

CANONICAL CONTACT STRUCTURES ON SOME SINGULARITY LINKS

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ABSTRACT. We identify the canonical contact structure on the link of a simple elliptic or cusp singularity by drawing a Legendrian handlebody diagram of one of its Stein fillings. We also show that the canonical contact structure on the link of a numerically Gorenstein surface singularity is trivial considered as a real plane bundle.

1. INTRODUCTION

There is a canonical (a.k.a. Milnor fillable) contact structure ξ_{can} on the link of an isolated complex surface singularity, which is unique up to isomorphism [6]. It is known that a Milnor fillable contact structure is Stein fillable [5] and universally tight [17] (note that a Stein fillable contact structure is not necessarily universally tight [12, page 670]). Moreover there are only finitely many isomorphism classes of Stein fillable contact structures on a closed orientable 3-manifold [2, Lemma 3.4]. In favorable circumstances, these facts are sufficient to pin down the canonical contact structure on a given singularity link.

In this article we determine the canonical contact structure on the link of a simple elliptic or cusp singularity by drawing a Legendrian handlebody diagram (cf. [12, 13]) of one of its Stein fillings. In addition, we convert this diagram into a surgery diagram of the canonical contact structure by replacing any 1-handle in the Stein filling by a contact (+1)-surgery along a Legendrian unknot in the standard contact S^3 (cf. [7, 8]).

As a byproduct of our constructions, we answer positively, for the classes of singularities above, the question posed by Caubel, Nemethi and Popescu-Pampu of whether all Milnor fillable contact structures on a Milnor fillable 3-manifold are not just isomorphic but *isotopic* (cf. [6, Remark 4.10]).

The classes of simple elliptic and cusp singularities are known to be numerically Gorenstein. Here we show that the canonical contact structure on the link of such a surface singularity is trivial considered as a real plane bundle.

The contact structures which appear in this paper are assumed to be positive and co-orientable. The reader is advised to turn to [19] for more on the canonical contact structures.

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2. NUMERICALLY GORENSTEIN SURFACE SINGULARITIES

The class of *numerically Gorenstein* surface singularities was introduced by A. Durfee in [9] as the class of singularities for which the holomorphic tangent bundle is smoothly trivial on a punctured neighborhood of the singular point. As Durfee observed (see Lemma 3), a surface singularity is numerically Gorenstein if and only if its canonical bundle is smoothly trivial.

By contrast, a normal surface singularity is *Gorenstein* if and only if its canonical bundle is holomorphically trivial. This shows that a Gorenstein normal surface singularity is numerically Gorenstein, but the converse is not true in general. However, every numerically Gorenstein normal surface singularity is homeomorphic to a Gorenstein one [24], which shows that, in order to study their canonical contact structures, it suffices to study those of normal Gorenstein ones.

Proposition 1. *The canonical contact structure of a numerically Gorenstein surface singularity is trivial, considered as a real plane bundle.*

Proposition 1 follows immediately from the following result that was communicated to us by the referee, whose proof is due to P. Popescu-Pampu.

Lemma 2. *Let Y be the link of a normal surface singularity (S, p) . Then the canonical contact structure on Y , viewed as an abstract bundle, is smoothly isomorphic to the restriction to Y of the anticanonical bundle K_S^* of S .*

Proof. Fix an auxiliary Riemannian metric on a neighborhood of $p \in S$ containing Y and consider the (real) normal bundle ν of Y in S . Observe that ν is trivial since Y is oriented. Hence its complexification $\nu^{\mathbb{C}}$ is trivial as a smooth complex bundle. Now since ξ is given by complex tangencies to Y , $TS|_Y = \xi \oplus \nu^{\mathbb{C}}$. Thus, as smooth complex bundles,

$$\Lambda^2 TS|_Y \simeq \xi \otimes_{\mathbb{C}} \nu^{\mathbb{C}} \simeq \xi.$$

The lemma now follows from the fact that $\Lambda^2 TS$ is the dual of $\Lambda^2 T^*S = K_S$. □

Proof of Proposition 1. Suppose that (S, p) is numerically Gorenstein. Then K_S restricted to Y is smoothly trivial as a complex line bundle. Hence ξ is also smoothly trivial as a real plane bundle by Lemma 2. □

In fact, for this dimension the converse of Proposition 1 is true, as stated in Lemma 1.1 of Durfee's paper quoted before:

Lemma 3 (Durfee). *Let ζ be a two-dimensional complex bundle over a CW complex X with $H^i(X) = 0$ for $i > 3$. Then the following conditions are equivalent:*

- (a) ζ is trivial.
- (b) ζ is stably trivial.
- (c) The first Chern class $c_1(\zeta)$ is zero.

(d) *The second exterior power $\Lambda^2\zeta$ is a trivial line bundle.*

In this article, we consider canonical contact structures on links of simple elliptic and cusp singularities. As every simple elliptic and cusp singularity is minimally elliptic, it follows by Laufer ([16]) that every such singularity is Gorenstein and hence *a fortiori* numerically Gorenstein.

