MILNOR OPEN BOOKS OF LINKS OF SOME RATIONAL SURFACE SINGULARITIES

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ABSTRACT. We determine Legendrian surgery diagrams for the canonical contact structures of links of rational surface singularities that are also small Seifert fibred 3-manifolds. Moreover, we describe an infinite family of Milnor fillable contact 3-manifolds so that, for each member of this family, the Milnor genus and Milnor norm are strictly greater than the support genus and support norm of the canonical contact structure. For some of these contact structures we construct supporting Milnor open books.

1. INTRODUCTION

The link of a normal complex surface singularity carries a canonical contact structure ξ_{can} (a.k.a. the Milnor fillable contact structure) which is supported by any Milnor open book on this link [7]. The canonical contact structure ξ_{can} is known to be Stein fillable [5] and therefore it is tight [11]. In fact, ξ_{can} is universally tight, i.e., the pullback to the universal cover is tight [25].

In [17], three numerical invariants of contact structures were defined in terms of open books supporting the contact structures. These invariants are the support genus $\mathrm{sg}(\xi)$ (the minimal genus of a page of a supporting open book for ξ), the binding number $\mathrm{bn}(\xi)$ (the minimal number of binding components of a supporting open book for ξ with minimal genus pages) and the support norm $\mathrm{sn}(\xi)$ (minus the maximal Euler characteristic of a page of a supporting open book for ξ).

Moreover, a new set of invariants were derived in [1], specifically for the canonical contact structure ξ_{can} on the link of a complex surface singularity by restricting the set of open books to only Milnor open books on the link at hand. In this article we will call these invariants the Milnor genus $\mathrm{Mg}(\xi_{can})$, the Milnor binding number $\mathrm{Mb}(\xi_{can})$,

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and the Milnor norm $\mathrm{Mn}(\xi_{can})$ of the canonical contact structure ξ_{can}. Note that the well-known *Milnor number* corresponds to the first Betti number of the page of the Milnor open book in our context.

It follows by definition that $\mathrm{sg}(\xi_{can}) \leq \mathrm{Mg}(\xi_{can})$ and $\mathrm{sn}(\xi_{can}) \leq \mathrm{Mn}(\xi_{can})$, since the set of Milnor open books is a subset of all open books on the link of a surface singularity. (No such inequality exists, however, between $\mathrm{bn}(\xi_{can})$ and $\mathrm{Mb}(\xi_{can})$, in general.) In Section 9, we show that for each positive integer k, there is a rational surface singularity whose canonical contact structure ξ_{can} satisfies

$$k \leq \mathrm{Mg}(\xi_{can}) - \mathrm{sg}(\xi_{can}) \ \text{ and } \ k \leq \mathrm{Mn}(\xi_{can}) - \mathrm{sn}(\xi_{can}).$$

An immediate consequence is the existence of links of surface singularities carrying open books which are not isomorphic to Milnor open books. As another consequence, we deduce that Milnor open books are neither norm nor genus minimizing although our aim originally was to show that the support genus of a Milnor fillable contact structure is realized by a Milnor open book. We find this result interesting since there are other instances in geometric-topology, where the "complex representatives" are minimizers. Most notably, the link of a complex *plane* curve singularity bounds a smooth complex curve of genus equal to its Seifert genus.

The aforementioned examples are the canonical contact structures on links of some rational surface singularities which are also small Seifert fibred 3-manifolds. In Section 8, we identify the canonical contact structures on *all* such manifolds via their Legendrian surgery diagrams. Note that a Legendrian surgery diagram is perhaps the most efficient way of describing a contact structure—from a topological point of view, since it also allows one to calculate many invariants of the contact structure (e.g., the Euler class of the underlying oriented plane field) by easily converting the diagram into a smooth handlebody diagram.

2. OPEN BOOKS AND CONTACT STRUCTURES

A complete exposition of the correspondence between open books and contact structures can be found in the lecture notes of Etnyre [15]. In this section, we would like to recall some basic definitions for the sake of completeness.

Suppose that for an oriented link B in a closed and oriented 3-manifold Y, the complement $Y \setminus B$ fibers over the circle as $p \colon Y \setminus B \to S^1$ such that $p^{-1}(t) = \Sigma_t$ is the interior of a compact surface with $\partial \Sigma_t = B$, for all $t \in S^1$. Then (B, p) is called an *open book decomposition* (or just an *open book*) of Y. For each $t \in S^1$, the surface Σ_t is called a *page*, while B is referred to as the *binding* of the open book.

The monodromy of the fibration p is defined as the diffeomorphism of a fixed page which is given by the first return map of a flow that is transverse to the pages and meridional near the binding. The isotopy class of this diffeomorphism is independent of the chosen flow and we will refer to that as the *monodromy* of the open book decomposition. In order to describe the monodromy of an open book explicitly, one usually writes it as a product of Dehn twists along some curves on the page. In this paper, we will denote a right-handed (resp. left-handed) Dehn twist along a curve α as α (resp. α^{-1}), for simplicity.

An open book can also be described as follows. First consider the mapping torus

$$\Sigma_\phi = [0, 1] \times \Sigma / (1, x) \sim (0, \phi(x))$$

where Σ is a compact oriented surface with r boundary components and ϕ is an element of the mapping class group Γ_Σ of Σ. Since ϕ is the identity map on $\partial \Sigma$, the boundary $\partial \Sigma_\phi$ of the mapping torus Σ_ϕ can be canonically identified with r copies of $T^2 = S^1 \times S^1$, where the first S^1 factor is identified with $[0, 1]/(0 \sim 1)$ and the second one comes from a component of $\partial \Sigma$. Now we glue in r copies of $D^2 \times S^1$ to cap off Σ_ϕ so that ∂D^2 is identified with $S^1 = [0, 1]/(0 \sim 1)$ and the S^1 factor in $D^2 \times S^1$ is identified with a boundary component of $\partial \Sigma$. Thus we get a closed 3-manifold $Y = \Sigma_\phi \cup_r D^2 \times S^1$ equipped with an open book decomposition whose binding is the union of the core circles of the $D^2 \times S^1$'s that we glue to Σ_ϕ to obtain Y. In conclusion, an element $\phi \in \Gamma_\Sigma$ determines a 3-manifold together with an *abstract* open book decomposition on it. Notice that by conjugating the monodromy ϕ of an open book on a 3-manifold Y by an element in Γ_Σ we get an isomorphic open book on a 3-manifold Y' which is diffeomorphic to Y.

