariant $c(\xi)$.

The next application of Giroux's correspondence is to symplectic fillings. Building on the work of many people, Eliashberg and, independently, the reviewer showed that any symplectic filling of a contact manifold can be embedded in a closed symplectic manifold. There has been a great deal of other work concerning symplectic cobordisms, but this result was a key ingredient in several applications to topology. Using this result and the work of Feehan and Leness on the relation between Seiberg-Witten theory and Donaldson theory, Kronheimer and Mrowka were able to prove that all non-trivial knots satisfy Property P. A knot has Property P if non-trivial surgery on it never yields a homotopy sphere. It had long been conjectured that non-trivial knots always have Property P, but it took an elegant confluence of many trends in modern topology to establish this much-studied conjecture. A second application of the above embedding result is the characterization of the unknot in S^3 by Ozsváth and Szabó. They proved that if p-surgery on a knot in S^3 yields $-L(p, 1)$ (this is the result of p-surgery on the unknot), then the knot must be the unknot. This conjecture of Gordon was originally established by Kronheimer, Mrowka, Ozsváth and Szabó without the use of the above symplectic embedding result, but in Ozsváth and Szabó's proof, many interesting and useful properties of Heegaard Floer homology are revealed (some of these properties need Eliashberg's version of the embedding theorem).

The theory of Lefschetz fibrations is a useful 4-dimensional tool related to open book decompositions. A Lefschetz fibration of a 4-manifold is a fibration of the 4-manifold over a surface with certain types of singularities. If the 4-manifold (and the fibers of the fibration) has boundary and the Lefschetz fibration is over a disk, then one obtains an open book decomposition of the boundary by restricting the Lefschetz fibration to the boundary. This observation has been very useful in understanding fillings of contact manifolds. In particular, Loi and Piergallini and, independently, Akbulut and Ozbagci have proven various results about open book decompositions associated to Stein fillable contact structures. In addition the structure of Lefschetz fibrations was essential to the applications of Heegaard Floer homology mentioned above.

The last trend in contact geometry we wish to discuss is contact surgery. Above it was mentioned that when constructing Stein manifolds, one can perform Legendrian surgery on a Legendrian knot. Topologically this amounts to performing a surgery on a Legendrian knot with framing one less than the contact framing. Though not related to the construction of Stein manifolds, one can also perform a surgery on a Legendrian knot with framing one more that the contact framing and get a natural contact structure. This is called $+1$-contact surgery (in this context, Legendrian surgery is called -1-contact surgery). There are other types of contact surgery, but they all amount to ± 1-contact surgery, so this is all we mention here. Though these notions have floated about the field for some time, Ding and Geiges were the first to systematically study contact surgery and show that all contact structures can be obtained from the standard one on S^3 by contact surgeries. While contact surgery is interesting in and of itself, it took a central role in contact geometry when it was shown that the Heegaard Floer contact invariant behaved nicely with respect to contact surgery. Lisca and Stipsicz, and others, have used this to great effect, significantly illuminating the world of tight contact structures.

While there are other exciting trends in contact geometry, such as the use of convex surfaces, holomorphic curves and, in particular, Symplectic Field Theory, the book under review is a user's guide to the ideas discussed in the previous few paragraphs. The book assumes very little background. Anyone with knowledge of basic topology, manifold theory and algebraic topology should have no trouble reading it. The authors have chosen several recent developments in contact geometry and have built the book with an idea toward taking someone with the above-mentioned modest background to these cutting-edge developments. Given the size of the book (281 pages), this means not fully developing much of the theory discussed in the book, but there is ample discussion for the reader to follow the train of arguments and appreciate the main results. Some of the main results at which the book aims are a topological characterization of Stein domains and Stein neighborhoods of surfaces in complex projective space, an application of contact geometry to the 4-ball genus of knots in S^3, the existence of tight but not symplectically fillable contact structures, and the topology of Stein fillings of a contact 3-manifold. After an introduction that outlines some of the aims of the book, there are three chapters that cover the basic theory of surgery on knots, symplectic 4-manifolds and contact 3-manifolds, respectively. The authors then discuss convex surfaces (a fundamental tool in contact geometry), spin^c structures on 3- and 4-manifolds, symplectic surgery and Stein manifolds. At this point the reader has the background to appreciate the breakthroughs in contact geometry starting around 2000. Specifically, the next two chapters discuss Giroux's correspondence between open book decompositions and contact structures, and the related theory of Lefschetz fibrations of 4-manifolds. The last two chapters cover contact surgery and fillings of contact manifolds. Throughout the book the authors use Seiberg-Witten theory and Heegaard Floer homology, so to make the book self-contained there are two appendices covering these topics. There is also an appendix on the mapping class group of surfaces, which plays a role in studying open book decompositions.

This book is an enjoyable introduction to many of the ideas that are driving contact geometry today. Though there are a few survey articles and a chapter or two in books that cover contact geometry, there are few places where someone can find an introduction to contact geometry, much less one that takes the reader to the frontiers of research. This book is highly recommended to any graduate student or researcher who would like to understand many of the exciting developments in the contact geometry of 3-manifolds and their application to low-dimensional topology.

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