3. SIMPLE ELLIPTIC AND CUSP SINGULARITIES

Part of the discussion in this article is based on the topological characterization of singularity links which fiber over the circle given by Neumann [20]: A singularity link fibers over the circle if and only if it is a torus bundle over the circle whose monodromy $A \in \mathrm{SL}(2, \mathbb{Z})$ is either parabolic, i.e., $\mathrm{tr}(A) = 2$ or hyperbolic with $\mathrm{tr}(A) \geq 3$. Moreover these links correspond precisely to simple elliptic and cusp singularities, respectively.

On the other hand, the classification of tight contact structures on torus bundles by Honda [14] coupled with a theorem of Gay [11] implies that on any torus bundle over the circle there is a unique Stein fillable universally tight contact structure up to isomorphism. We conclude that on a singularity link which fibers over the circle, the canonical contact structure is the unique universally tight Stein fillable contact structure.

In the following subsections we determine the canonical contact structures on simple elliptic and cusp singularities—each of which requires a separate treatment.

3.1. Simple elliptic singularities. For a positive integer n , let (S_n, p) denote the complex surface singularity whose minimal resolution consists of a single elliptic curve of *negative* self-intersection number $-n$. Such singularities are known as *simple elliptic singularities*. The link Y_n of such a singularity is a torus bundle over the circle with parabolic monodromy $A \in \mathrm{SL}(2, \mathbb{Z})$. Moreover, the 3-manifold Y_n also admits an oriented circle fibration over the torus with Euler number $-n$.

An open book decomposition \mathcal{OB}_n of Y_n transverse to the circle fibration was constructed in [10] such that the binding consists of n distinct *positively oriented* circle fibres, the page is a torus with n boundary components and the monodromy is the product of n boundary parallel right-handed Dehn twists. According to [21, Theorem 2.1], there is a Milnor open book $\widehat{\mathcal{OB}}_n$ on Y_n whose binding agrees with the binding of \mathcal{OB}_n . On the other hand, by [6, Theorem 4.6], any two horizontal open books on Y_n with the same binding are isomorphic. It follows that \mathcal{OB}_n is in fact a Milnor open book and hence it supports the canonical contact structure which is certainly Stein fillable and universally tight.

Proposition 4. *There are precisely two distinct isotopy classes of Stein fillable universally tight contact structures on Y_n . These are given as the boundaries of the two distinct Stein surfaces shown in Figure 1. Both of these contact structures represent the isomorphism class of the canonical contact structure ξ_{can} on Y_n .*

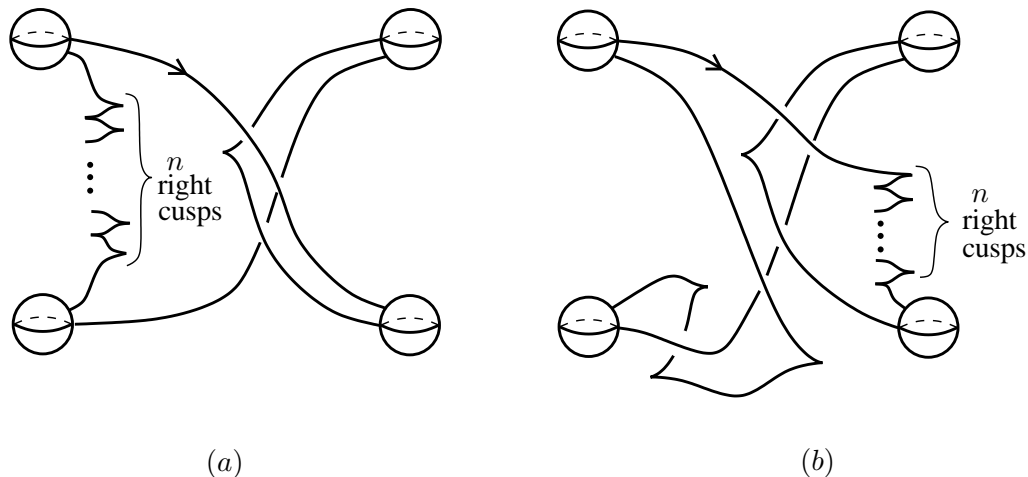


FIGURE 1. Legendrian handlebody diagrams for two distinct Stein structures on X_n , both inducing the canonical contact structure ξ_{can} on Y_n , up to isomorphism. (We identify the spheres aligned *horizontally* in both diagrams (a) and (b).)

Proof. There are $n + 1$ distinct isotopy classes of Stein fillable contact structures on Y_n , two of which are universally tight according to Honda's classification [14]. Let X_n denote the oriented D^2 -bundle over T^2 with Euler number $-n$, a smooth handlebody diagram of which is depicted in [12, Figure 36 (a)]. In order to obtain Legendrian handlebody diagrams of distinct Stein surfaces inducing the aforementioned distinct contact structures on Y_n , we use Gompf's trick illustrated in [12, Figure 36 (b), (c)]: We just convert the standard smooth handlebody diagram of X_n (which consists of two 1-handles and one 2-handle) into a Legendrian handlebody diagram by Legendrian realizing the attaching circle of the 2-handle. It is easy to see that any integer which belongs to the set

$$\{-n, -(n-2), \dots, n-2, n\}$$

appears as the rotation number of some Legendrian realization.