It has been known for a long time that every closed and oriented 3-manifold admits an open book decomposition. A new interest in open books on 3-manifolds arose recently from their connection to contact structures, which we will describe very briefly.

Recall that a (positive) contact structure ξ on an oriented 3-manifold is locally the kernel of a 1-form α such that $\alpha \wedge d\alpha > 0$. In this paper we assume that ξ is coorientable, i.e., α is a global 1-form.

Definition 2.1. An open book decomposition (B, p) of a 3-manifold Y is said to *support* a contact structure ξ on Y if ξ can be represented by a contact form α such that α evaluates positively on B and $d\alpha$ is a symplectic form on every page.

In [34], Thurston and Winkelnkemper associated a contact structure to every open book. It turns out that the contact structure they constructed is in fact supported by the underlying open book. (Definition 2.1 was not available at the time.) To state the converse we need a little digression.

Suppose that an open book decomposition with page Σ is specified by $\phi \in \Gamma_\Sigma$. Attach a 1-handle to the surface Σ connecting two points on $\partial\Sigma$ to obtain a new surface Σ'. Let γ be a closed curve in Σ' going over the new 1-handle exactly once. Define a new open book decomposition with $\phi' = \phi \circ t_\gamma \in \Gamma_{\Sigma'}$, where t_γ denotes the right-handed Dehn twist along γ. The resulting open book decomposition is called a *positive stabilization* of the one defined by ϕ. Notice that although the resulting monodromy depends on the chosen curve γ, the 3-manifold specified by (Σ', ϕ') is diffeomorphic to the 3-manifold specified by (Σ, ϕ). A converse to the aforementioned result of Thurston and Winkelnkemper is given by

Theorem 2.2 (Giroux [20])**.** *Every contact structure on a 3-manifold is supported by an open book. Two open books supporting the same contact structure admit a common positive stabilization. Moreover two contact structures supported by the same open book are isotopic.*

3. LEGENDRIAN SURGERY DIAGRAMS

Recall that a knot in a contact 3-manifold is called Legendrian if it is everywhere tangent to the contact planes. In order to have a better understanding of the topological constructions in the later sections, we discuss a standard way to visualize Legendrian knots in $\mathbb{R}^3$ (and therefore S^3) equipped with the standard contact structure $\xi_{st} = \ker(dz + x\, dy)$.

Now consider a Legendrian knot $L \subset (\mathbb{R}^3, \xi_{st})$ and take its *front* projection, i.e., its projection to the yz-plane. Notice that the projection has no vertical tangencies (since $-\frac{dz}{dy} = x \neq \infty$), and for the same reason at a crossing the strand with smaller slope is in front. It turns out that L can be C^2-approximated by a Legendrian knot for which the projection has only transverse double points and *cusp* singularities (see [18], for example). Conversely, a knot projection with these properties gives rise to a unique Legendrian knot in $(\mathbb{R}^3, \xi_{st})$ by defining x from the projection as $-\frac{dz}{dy}$. Since any projection can be isotoped to satisfy the above properties, every knot in S^3 can be isotoped (non uniquely) to a Legendrian knot.

There are two classical invariants of a Legendrian knot L: The Thurston–Bennequin number $\mathrm{tb}(L)$ and the rotation number $\mathrm{rot}(L)$. Recall that $\mathrm{tb}(L)$ is the contact framing of L (measured with respect to the Seifert framing in S^3), which can be easily computed from a front projection of L. Define $w(L)$ (the *writhe* of L) as the sum of signs of the double points. For this to make sense we need to fix an orientation on the knot, but the result is independent of this choice. If $c(L)$ is the number of cusps, then $\mathrm{tb}(L) = w(L) - \frac{1}{2}c(L)$.

The rotation number $\mathrm{rot}(L)$ is defined by trivializing ξ_{st} along a Seifert surface Σ whose oriented boundary is L and then taking the winding number of TL with respect to this trivialization. For this invariant to make sense we need to orient L, and the result changes sign by reversing the orientation. Since $H^2(S^3; \mathbb{Z}) = 0$, this number is independent of the chosen trivialization. If $c_d(L)$ (resp. $c_u(L)$) denotes the number of down (resp. up) cusps in the projection, then $\mathrm{rot}(L) = \frac{1}{2}(c_d(L) - c_u(L))$.

To describe the Stein fillable contact structures that we deal with in this paper we use Legendrian knots (actually their front projections) as follows: Consider the standard Stein 4-ball B^4 with the induced standard contact structure on its boundary. Then attach Weinstein 2-handles [35] along an arbitrary Legendrian link in $\partial B^4 = S^3$ to this ball. By the work of Eliashberg [10] the Stein structure on B^4 extends over the 2-handles as long as the attaching framing of each 2-handle is one less than the Thurston–Bennequin number. The resulting Stein domain has an induced contact structure on its boundary which can be represented by the front projection of the Legendrian link along which we attach the 2-handles. Such a front projection is called a *Legendrian surgery diagram* (see [21] for a thorough discussion). Notice that Legendrian surgery is equivalent to performing contact (-1)-surgery along the given Legendrian link in

the standard contact S^3 [9]. To describe all Stein fillable contact structures in general, one needs 1-handles as well, but those will not appear in our discussion.

4. MILNOR OPEN BOOKS AND CANONICAL CONTACT STRUCTURES

Let (X, x) be an isolated normal complex surface singularity (cf. [28]). Fix a local embedding of (X, x) in $(\mathbb{C}^N, 0)$. Then a small sphere $S_\epsilon^{2N-1} \subset \mathbb{C}^N$ centred at the origin intersects X transversely, and the complex hyperplane distribution ξ_{can} on $M = X \cap S_\epsilon^{2N-1}$ induced by the complex structure on X is called the *canonical* contact structure. It is known that, for sufficiently small radius ϵ, the contact manifold is independent of ϵ and the embedding, up to isomorphism. The 3-manifold M is called the link of the singularity and (M, ξ_{can}) is called the *contact boundary* of (X, x). Note that while Y denotes a general 3-manifold, we use M for those which are Milnor-filled.

Definition 4.1. A contact manifold (Y, ξ) is said to be *Milnor fillable* and the germ (X, x) is called a *Milnor filling* of (Y, ξ) if (Y, ξ) is isomorphic to the contact boundary (M, ξ_{can}) of some isolated complex surface singularity (X, x).