According to [12, Proposition 2.3], the first Chern class of the resulting Stein structure on the smooth 4-manifold X_n is represented by a cocycle whose value on the homology class induced by the oriented Legendrian knot (generating $H_2(X_n; \mathbb{Z}) \cong \mathbb{Z}$) is given by the rotation number of this Legendrian knot. Therefore by [18, Theorem 1.2], the induced contact structures on Y_n are all pairwise nonisotopic for all distinct rotation numbers that are listed above. Moreover, all of these contact structures except the two that are induced by the Stein surfaces depicted in Figure 1 are virtually overtwisted by [12, Proposition 5.1].

The rotation numbers of the Legendrian knots in Figure 1(a) and Figure 1(b) are $-n$ and n , respectively, with the indicated orientations. Here if one reverses the orientation

of the Legendrian knot, the sign of the rotation number as well as the sign of the second homology class induced by this knot in $H_2(X_n; \mathbb{Z})$ gets reversed. In conclusion, the two extreme cases where the rotation number of the Legendrian realization of the attaching circle of the 2-handle takes its minimal possible value $-n$ and maximum possible value n , respectively, must induce the two universally tight Stein fillable contact structures on Y_n in Honda's classification.

To prove the last statement in the proposition, we first observe that one of two nonisotopic Stein fillable universally tight contact structures on Y_n is ξ_{can} . Let $\bar{\xi}_{can}$ denote the 2-plane field ξ_{can} with the opposite orientation. Recall that ξ_{can} is supported by the Milnor open book \mathcal{OB}_n . By reversing the orientation of the page (and hence the orientation of the binding) of \mathcal{OB}_n we get another open book $\overline{\mathcal{OB}}_n$ on Y_n . The open book $\overline{\mathcal{OB}}_n$ is in fact isomorphic to \mathcal{OB}_n , since they have identical pages and the same monodromy map measured with the respective orientations. To see that $\overline{\mathcal{OB}}_n$ is also horizontal we simply reverse the orientation of the fibre (to agree with the orientation of the binding) as well as the orientation of base T^2 of the circle bundle Y_n , so that we do not change the orientation of Y_n . In addition we observe that the contact structure supported by $\overline{\mathcal{OB}}_n$ can be obtained from ξ_{can} by changing the orientations of the contact planes since $\overline{\mathcal{OB}}_n$ is obtained from \mathcal{OB}_n by changing the orientations of the pages. We conclude that $\bar{\xi}_{can}$ is a contact structure on Y_n that is isomorphic (but *nonisotopic*) to ξ_{can} . \square

Notice that we can trade the 1-handles in the handlebody diagram depicted in Figure 1 with contact $(+1)$ -surgeries as described in [8] and obtain the contact surgery diagram of ξ_{can} on Y_n as shown in Figure 2.

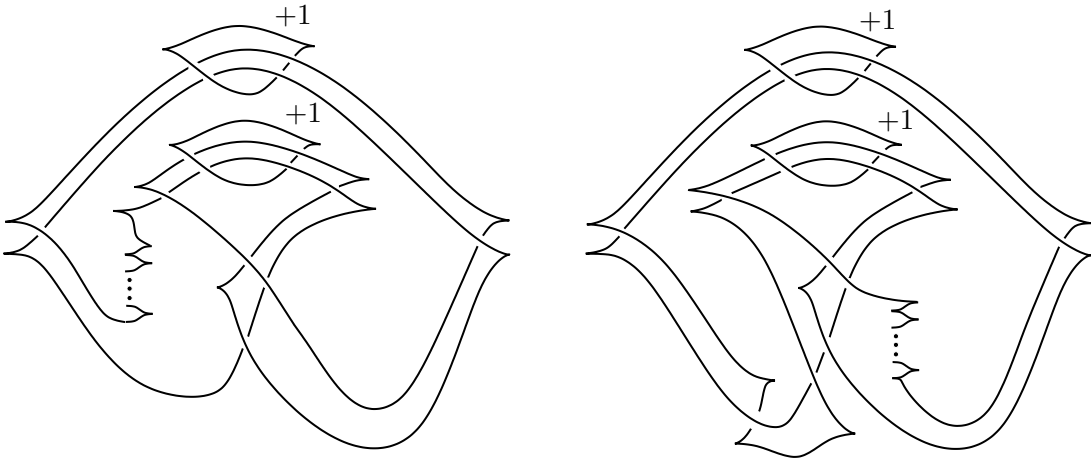


FIGURE 2. Two contact surgery diagrams inducing non-isotopic contact structures that are isomorphic to the canonical contact structure ξ_{can} on Y_n .

Remark 5. *Simple elliptic singularities are Gorenstein, and by a result of Seade [25], the canonical class of any smoothing of S_n is trivial. Combining this with a result of Pinkham [23], which states that S_n admits a smoothing if and only if $n \leq 9$, we see that the Euler class of the canonical contact structure on Y_n vanishes if $n \leq 9$. Proposition 1 implies that the Euler class of the canonical contact structure on Y_n vanishes in the case $n > 9$ as well.*

We would like to point out that the first Chern classes of both Stein fillings of the contact 3-manifold (Y_n, ξ_{can}) depicted in Figure 1 are non-vanishing, although their restriction to the boundary—the Euler class of ξ_{can} —vanishes.

Remark 6. *Ohta and Ono [22] showed that (Y_n, ξ_{can}) admits a strong symplectic filling with vanishing first Chern class if and only if $n \leq 9$. Moreover they proved that any such filling is diffeomorphic to a smoothing of the singularity—which is unique unless $n = 8$. For $1 \leq n \leq 9$, a Stein filling of (Y_n, ξ_{can}) with vanishing first Chern class can be constructed as a PALF (positive allowable Lefschetz fibration [1]) using the n -holed torus relation discovered in [15].*

3.2. Cusp singularities. A normal surface singularity having a resolution with exceptional divisor consisting of a cycle of smooth rational curves, or with exceptional divisor a single rational curve with a node is called a *cusp singularity*. The link of a cusp singularity also fibers over the circle with torus fibers and hyperbolic monodromy $A \in \mathrm{SL}(2, \mathbb{Z})$. Based on a factorization of the monodromy $A = A(n_1, \dots, n_k)$, a description of this torus bundle is given as a circular plumbing graph in Neumann’s paper [20]. For $k > 1$, the Euler number of the i th vertex in the plumbing is equal to $-n_i$, as depicted in Figure 3(a), where $n_i \geq 2$ for all i , and $n_i \geq 3$ for some i . For $k = 1$, the plumbing graph consists of a single vertex decorated with an integer $-n_1 \leq -3$ and a loop at this vertex as shown in Figure 3(b)—which corresponds to a self-plumbing of an oriented circle bundle over S^2 with Euler number $-n_1$. To simplify the notation, we denote the total space of this torus bundle by $Y_{\bar{n}}$, where $\bar{n} = (n_1, \dots, n_k)$.

Lemma 7. *There is a horizontal open book decomposition $\mathcal{OB}_{\bar{n}}$ on $Y_{\bar{n}}$ with page-genus one, whose monodromy is given explicitly as a product of some right-handed Dehn twists.*

Proof. We will construct a horizontal open book $\mathcal{OB}_{\bar{n}}$ with page-genus one using the methods in [10]. We first consider the case $k > 1$. Notice that the i th vertex in the plumbing is a circle bundle over the sphere with Euler number $-n_i \leq -2$. For such a circle bundle there is a horizontal open book \mathcal{OB}_i where the page is a sphere with n_i boundary components and the monodromy is the product of n_i boundary parallel right-handed Dehn twists.

When two consecutive circle bundles with Euler numbers $-n_i$ and $-n_{i+1}$ are connected by an edge in the plumbing graph $\Gamma_{\bar{n}}$, we can “glue” the corresponding open books together as follows: First of all, for $i = 1, \dots, k - 1$, we glue a page of \mathcal{OB}_i with a page of \mathcal{OB}_{i+1} using precisely one boundary component from each page. The Dehn twists along

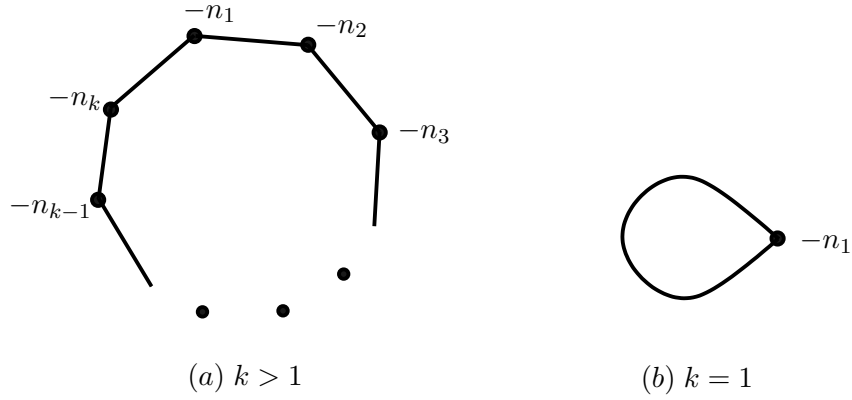


FIGURE 3. The circular plumbing graph $\Gamma_{\bar{n}}$ for the singularity link $Y_{\bar{n}}$. The vertex labeled by $-n_i$ represents a circle bundle over S^2 with Euler number $-n_i$.