In addition, we say that a closed and oriented 3-manifold Y is Milnor fillable if it carries a contact structure ξ so that (Y, ξ) is Milnor fillable. Such a contact structure ξ is called a Milnor fillable contact structure.

By a theorem of Mumford [27], if a contact 3-manifold is Milnor fillable, then it can be obtained by plumbing oriented circle bundles over surfaces according to a weighted graph with negative definite intersection matrix. Conversely, it follows from a well-known theorem of Grauert [22] that any 3-manifold that is given by plumbing oriented circle bundles over surfaces according to a weighted graph with negative definite intersection matrix is Milnor fillable. As for the uniqueness of Milnor fillable contact structures, we have the following fundamental result.

Theorem 4.2 ([7])**.** *Any closed and oriented 3-manifold has at most one Milnor fillable contact structure up to isomorphism.*

In summary, Milnor fillability of a closed and oriented 3-manifold Y is determined entirely by its topology and if Y is Milnor fillable, then it carries a canonical contact structure ξ_{can} which is unique up to isomorphism.

Since the ground-breaking result of Giroux [20], the geometry of contact structures is often studied via their topological counterparts, namely open book decompositions. In the realm of surface singularities this fits nicely with some work of Milnor [26].

Definition 4.3. Given an analytic function $f\colon (X,x) \to (\mathbb{C},0)$ vanishing at x, with an isolated singularity at x, the open book decomposition $\mathcal{OB}_f$ of the link M of (X,x) with binding $L = M \cap f^{-1}(0)$ and projection $\pi = \frac{f}{|f|}\colon M \setminus L \to S^1 \subset \mathbb{C}$ is called the *Milnor open book* induced by f.

Such functions f exist and one can talk about many Milnor open books on the singularity link M. Therefore, there are many Milnor open books on any given Milnor fillable contact 3-manifold (Y,ξ), since, by definition, it is isomorphic to the link (M,ξ_{can}) of some isolated complex surface singularity (X,x).

A Milnor open book on a Milnor fillable 3-manifold Y has two essential features as shown in [7]:

(i) It supports the canonical contact structure ξ_{can} and

(ii) it is horizontal when considered on the plumbing description of Y.

Suppose that the 3-manifold Y is obtained by plumbing oriented circle bundles $M_i \to S_i$, for $i = 1, 2, \ldots, r$. For any r-tuple of nonnegative integers $\underline{n} = (n_1, n_2, \ldots, n_r)$ a vertical link of type $\underline{n}$ consists of a disjoint union of n_i generic fibers from each bundle $M_i \to S_i$. An open book on Y is called *horizontal* if its binding is a vertical link and its pages are transverse to the fibres. We also require that the orientation induced on the binding by the pages coincides with the orientation of the fibres induced by the fibration.

5. RATIONAL SURFACE SINGULARITIES

Let (X,x) be a germ of a normal complex surface having a singularity at x. Recall that (X,x) is called rational [2] if the geometric genus $p_g := \dim_{\mathbb{C}} H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is equal to zero, where $\tilde{X} \to X$ is a resolution of the singular point $x \in X$. This definition does not depend on the resolution.

Now fix a resolution $\pi : \tilde{X} \to X$ and denote the irreducible components of the exceptional divisor $E = \pi^{-1}(x)$ by $\bigcup_{i=1}^r E_i$. The *fundamental cycle* of E is by definition the

componentwise smallest nonzero effective divisor $Z = \sum z_i E_i$ satisfying $Z \cdot E_i \leq 0$ for all $1 \leq i \leq r$.

It turns out that the singularity (X, x) is rational if each irreducible component E_i of the exceptional divisor E is isomorphic to $\mathbb{C}P^1$ and

$$Z \cdot Z + \sum_{i=1}^{r} z_i(-E_i^2 - 2) = -2,$$

where $Z = \sum z_i E_i$ is the fundamental cycle of E. Note that once a dual resolution graph of a surface singularity is given, then the so-called Laufer algorithm [24] can be applied to calculate the corresponding fundamental cycle. Therefore the criterion above makes it particularly simple to identify the given singularity as rational.

Suppose that (X, x) is a germ of a normal complex surface having a rational singularity at x. Then we have the following

Theorem 5.1 ([1]). *Both the page-genus and the page-genus plus the number of binding components of the Milnor open book $\mathcal{OB}_f$ are minimized when f is taken to be the restriction of a generic linear form on $\mathbb{C}^N$ to (X, x) for some / any local embedding of (X, x) in $(\mathbb{C}^N, 0)$.*

If $\mathcal{OB}_{min}$ denotes the Milnor open book given by taking the restriction of a generic linear form on $\mathbb{C}^N$ to (X, x) for some local embedding of (X, x) in $(\mathbb{C}^N, 0)$, then Theorem 5.1 implies that $\mathrm{Mg}(\xi_{can}) = g(\mathcal{OB}_{min})$ and $\mathrm{Mb}(\xi_{can}) = \mathrm{bc}(\mathcal{OB}_{min})$, where $g(\mathcal{OB})$ (resp. $\mathrm{bc}(\mathcal{OB})$) denotes the page-genus (resp. the number of binding components) of the open book $\mathcal{OB}$. We will call $\mathcal{OB}_{min}$ the *minimal* Milnor open book. For the Milnor norm, note that, from the definition,

$$\mathrm{Mn}(\xi_{can}) = \min\{2g(\mathcal{OB}) - 2 + \mathrm{bc}(\mathcal{OB})\},$$

where the minimum is taken over all supporting Milnor open books $\mathcal{OB}$. Hence it also follows from Theorem 5.1 that

$$\mathrm{Mn}(\xi_{can}) = 2g(\mathcal{OB}_{min}) - 2 + \mathrm{bc}(\mathcal{OB}_{min})$$
$$= 2\,\mathrm{Mg}(\xi_{can}) - 2 + \mathrm{Mb}(\xi_{can}).$$

Remark 5.2. The equation $\mathrm{sn}(\xi) = 2\,\mathrm{sg}(\xi) - 2 + \mathrm{bn}(\xi)$ is not necessarily true for an arbitrary contact structure ξ, as illustrated in [3, 12].