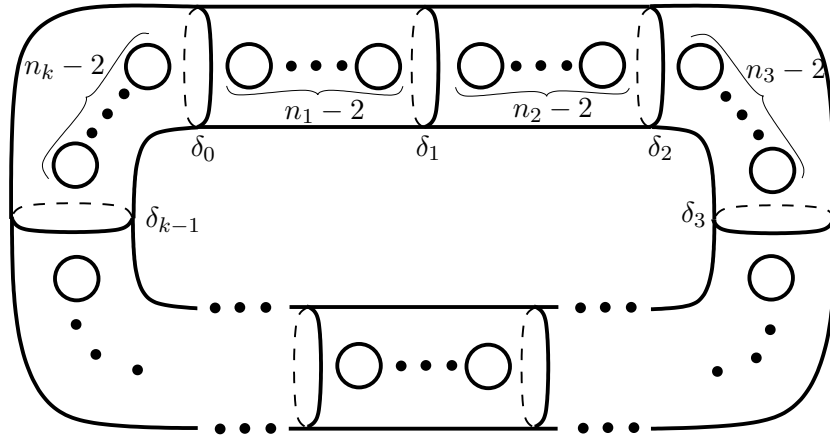


FIGURE 4. The page of the open book $\mathcal{OB}_{\bar{n}}$ of $Y_{\bar{n}}$

the identified boundary components merge into a single right-handed Dehn twist along the resulting curve δ_i (see Figure 4) after gluing the pages. By gluing all the open books corresponding to the vertices $i = 1, \dots, k - 1$, we get a planar open book. But since the plumbing graph is circular we need to glue the \mathcal{OB}_k with \mathcal{OB}_1 along δ_0 so that the resulting page of the open book $\mathcal{OB}_{\bar{n}}$ is a torus with $\sum_{i=1}^k (n_i - 2)$ many boundary components as illustrated in Figure 4. Let $\gamma_{i,1}, \gamma_{i,2}, \dots, \gamma_{i,n_i-2}$ be the boundary-parallel curves between δ_{i-1} and δ_i for $1 \leq i \leq k - 1$, and let $\gamma_{k,1}, \gamma_{k,2}, \dots, \gamma_{k,n_k-2}$ be the boundary-parallel curves

between δ_{k-1} and δ_0 . Then the monodromy of $\mathcal{OB}_{\bar{n}}$ is given by

$$\prod_{i=0}^{k-1} D(\delta_i) \prod_{i=1}^k \prod_{j=1}^{n_i-2} D(\gamma_{i,j}),$$

where $D(\cdot)$ denotes a right-handed Dehn twist.

We now consider the case $k = 1$. Start off with a circle bundle over a sphere with Euler number $-n_1 \leq -3$. As before, for such a circle bundle there is a horizontal open book \mathcal{OB}_1 where the page is a sphere with n_1 boundary components and the monodromy is the product of n_1 boundary parallel right-handed Dehn twists. Adding a self-plumbing corresponds to gluing together two boundary components of \mathcal{OB}_1 and replacing the corresponding two boundary-parallel Dehn twists by a single right-handed Dehn twist along the resulting curve δ_0 . Let $\gamma_1, \gamma_2, \dots, \gamma_{n_1-2}$ denote curves parallel to the remaining boundary curves. The monodromy of the resulting horizontal open book $\mathcal{OB}_{(n_1)}$ is then given by

$$D(\delta_0) \prod_{i=1}^{n_1-2} D(\gamma_i),$$

as shown in Figure 5.

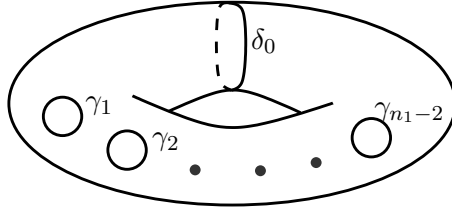


FIGURE 5. The page of the open book $\mathcal{OB}_{(n_1)}$ of $Y_{(n_1)}$

□

Remark 8. *The case $k = 1$ corresponds to a normal surface singularity having a resolution with exceptional divisor consisting of a single rational curve with a node and having self-intersection $-n_1 + 2 \leq -1$. Blowing up gives the minimal good resolution with exceptional divisor consisting of two nonsingular irreducible rational curves with self-intersections -1 and $-n_1 - 2 \leq -5$ and dual graph a cycle. For the minimal good resolution, the method in [10] is no longer applicable; however, the method in [3] is still applicable and gives an alternative way of constructing the open book decomposition $\mathcal{OB}_{(n_1)}$ on $Y_{(n_1)}$.*

Remark 9. *According to [26, Thm 4.3.1], the open book $\mathcal{OB}_{\bar{n}}$ we constructed on $Y_{\bar{n}}$ is compatible with a universally tight contact structure. This fact also follows by Lemma 10 below.*

Lemma 10. *The open book $\mathcal{OB}_{\bar{n}}$ on $Y_{\bar{n}}$ is a Milnor open book.*

Proof. According to [21, Theorem 2.1], there is an analytic structure (Z, p) on the cone over $Y_{\bar{n}}$ and a corresponding Milnor open book $\widehat{\mathcal{OB}}_{\bar{n}}$ on $Y_{\bar{n}}$ whose binding agrees with the binding of $\mathcal{OB}_{\bar{n}}$. Now the circular plumbing graph $\Gamma_{\bar{n}}$ provides a decomposition of $Y_{\bar{n}}$ into a union $\bigcup V_i$, where V_i is an S^1 -bundle over S^2 with 2 discs removed. Since any page of $\mathcal{OB}_{\bar{n}}$ intersects in V_i in exactly one component for each i , by the argument of the proof of Theorem 4.6 of [6], any horizontal open book whose binding agrees with the binding of $\widehat{\mathcal{OB}}_{\bar{n}}$ must be isomorphic to $\widehat{\mathcal{OB}}_{\bar{n}}$. Thus $\mathcal{OB}_{\bar{n}}$ is indeed a Milnor open book. \square

Proposition 11. *There are precisely two distinct isotopy classes of Stein fillable universally tight contact structures on $Y_{\bar{n}}$. One of them is the contact structure induced on the boundary of the Stein surface depicted in Figure 6 where each Legendrian 2-handle attains its minimal possible rotation number. The other one is obtained, similarly, by achieving the maximal possible rotation number for each Legendrian 2-handle. Both of these contact structures represent the isomorphism class of the canonical contact structure ξ_{can} on $Y_{\bar{n}}$.*

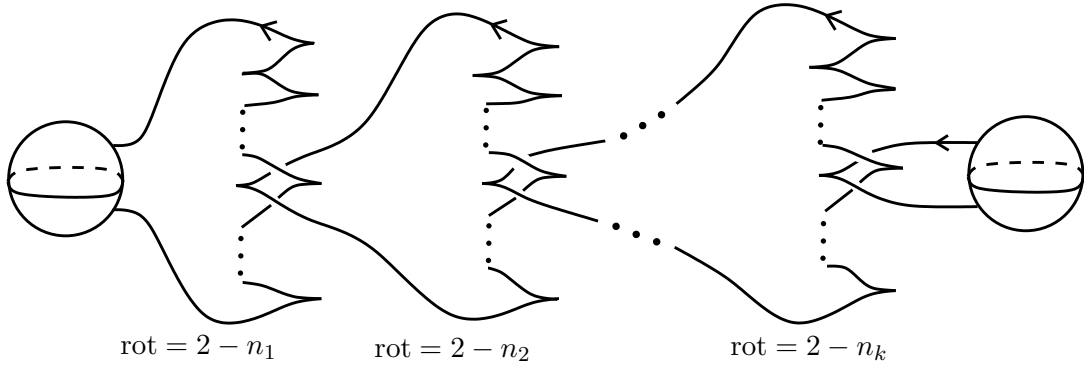
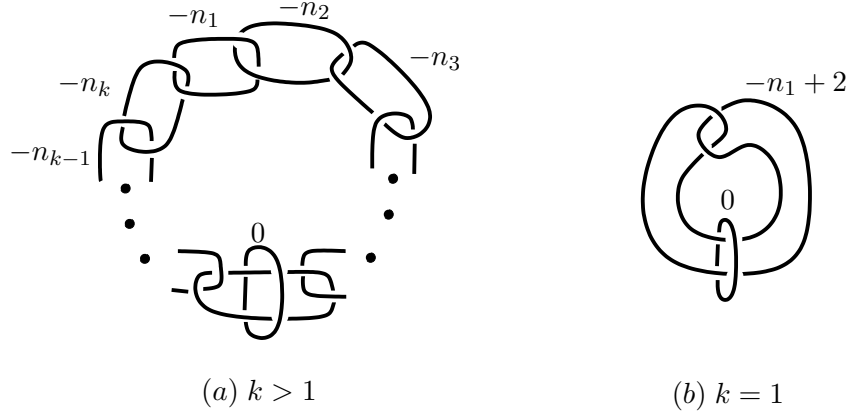


FIGURE 6. Legendrian handlebody diagram of a Stein filling of the contact 3-manifold $(Y_{\bar{n}}, \xi_{can})$.

Proof. On the torus bundle $Y_{\bar{n}}$, there are $(n_1 - 1)(n_2 - 1) \cdots (n_k - 1)$ distinct isotopy classes of Stein fillable contact structures two of which are universally tight according to Honda's classification [14]. Each of these contact structures can be realized as the boundary of some Stein surface as follows: Consider the Dehn surgery description of the 3-manifold $Y_{\bar{n}}$ depicted in Figure 7(a) and (b), corresponding to the cases $k > 1$ and $k = 1$, respectively (cf. [20]). By replacing the 0-framed unknot linking the chain with a dotted circle—which corresponds to surgering the corresponding 2-handle into a 1-handle (see, for example, page 168 in [13])—one obtains a handlebody diagram of a smooth 4-manifold whose boundary is diffeomorphic to $Y_{\bar{n}}$. We opt to represent the unique 1-handle in the diagram by a pair of spheres aligned horizontally.