Suppose that $\pi\colon \tilde{X} \to X$ is a good resolution of (X, x) and let $E_1, \ldots, E_r$ denote the irreducible components of the exceptional divisor E. Given an analytic function $f\colon (X, x) \to (\mathbb{C}, 0)$ vanishing at x, with an isolated singularity at x, the open book decomposition $\mathcal{OB}_f$ is a horizontal open book with binding a vertical link of type $\underline{n} = (n_1, \ldots, n_r)$, where the n_i are defined as follows: Consider the decomposition $(f \circ \pi) = (f \circ \pi)_e + (f \circ \pi)_s$ of the divisor $(f \circ \pi) \in \mathrm{Div}(\tilde{X})$ into its exceptional and strict parts such that $(f \circ \pi)_e$ is supported on E and $\dim(|(f \circ \pi)_s \cap E| < 1$. Then n_i is the number of components of $(f \circ \pi)_s$ which cut E_i. It is known that the r-tuple $\underline{n} = (n_1, \ldots, n_r)$ satisfies

$$(5.1) \qquad\qquad I(\Gamma(\pi))\underline{m}^t = -\underline{n}^t$$

for some r-tuple $\underline{m} = (m_1, \ldots, m_r)$ of positive integers, where $I(\Gamma(\pi))$ denotes the intersection matrix of the dual resolution graph $\Gamma(\pi)$ associated to π and t is used for transpose. In [1, Lemma 3.1] it is proved that

$$(5.2) \qquad\qquad g(\mathcal{OB}_f) = 1 + \sum_{i=1}^{r} \frac{(v_i - 2)m_i + (m_i - 1)n_i}{2},$$

where v_i denotes the number of irreducible curves E_j, $j \neq i$, in E intersecting E_i for $i = 1, \ldots, r$. Also

$$\mathrm{bc}(\mathcal{OB}_f) = \sum_{i=1}^{r} n_i.$$

On the other hand, it follows from the work of Artin [2] that for any r-tuple $\underline{n}$ of nonnegative integers which satisfies (5.1) for some r-tuple $\underline{m}$ of positive integers there is a Milnor open book decomposition of the boundary of (X, x) whose binding is equivalent to a vertical link of type $\underline{n}$.

The upshot is that if $Z = \sum_{i=1}^{r} z_i E_i$ is the fundamental cycle of the resolution π, then the above construction for the r-tuple $\underline{m} = (z_1, \ldots, z_r)$ gives the minimal Milnor open book $\mathcal{OB}_{min}$, which we will also denote by $\mathcal{OB}(\underline{m})$.

Remark 5.3. A generalization of Theorem 5.1 is given for all Milnor fillable rational homology 3-spheres in [31] (see also [30]), where the following formulas are proven:

$$\mathrm{Mg}(\xi_{can}) = Z \cdot E - Z \cdot Z \quad \text{and} \quad \mathrm{Mb}(\xi_{can}) = -Z \cdot E.$$

6. TIGHT CONTACT STRUCTURES ON SMALL SEIFERT FIBRED 3-MANIFOLDS

A *small* Seifert fibred 3-manifold Y is a closed and oriented 3-manifold which admits a Seifert fibration over S^2 with at most three singular fibres. Equivalently, such a manifold $Y = Y(e_0; r_1, r_2, r_3)$ can be described by the rational surgery diagram depicted in Figure 1, where $e_0 \in \mathbb{Z}$ and $r_i \in (0,1) \cap \mathbb{Q}$, for $i = 1, 2, 3$.
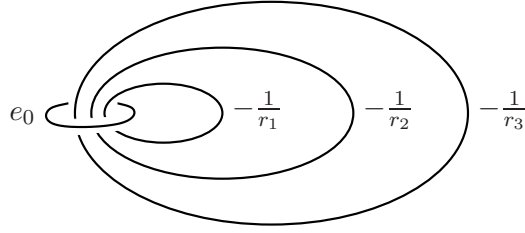

FIGURE 1. Rational surgery diagram for the small Seifert fibred 3-manifold $Y(e_0; r_1, r_2, r_3)$

One can also obtain an integral surgery description of Y as follows. Consider the continued fraction expansion of $-\frac{1}{r_i}$:

$$-\frac{1}{r_i} = a_1^{(i)} - \cfrac{1}{a_2^{(i)} - \cfrac{1}{\ddots - \cfrac{1}{a_{n_i}^{(i)}}}}, \quad i = 1, 2, 3,$$

for some uniquely determined integers $a_1^{(i)}, \ldots, a_{n_i}^{(i)} \leq -2$. Let $K_0^{(i)}$ denote the unknot with framing e_0 in Figure 1, for $i = 1, 2, 3$. Now replace the unknot with coefficient $-\frac{1}{r_i}$, by a chain of unknots $K_1^{(i)}, \ldots, K_{n_i}^{(i)}$ with integral framings $a_1^{(i)}, \ldots, a_{n_i}^{(i)}$, respectively, so that

$$\mathrm{lk}(K_j^{(i)}, K_k^{(i)}) = \begin{cases} \pm 1 & if \ |j - k| = 1 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq j, k \leq n_i$ and $i = 1, 2, 3$. As we discuss below, this "star shaped" integral surgery presentation is very convenient in terms of describing Stein fillable contact structures on Y.

In [36], all tight contact structures on Y are classified up to isotopy under the assumption that $e_0 \leq -3$. It turns out that they are all Stein fillable and can be represented by Legendrian surgery diagrams which are obtained by all possible Legendrian realizations (without double points) of the unknots imposed by the surgery coefficients in the integral surgery description of Y. Moreover, in [19], it was shown that the same classification scheme works for the case $e_0 = -2$, as long as Y is assumed to be an L-space. Recall that a rational homology sphere is called an L-space if its Heegaard Floer homology is as simple as possible, i.e., $\mathrm{rk}\, \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$.

Note that the link of any rational surface singularity is an L-space by a theorem of Némethi [29]. A necessary condition for the 3-manifold $Y = Y(e_0; r_1, r_2, r_3)$ to be the link of a rational singularity is that $e_0 \leq -2$; however, it is not sufficient.

7. PLANAR MILNOR OPEN BOOKS

By a weighted plumbing graph we mean a graph such that each vertex is decorated by some integer "weight". Such a graph naturally represents a closed oriented 3-manifold—called a graph manifold, which can be described as follows. For each vertex of the graph, one takes an oriented circle bundle over S^2 whose Euler number is equal to the weight of that vertex and plumbs these bundles together according to the given graph. This means that if there is an edge connecting two vertices in Γ, then one plumbs the circle bundles corresponding to these vertices. More precisely, one first removes a neighborhood of a circle fibre on each circle bundle which is given by the preimage of a disk on the base sphere. The resulting boundary torus on each circle bundle can be identified with $S^1 \times S^1$ using the natural trivialization of the circle fibration over the disk that is removed. Now one glues these bundles together using the diffeomorphism that exchanges the two circle factors on the boundary tori.

If the graph is a tree, in particular, then an integral surgery presentation of the 3-manifold is readily available by replacing each vertex by an unknot framed by the weight of that vertex such that any two of these unknots are linked once if there is an edge between the vertices they represent and they are unlinked otherwise.

Recall that the degree of a vertex in a graph is the number of edges emanating from that vertex. A vertex in a weighted plumbing graph is called a bad vertex if the sum of the weight (the Euler number) and the degree of that vertex is positive.

Proposition 7.1. *If Y is the link of a rational surface singularity presented by a plumbing tree without any bad vertices, then Y carries a planar Milnor open book.*

Proof. Let Γ be a plumbing tree for Y with r vertices $v_1, \ldots, v_r$ so that v_i has Euler number e_i and degree d_i. Suppose that Γ has no bad vertices, i.e., $e_i + d_i \leq 0$ for $i = 1, \ldots, r$. An explicit planar horizontal open book $\mathcal{OB}$ with binding a vertical link of type

$$\underline{n} = (-e_1 - d_1, -e_2 - d_2, \ldots, -e_r - d_r)$$

was constructed on such a graph manifold Y in [13]. Now it is easy to check that for $\underline{m} = (1, 1, \ldots, 1)$, we have

$$I(\Gamma)\underline{m}^t = -\underline{n}^t,$$

where $I(\Gamma)$ denotes the intersection matrix of the tree Γ which defines Y. Suppose that Y is the link of some rational surface singularity (X, x). Then $\underline{m}$ corresponds to the fundamental cycle of the minimal resolution of (X, x). As we indicated in the last paragraph in Section 5, the binding of the minimal Milnor open book $\mathcal{OB}_{min}$ is a vertical link of type $\underline{n}$ as well. Since the open books $\mathcal{OB}$ and $\mathcal{OB}_{min}$ on the rational homology 3-sphere Y have equivalent bindings, it follows, by a result of Caubel and Popescu-Pampu [8], that $\mathcal{OB}$ is isotopic to $\mathcal{OB}_{min}$. This proves that Y carries a planar Milnor open book, i.e., $\mathrm{Mg}(\xi_{can}) = 0$. Moreover, since the binding of $\mathcal{OB}$ is a vertical link of type $\underline{n}$, we have $\mathrm{Mb}(\xi_{can}) = -\sum_{i=1}^{r}(e_i + d_i)$ and hence $\mathrm{Mn}(\xi_{can}) = \mathrm{Mb}(\xi_{can}) - 2$. $\square$

In the proof of Proposition 7.1 we showed that the horizontal open book $\mathcal{OB}$ constructed in [13] is isotopic to a Milnor open book, which implies that the horizontal contact structure supported by $\mathcal{OB}$ is isomorphic to ξ_{can} on such rational singularity links. On the other hand, Legendrian surgery diagrams of such horizontal contact structures (which are known to be Stein fillable) were studied in [32]. A lens space $L(p, q)$, for example, is given by a linear plumbing diagram without any bad vertices. Equivalently an integral surgery diagram of $L(p, q)$ is given by a chain of unknots so that linking number between every two consecutive unknots is equal to ± 1 and the

framing of every unknot is less than or equal to -2. The next result is a rewording of Proposition 3.2 in [32].

Proposition 7.2. *Orient the unknots in the linear integral surgery diagram giving $L(p,q)$ so that the linking number is $+1$ between any two consecutive unknots. A Legendrian surgery diagram for ξ_{can} on $L(p,q)$ is obtained by Legendrian realizing each unknot with maximum possible rotation number imposed by its surgery coefficient.*

Note that a Legendrian realization of an unknot with maximum possible rotation number imposed by its surgery coefficient is given by a front projection without any double points and with a single up cusp or a single down cusp, depending on the chosen orientation. Once we orient any knot in the chain describing $L(p,q)$, the orientations of the other knots are determined uniquely, by the hypothesis in Proposition 7.2. Hence there are two choices of overall orientations inducing two Legendrian surgery diagrams for ξ_{can} which are mirror images of each other. In other words, ξ_{can} is represented by a Legendrian surgery diagram where all the zigzags of all the Legendrian unknots are on the left or all on the right (see Figure 4 in [32]). Nevertheless, these two diagrams induce isomorphic contact structures, where the underlying plane fields are obtained from each other by simply reversing the orientations.

There are two key properties used to prove Proposition 7.2: (i) the linear plumbing diagram of $L(p,q)$ does not have any bad vertices and (ii) all the tight contact structures on $L(p,q)$ are Stein fillable and given by all possible Legendrian realizations (without double points) of the unknots in the plumbing diagram. In Section 6, we described a star shaped plumbing diagram of a small Seifert fibred 3-manifold $Y = Y(e_0; r_1, r_2, r_3)$ which does not include any bad vertices as long as $e_0 \leq -3$. Therefore it is straightforward to generalize Proposition 7.2 to all rational singularity links which are small Seifert fibred spaces with $e_0 \leq -3$, using the methods in [32] coupled with Wu's classification [36] of tight contact structures on such manifolds. This generalization is included in the statement of Theorem 8.1 for which we present a more conceptual proof.