FIGURE 7. Surgery description of $Y_{\bar{n}}$

Next we Legendrian realize all the unknots in this diagram including the one that goes over the 1-handle twice with zero linking. For the case $k > 1$, it is easy to see that the unknot with framing $-n_i \leq -2$ can be realized as a Legendrian unknot whose rotation number belongs to the set

$$\{2 - n_i, 4 - n_i, \dots, n_i - 4, n_i - 2\}$$

of $n_i - 1$ elements, by putting zigzags to the right and left alternatively. Similarly, for the case $k = 1$, one can check that the unknot with framing $-n_1 + 2$ which goes over the 1-handle twice with zero linking can be realized as a Legendrian unknot whose rotation number belongs to the set

$$\{2 - n_1, 4 - n_1, \dots, n_1 - 4, n_1 - 2\}$$

of $n_1 - 1$ elements, again by putting zigzags to the right and left alternatively. As a consequence, the finite number of isotopy classes of Stein fillable contact structures on $Y_{\bar{n}}$ can be realized as boundaries of certain Stein surfaces by considering all possible rotation numbers for each Legendrian unknot in the above described surgery diagram.

We claim that the canonical contact structure must be the one where the rotation numbers of all the Legendrian unknots are minimized as in Figure 6 (or the one where the rotation numbers of all the Legendrian unknots are maximized). This claim can be proved using the argument we used in [4, Proposition 9.1]; we provide details below for the convenience of the reader.

Fix an analytic structure (X, p) on the cone over $Y_{\bar{n}}$ and note that the minimal resolution $\pi: \tilde{X} \rightarrow X$ provides a holomorphic filling (W, J) of $(Y_{\bar{n}}, \xi_{can})$. In particular, W is a regular neighborhood of the exceptional divisor $E = \bigcup E_j$ of π , the dual graph of which is just the circular plumbing graph $\Gamma_{\bar{n}}$. Since the curves E_j are holomorphic, by the adjunction

formula, we have

$$(1) \quad \langle c_1(J), [E_j] \rangle = E_j \cdot E_j - 2 \operatorname{genus}(E_j) + 2 = E_j \cdot E_j + 2.$$

Now using a result of Bogomolov [5], deform the complex structure J so that (W, J') becomes a Stein surface, possibly after blowing down some (-1) -curves. Since W contains no topologically embedded spheres of self-intersection number -1 , (W, J') itself must be Stein. Note that (1) must continue to hold for J' even though the curves E_j are no longer holomorphic, since J and J' are homotopic to one another.

Now let $\{(W^i, J^i)\}$, for $i = 1, \dots, (n_1 - 1)(n_2 - 1) \cdots (n_k - 1)$, denote the finite set of Stein fillable contact structures on $Y_{\bar{n}}$ considered above by taking Legendrian realizations of the diagram in Figure 7. Denote by U_j^i a component of the corresponding Legendrian link and let S_j^i denote the associated surface in the Stein filling (W^i, J^i) obtained by pushing a Seifert surface for U_j^i into the 4-ball union 1-handle and capping off by the core of the corresponding 2-handle (see [12]). Notice that each W^i is diffeomorphic to W by a diffeomorphism which carries S_j^i to E_j for each j (see [13]).

Now, using the well-known identities

$$S_j^i \cdot S_j^i = \operatorname{tb}(U_j^i) - 1, \quad \langle c_1(J^i), [S_j^i] \rangle = \operatorname{rot}(U_j^i)$$

(see [12] for the second), observe that $\langle c_1(J^i), [S_j^i] \rangle = S_j^i \cdot S_j^i + 2$ precisely when $\operatorname{rot}(U_j^i) = \operatorname{tb}(U_j^i) + 1$. Since the latter equality holds exactly when all the cusps of U_j^i except one are up cusps, it follows that $\langle c_1(J), [E_j] \rangle = \langle c_1(J^i), [S_j^i] \rangle$ for each j precisely when all the extra zigzags are chosen so that the additional cusps are all up cusps, that is, when all the extra zigzags are chosen on the same fixed side (which is determined by the orientation of the Legendrian unknots). The proof is now completed by appealing to Lisca–Matić [18] and noting that in the finite list of Stein fillable contact structures on $Y_{\bar{n}}$ there is only one Stein fillable contact structure up to isomorphism that comes from a Legendrian realization as above where all the extra zigzags are on the same fixed side. □

The 1-handle in the handlebody diagram depicted in Figure 6 can be replaced by a contact $(+1)$ -surgery along a Legendrian unknot as described in [8] to obtain a contact surgery diagram of ξ_{can} on $Y_{\bar{n}}$ as shown in Figure 8. Notice that the Euler class of ξ_{can} vanishes by Proposition 1 although its Stein filling depicted in Figure 6 has non-vanishing first Chern class.