8. LEGENDRIAN SURGERY DIAGRAMS FOR

CANONICAL CONTACT STRUCTURES

Theorem 8.1. *Let $Y = Y(e_0; r_1, r_2, r_3)$ be a small Seifert fibred 3-manifold which is diffeomorphic to the link of some rational surface singularity. Orient the unknots in the star-shaped integral surgery diagram giving Y, so that the linking number is $+1$ between any two consecutive unknots in every chain. A Legendrian surgery diagram for ξ_{can} on Y is obtained by Legendrian realizing each unknot with maximum possible rotation number imposed by its surgery coefficient.*

Proof. Suppose that (Y, ξ_{can}) is diffeomorphic to the link of the rational surface singularity (X, x). Then the minimal resolution $\pi \colon \tilde{X} \to X$ provides a holomorphic filling (W, J) of (Y, ξ_{can}). In particular, W is a regular neighborhood of the exceptional divisor $E = \bigcup E_j$ of π. Since the curves E_j are holomorphic, by the adjunction formula, we have

$$\langle c_1(J), [E_j]\rangle = E_j \cdot E_j - 2 \operatorname{genus}(E_j) + 2 = E_j \cdot E_j + 2.$$

Recall that in Section 6 we discussed the classification of tight contact structures on a small fibred 3-manifold Y under the assumption that Y is an L-space for the case $e_0 = -2$, which is satisfied for a rational singularity link. For each such 3-manifold Y, there are finitely many tight contact structures $\xi_1, \ldots, \xi_m$, all of which are Stein fillable. Moreover, for $i = 1, \ldots, m$, a Stein filling (W^i, J^i) of ξ_i is given by taking a Legendrian surgery diagram, obtained from the plumbing tree describing Y, with the zigzags chosen in a certain way. Denote by U_j^i a component of the corresponding Legendrian link and let S_j^i denote the associated surface in the Stein filling (W^i, J^i) obtained by pushing a Seifert surface for U_j^i into to the 4-ball and capping off by the core of the corresponding 2-handle (see [21]). Notice that each W^i is diffeomorphic to W by a diffeomorphism which carries S_j^i to E_j for each j.

Now, using the well-known identities

$$S_j^i \cdot S_j^i = \operatorname{tb}(U_j^i) - 1, \qquad \langle c_1(J^i), [S_j^i]\rangle = \operatorname{rot}(U_j^i)$$

(see [21] for the second), observe that $\langle c_1(J^i), [S_j^i]\rangle = S_j^i \cdot S_j^i + 2$ precisely when $\operatorname{rot}(U_j^i) = \operatorname{tb}(U_j^i) + 1$. Since the latter equality holds exactly when all the cusps of U_j^i except one are up cusps, it follows that $\langle c_1(J), [E_j]\rangle = \langle c_1(J^i), [S_j^i]\rangle$ for each j precisely when all the extra zigzags are chosen so that the additional cusps are all up cusps, that is, when all the extra zigzags are chosen on the same fixed side (which is

determined by the orientation of the Legendrian unknots). The proof is now completed by noting that in the finite list of tight contact structures on Y there is only one such Stein fillable contact structure up to isomorphism (cf. [19, 36]).   □

9. MILNOR VERSUS SUPPORT GENUS

In this section, we describe an infinite family of Milnor fillable contact 3-manifolds so that for each member of this family, the support genus (resp. norm) of the canonical contact structure is *strictly* less than its Milnor genus (resp. norm).

Consider the small Seifert fibred 3-manifold $Y_p = Y(-2; \frac{1}{3}, \frac{2}{3}, \frac{p}{p+1})$, for $p \geq 2$. First, we observe that Y_p, whose dual resolution graph Γ_p is shown in Figure 2, is the link of a rational complex surface singularity. By the classification of the tight contact structures on Y_p given by Ghiggini [19], there are exactly two nonisotopic tight contact structures ξ_1 and ξ_2 on Y_p, both of which are Stein fillable. A Legendrian surgery diagram of ξ_1 is depicted on the bottom right in Figure 2. By putting the extra zigzag on the blue curve on the opposite side, we get a diagram for ξ_2.
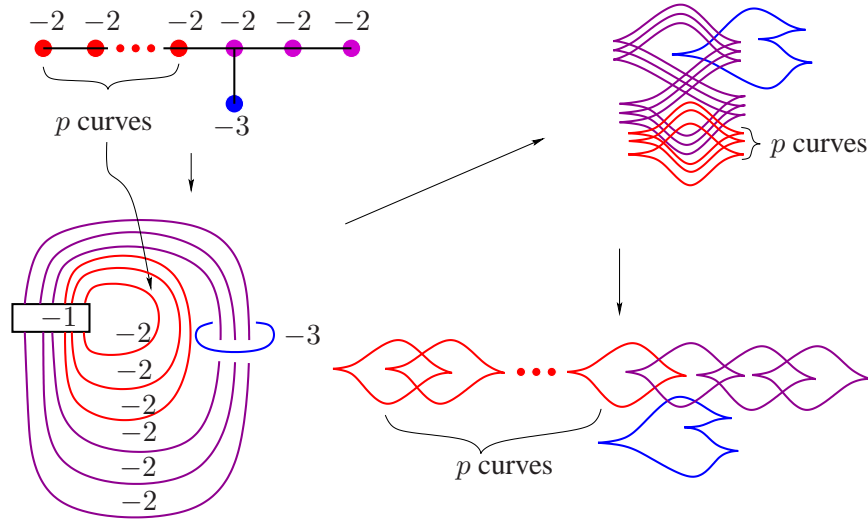

FIGURE 2. The contact structure $\xi \cong \xi_1 \cong \xi_{can}$ on Y_p

Proposition 9.1. *For $i = 1, 2$, we have* $\mathrm{sg}(\xi_i) \leq 1$ *and* $\mathrm{sn}(\xi_i) = 2$.

Proof. We first construct a supporting elliptic (i.e. genus one) open book with two binding components for some Stein fillable contact structure ξ on Y_p following the recipe in [16]. We start from the plumbing diagram on the top left in Figure 2 (which is equivalent to a smooth surgery diagram including only unknots linked according to the given tree) and "roll up" this diagram by appropriately sliding handles to obtain the surgery diagram of Y_p on the bottom left. Next we Legendrian realize the given surgery curves in a certain way (as depicted on the top right in Figure 2) to obtain the Legendrian surgery diagram for some Stein fillable contact structure ξ on Y_p, which is isomorphic to ξ_1 depicted on the bottom right. We refer to [16] for the justification of such statements.
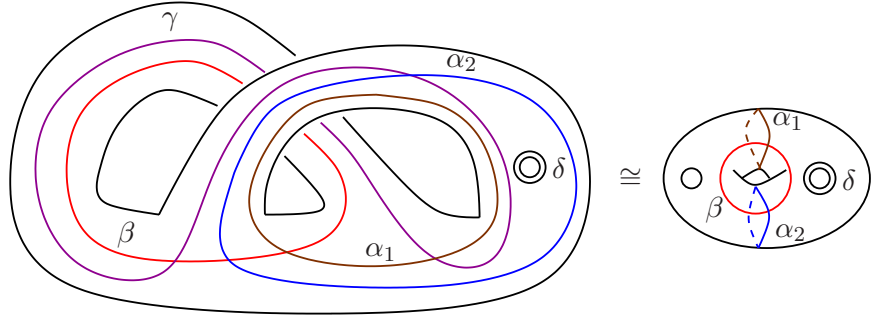

FIGURE 3. The page of an open book compatible with ξ

In order to construct an open book of Y_p supporting ξ, we start from an open book of S^3 and then embed the surgery curves onto the pages as depicted on the left in Figure 3. Note that we used a color coding to make it easier for the reader to follow how the surgery curves are embedded on the page. The initial page is a torus with one boundary component and the monodromy of this open book of S^3 before the surgery is given by $\beta\alpha_1$, where α_1 and β generate the first homology group of the page. Now we apply Legendrian surgeries along the given curves to get an open book of Y_p with monodromy $\phi_p = \alpha_2\gamma^3\beta^p\beta\alpha_1\delta$, where δ is parallel to the small puncture on the torus, which occurs as a result of stabilizing the page appropriately. Next we move β over γ^3 to the left and use the fact that $\gamma\beta = \beta\alpha_1$ to get $\phi_p = \alpha_2\beta\alpha_1^3\beta^p\alpha_1\delta$. Then we use the well-known braid relations and some simple overall conjugations to obtain a more symmetrical presentation of the monodromy as

$$\phi_p = (\alpha_2\beta)^2(\alpha_1\beta)^2\beta^{p-2}\delta.$$

What we described here is an *abstract* open book which is compatible with ξ, where the page is a torus with two boundary components and monodromy is ϕ_p. Note that ξ is isomorphic to ξ_1 (which is isomorphic to ξ_2, since one can not distinguish the abstract open books corresponding to ξ_1 and ξ_2). It follows that $\mathrm{sg}(\xi_i) \leq 1$, since we have already constructed a genus one open book compatible with ξ.

Using the handlebody diagram of Y_p depicted on the bottom left in Figure 2, it is straightforward to calculate that

$$H_1(Y_p, \mathbb{Z}) = \begin{cases} \mathbb{Z}_3 \bigoplus \mathbb{Z}_3 & p = 2 \bmod 3, \\ \mathbb{Z}_9 & \text{otherwise.} \end{cases}$$

Moreover, one can show that the Poincaré dual $PD(e(\xi_i)) \in H_1(Y_p, \mathbb{Z})$ of the Euler class $e(\xi_i)$ is a generator of one of the $\mathbb{Z}_3$-factors when p is congruent to $2 \bmod 3$. Similarly $PD(e(\xi_i))$ is a generator of $H_1(Y_p, \mathbb{Z})$ when p is not congruent to $2 \bmod 3$. Therefore the contact structure ξ_i can not be compatible with an elliptic open book with connected binding by Lemma 6.1 in [17], since $e(\xi_i) \neq 0$. Note that $e(\xi_1) = -e(\xi_2)$, which implies that ξ_1 is not homotopic to ξ_2 as oriented plane fields, although they are isomorphic to each other. In fact, ξ_2 is obtained from ξ_1 by reversing the orientation of the underlying plane field.

Now we claim that $\mathrm{sn}(\xi_i) = 2$. To prove our claim we need to exclude the possibility that ξ_i is compatible with a planar open book with less than four binding components. Suppose that ξ_i is compatible with a planar open book, i.e, $\mathrm{sg}(\xi_i) = 0$. If $\mathrm{bn}(\xi_i) \leq 2$, then ξ_i is the unique tight contact structure on the lens space $L(n, n-1)$ for some $n \geq 0$ (cf. [17]) which is indeed impossible since Y_p is not a lens space.

Next we rule out the possibility that $\mathrm{bn}(\xi_i) = 3$. Let Σ be the planar surface with three boundary components. Any diffeomorphism of Σ is determined by three numbers q, r, s, that give the number of Dehn twists on curves τ_1, τ_2, τ_3 parallel to each boundary component. It is easy to see that the 3-manifold determined by the open book with page Σ and monodromy given by $\tau_1^q \tau_2^r \tau_3^s$ is the Seifert fibred 3-manifold $Y(0, -\frac{1}{q}, -\frac{1}{r}, -\frac{1}{s})$. Note that the first homology group of $Y(0, -\frac{1}{q}, -\frac{1}{r}, -\frac{1}{s})$ has order $qr + qs + rs$.

Suppose that ξ_i is compatible with an open book with page Σ and monodromy $\tau_1^q \tau_2^r \tau_3^s$. The tightness of ξ_i implies that the integers q, r, and s are all nonnegative, because otherwise $\tau_1^q \tau_2^r \tau_3^s$ is not right-veering [23]. Moreover, since the order of the first homology group of Y_p is 9, for all $p \geq 2$, we conclude that (q, r, s) is equal to either $(0, 1, 9)$, $(0, 3, 3)$ or $(1, 1, 4)$. Hence Y_p is diffeomorphic to either $L(9, 8)$, $L(3, 2)\#L(3, 2)$ or $L(9, 4)$, which is a contradiction. Hence, $\mathrm{bn}(\xi_i) \geq 4$. This finishes the proof of our claim that $\mathrm{sn}(\xi_i) = 2$. $\square$

One can ask whether or not $\mathrm{sg}(\xi_i) = 1$, although it is not essential for the purposes of this paper. There are two known methods (cf. [14, 33]) of finding obstructions to planarity of a contact structure, but unfortunately both fail in our case. That is because Y_p is an L-space and it is not an integral homology sphere.