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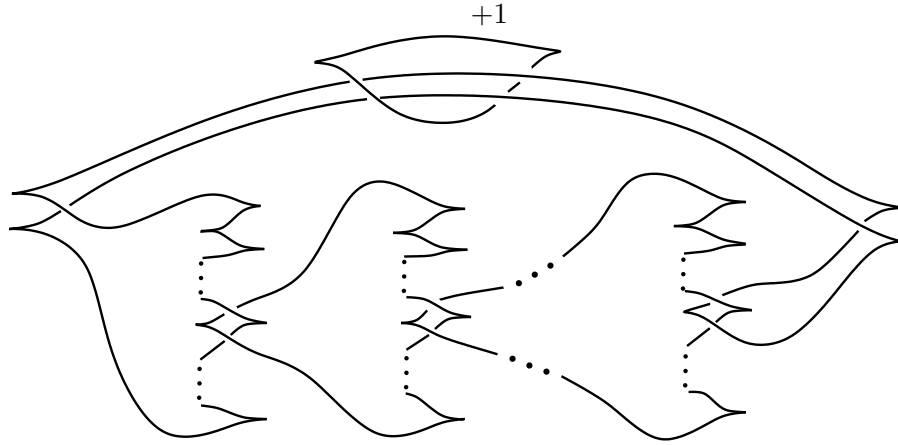


FIGURE 8. A contact surgery diagram for the canonical contact structure ξ_{can} on the link $Y_{\bar{n}}$ of a cusp singularity.

REFERENCES

- [1] S. Akbulut and B. Ozbagci, *Lefschetz fibrations on compact Stein surfaces*, *Geom. Topol.* 5 (2001), 319–334.
- [2] S. Akbulut and K. Yasui, *Infinitely many small exotic Stein fillings*, preprint, arXiv:1208.1053
- [3] M. Bhupal, *Open book decompositions of links of simple surface singularities*, *Internat. J. Math.*, 20 (12) (2009), 1527–1545.
- [4] M. Bhupal and B. Ozbagci, *Milnor open books of links of some rational surface singularities*, *Pacific. J. Math.* 254 (2011), 47–65.
- [5] F. A. Bogomolov and B. de Oliveira, *Stein small deformations of strictly pseudoconvex surfaces*, *Birational algebraic geometry (Baltimore, MD, 1996)*, 25–41, *Contemp. Math.*, 207, Amer. Math. Soc., Providence, RI, 1997.
- [6] C. Caubel, A. Némethi, and P. Popescu-Pampu, *Milnor open books and Milnor fillable contact 3-manifolds*, *Topology* 45 (2006), no. 3, 673–689.
- [7] F. Ding and H. Geiges, *A Legendrian surgery presentation of contact 3-manifolds*, *Proc. Cambridge Philos. Soc.* 136 (2004), 583–598.
- [8] F. Ding and H. Geiges, *Handle moves in contact surgery diagrams*, *J. Topol.* 2 (2009), no. 1, 105–122.
- [9] A. Durfee, *The signature of smoothings of complex surface singularities*, *Math. Ann.* 232 (1978), 85–98.
- [10] T. Etgü and B. Ozbagci, *Explicit horizontal open books on some plumbings*, *Internat. J. Math.*, 17 (9) (2006), 1013–1031.
- [11] David T. Gay, *Four-dimensional symplectic cobordisms containing three-handles*, *Geom. Topol.* 10 (2006), 1749–1759 (electronic).
- [12] R. E. Gompf, *Handlebody construction of Stein surfaces*, *Ann. of Math. (2)*, 148 (1998) no. 2, 619–693.

- [13] R. E. Gompf, A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999.
- [14] K. Honda, *On the classification of tight contact structures. II*, J. Differential Geom. 55 (2000), no. 1, 83–143.
- [15] M. Korkmaz and B. Ozbagci, *On sections of elliptic fibrations*, Michigan Math. J. 56 (2008), 77–87.
- [16] H. Laufer, *On minimally elliptic singularities*, Amer. J. Math. 99 (1977), 1257–1295.
- [17] Y. Lekili and B. Ozbagci, *Milnor fillable contact structures are universally tight*, Math. Res. Lett. 17 (2010), no. 6, 1055–1063.
- [18] P. Lisca and G. Matić, *Tight contact structures and Seiberg-Witten invariants*, Invent. Math. 129 (1997), no. 3, 509–525.
- [19] A. Némethi, *On the canonical contact structure of links of complex surface singularities*, Proceedings of the Geometry of Singularities and Manifolds, Kusatsu 2008, (2009), 99–118.
- [20] W. D. Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. 268 (1981), no. 2, 299–344.
- [21] W. D. Neumann and A. Pichon, *Complex analytic realization of links*, in Intelligence of Low Dimensional Topology, Eds. J. Scot Carter *et al.* World Scientific Publishing Co. (2006), 231–238.
- [22] H. Ohta and K. Ono, *Symplectic fillings of the link of simple elliptic singularities*, J. Reine und Angew. Math. 565 (2003), 183–205.
- [23] H. Pinkham, *Deformations of algebraic varieties with G_m action*, Astérisque, No. 20. Société Mathématique de France, Paris, 1974.
- [24] P. Popescu-Pampu, *Numerically Gorenstein surface singularities are homeomorphic to Gorenstein ones*, Duke Math. J. 159 (2011), 539–559.
- [25] J. Seade, *A cobordism invariant for surface singularities*, Proc. Symp. Pure Math. 40, Part 2 (1983), 479–484.
- [26] J. Van Horn Morris, *Constructions of open book decompositions*, Ph.D Dissertation, UT Austin, 2007.

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