Proposition 9.2. *For the canonical contact structure ξ_{can} on the rational singularity link Y_p we have $\mathrm{Mg}(\xi_{can}) = 2$ and $\mathrm{Mn}(\xi_{can}) = 3$.*

Proof. Enumerate the vertices of the plumbing graph of Y_p depicted in Figure 2 from left to right along the top row with the bottom vertex coming last. It is then easy to check that the $(p + 4)$-tuple of positive integers $\underline{m}$ corresponding to the fundamental cycle of the minimal resolution of the singularity of which Y_p is the link is given by $\underline{m} = (1, 2, 3, 3, \ldots, 3, 3, 2, 1, 1)$. The construction in [4] now gives an open book decomposition $\mathcal{OB}(\underline{m}) = \mathcal{OB}_{min}$ of Y_p with binding a vertical link of type $\underline{n} = (0, 0, 1, 0, \ldots, 0)$, where $\underline{m}$ and $\underline{n}$ are related by

$$I(\Gamma_p)\underline{m}^t = -\underline{n}^t.$$

Using formula (5.2) with $r = p + 4$, one has $\mathrm{Mg}(\xi_{can}) = g(\mathcal{OB}(\underline{m})) = 2$ for the canonical contact structure ξ_{can} on Y_p. Also one has $\mathrm{Mb}(\xi_{can}) = \mathrm{bc}(\mathcal{OB}(\underline{m})) = \sum_{i=1}^{p+4} n_i = 1$ and therefore $\mathrm{Mn}(\xi_{can}) = 3$, completing the proof of the proposition. $\square$

Corollary 9.3. *For $p \geq 2$, we have $\mathrm{sg}(\xi_{can}) < \mathrm{Mg}(\xi_{can})$ and $\mathrm{sn}(\xi_{can}) < \mathrm{Mn}(\xi_{can})$ for the canonical contact structure ξ_{can} on the singularity link Y_p.*

Proof. Since any Milnor fillable contact structure is Stein fillable, ξ_{can} is isomorphic to ξ_i by Ghiggini's classification [19]. Note that it does not make sense to distinguish
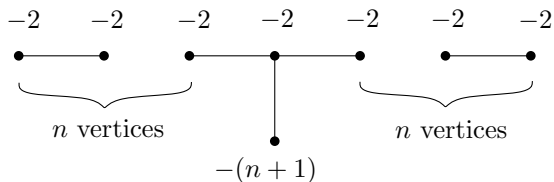
FIGURE 4. The plumbing graph for P_n

ξ_1 and ξ_2 here since they are isomorphic to each other. Thus Proposition 9.1 coupled with Proposition 9.2 clearly implies the corollary. $\qquad \square$

Remark 9.4. Note, however, that $\mathrm{Mb}(\xi_{can}) = 1$ while $\mathrm{bn}(\xi_{can}) \geq 2$, which shows that the binding number is not necessarily less than or equal to the Milnor binding number.

We can improve Corollary 9.3 as follows.

Theorem 9.5. *For each positive integer k, there is a Milnor fillable contact 3-manifold such that $\mathrm{Mg}(\xi_{can}) - \mathrm{sg}(\xi_{can}) \geq k$ and $\mathrm{Mn}(\xi_{can}) - \mathrm{sn}(\xi_{can}) \geq k$.*

Proof. The small Seifert fibred 3-manifold $P_n = Y(-2; \frac{1}{n+1}, \frac{n}{n+1}, \frac{n}{n+1})$, for $n \geq 2$, is a rational singularity link, whose dual resolution graph is depicted in Figure 4. There are exactly n nonisotopic tight contact structures $\xi_1, \ldots, \xi_n$ on P_n, each of which is Stein fillable [19]. On the other hand, ξ_i is supported by an elliptic open book with n binding components, for $1 \leq i \leq n$ [16]. This proves that $\mathrm{sg}(\xi_{can}) \leq 1$ and $\mathrm{sn}(\xi_{can}) \leq n$ since the canonical contact structure ξ_{can} on P_n is isomorphic to ξ_i for some $1 \leq i \leq n$.

Now enumerate the vertices of the graph in Figure 4 from left to right along the top row with the bottom vertex coming last and consider the $(2n + 2)$-tuple $\underline{m} = (1, 2, 3, \ldots, n-1, n, n+1, n, n-1, \ldots, 3, 2, 1, 1)$ of positive integers. This corresponds to the fundamental cycle of the minimal resolution of the singularity of which P_n is the link. It follows that $\mathrm{Mg}(\xi_{can}) = n$ and $\mathrm{Mn}(\xi_{can}) = 2n-1$. Taking $k = n-1$ now proves the theorem. $\qquad \square$
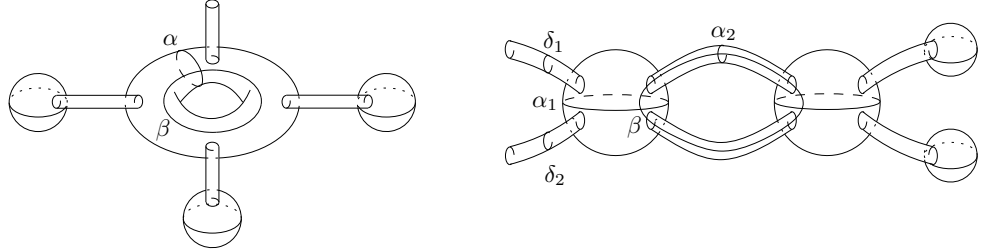
FIGURE 5. Pages of two different Milnor open books on $Y(-2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

10. FINAL REMARKS

The minimal Milnor open book $\mathcal{OB}_{min}$ on $Y = Y(e_0; r_1, r_2, r_3)$ realizes $\mathrm{Mg}(\xi_{can})$, $\mathrm{Mb}(\xi_{can})$ and $\mathrm{Mn}(\xi_{can})$. In fact, it follows from the proof of Theorem 5.1 given in [1] that $\mathcal{OB}_{min}$ is the unique Milnor open book that realizes $\mathrm{Mg}(\xi_{can})$, $\mathrm{Mb}(\xi_{can})$ and $\mathrm{Mn}(\xi_{can})$. Thus any other Milnor open book on Y that realizes $\mathrm{Mg}(\xi_{can})$ cannot realize $\mathrm{Mb}(\xi_{can})$ and $\mathrm{Mn}(\xi_{can})$. For example, consider the 3-manifold $Y = Y(-2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which is the link of the singularity D_4. The pages of two Milnor open books on Y are given in Figure 5. The first one is the minimal Milnor open book $\mathcal{OB}_{min} = \mathcal{OB}((1, 2, 1, 1))$ with page a once-punctured torus and monodromy $\phi = (\alpha\beta)^3$; the second one is the Milnor open book $\mathcal{OB} = \mathcal{OB}((2, 2, 1, 1))$ with page a twice-punctured torus and monodromy ψ satisfying $\psi^2 = \delta_1\delta_2\alpha_2^2$. Using the uniqueness result from [6] and the two-holed torus relation one can check that $\psi = \alpha_1\alpha_2\beta\alpha_2^2\beta\alpha_2$. It is easy to see that $\mathcal{OB}$ is related to $\mathcal{OB}_{min}$ by a single positive stabilization.

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